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Logarithmic Interpolation Spaces between Quasi-Banach Spaces

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To the memory of Professor Miguel de Guzmán

Abstract. Let A_0 and A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. By means of a direct approach, we show that the interpolation spaces on (A_0, A_1) generated by the function parameter $t^\theta(1 + |\log t|)^{-b}$ can be expressed in terms of classical real interpolation spaces. Applications are given to Zygmund spaces $L_p(\log L)_b(\Omega)$, Lorentz-Zygmund function spaces and operator spaces defined by using approximation numbers.

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1. Introduction

In 1993, Triebel [31] studied the degree of compactness of the embedding from the (fractional) Sobolev space $H_p^{n/p}(\Omega)$ into the Orlicz space $L_\infty(\log L)_b(\Omega)$. Here Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $1 < p < \infty$ and $b < \frac{1}{p} - 1$. The investigation of this limiting case of the well known Sobolev embedding theorem goes back to Trudinger [33] and Strichartz [29]. The "L_p-counterpart" to the "L_∞-case" considered by Triebel, was studied by Edmunds

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and Triebel [9, 10], where they determined the behaviour of entropy numbers of the embedding from $H^s_{np/(n+sp)}(\Omega)$ into the Zygmund space $L_p(\log L)_b(\Omega)$.

A basic tool in the approach of Edmunds and Triebel is a representation theorem of Zygmund spaces $L_p(\log L)_b(\Omega)$ in terms of $L_p(\Omega)$ spaces. This characterization has intrinsic interest and has led Edmunds and Triebel to introduce in [9, 10] the so-called logarithmic Sobolev spaces, and to study in [11] the abstract construction that comes up replacing in the representation spaces $L_p(\Omega)$ by complex interpolation spaces. They called logarithmic interpolation spaces to the spaces defined in this way.

More recently, Triebel and the first two present authors [7] have investigated a similar construction but now based on the real interpolation spaces $(A_0, A_1)_{\theta,q}$. In this case, it turns out that logarithmic spaces coincide with those spaces obtained by real interpolation with the function parameter $t^\theta(1 + |\log t|)^{-b}$. As a consequence they have established representation theorems for Zygmund spaces $L_p(\log L)_b(\Omega)$ in terms of Lorentz spaces $L_{r,s}(\Omega)$, and characterizations for Lorentz-Zygmund operator spaces $\mathcal{L}_{p,q,b}(H)$ in terms of Lorentz operator spaces $\mathcal{L}_{r,s}(H)$. Here H is a Hilbert space.

The results of [7] refer to the Banach case. They do not apply to spaces $L_p(\log L)_b(\Omega)$ for $0 < p < 1$, and they do not cover the extension of $\mathcal{L}_{p,q,b}(H)$ to operator spaces on Banach spaces, because operator spaces defined in terms of approximation numbers are only quasi-Banach spaces, even if $1 < p, q < \infty$. To accomplish these results one should study logarithmic interpolation spaces in the class of quasi-Banach spaces.

From the point of view of extrapolation theory, logarithmic spaces are special cases of the more general notion of "one-sided" $\Sigma^{(p)-}$ and $\delta^{(p)-}$ spaces in the sense of Karadzhov and Milman [19]. In that recent paper (see also [13]) it is given an extensive study of the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods of extrapolation, complementing the previous work of Jawerth and Milman [18, 22] which deals mainly with the $\Sigma^{(1)}$ and $\Delta^{(\infty)}$ methods. Since results of [19] work for quasi-Banach spaces, one can apply them to derive results on abstract and concrete logarithmic interpolation spaces. In particular, Theorems 4.4 and 4.7 of [19] show that representation theorems for $L_p(\log L)_b(\Omega)$ in terms of spaces $L_{r,s}(\Omega)$ hold for the full range of parameters.

In this paper we study quasi-Banach logarithmic interpolation spaces by following a direct approach, based on ideas of [7]. We start by showing that logarithmic spaces generated by quasi-Banach couples coincide also with interpolation spaces obtained by using function parameters. Then we investigate the role of the scalar parameter q of real interpolation in logarithmic spaces. The value of q is the same for all real interpolation spaces that appear in the definition of logarithmic spaces (see Definition 2.1 below) and it coincides with the power of the summation over j as well. This is a help for computations but it is also the reason why representation theorems of [7] for $L_p(\log L)_b(\Omega)$ are given in terms of Lorentz spaces, instead of the simpler Lebesgue spaces. We show here that the construction of logarithmic spaces is sufficiently flexible to allow certain changes of q with the summing index j. As a consequence, applying the abstract results to Zygmund spaces $L_p(\log L)_b(\Omega)$, we derive representations that only require Lebesgue spaces and that work for $0 < p < 1$ as well.

Moreover, we apply the abstract results to spaces $\mathcal{L}_{p,q,b}(E,F)$, formed by all operators T acting between the quasi-Banach spaces E and F, whose approximation numbers $\{a_m(T)\}$ lie in the Lorentz-Zygmund sequence space $\ell_{p,q}(\log \ell)_b$ (see [5] and [6]). Spaces $\mathcal{L}_{p,q,b}(E,F)$ are the natural extension of $\mathcal{L}_{p,q,b}(H)$. Some results on bounded linear maps between spaces $\mathcal{L}_{p,q,b}(E,F)$ are also established. This kind of application is not considered in [19]. It is also not covered by the results of [13].

The organization of the paper is as follows. In Section 2 we study logarithmic interpolation spaces in the quasi-Banach setting. Section 3 deals with the applications to function spaces. Finally, in Section 4, we give the applications to operator spaces.

2. Logarithmic interpolation spaces

Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$, where the notation $\hookrightarrow$ means continuous inclusion. The *Peetre's K-functional* and *J-functional* are defined by

$$\begin{aligned} K(t,a) &= K(t,a;A_0,A_1) \\ &= \inf\left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} \,:\, a = a_0 + a_1,\ a_j \in A_j\right\}, \quad t>0,\ a \in A_1, \end{aligned}$$

and

$$J(t,a) = J(t,a;A_0,A_1) = \max\left\{\|a\|_{A_0}, t\|a\|_{A_1}\right\}, \qquad t>0,\ a\in A_0.$$

For $0<\theta<1$ and $0<q\le\infty$, the real interpolation space $A_{\theta,q} = (A_0,A_1)_{\theta,q}$ is formed by all those elements $a \in A_1$ having a finite quasi-norm

$$\|a\|_{A_{\theta,q}} = \begin{cases} \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0<q<\infty \\ \sup\limits_{t>0}\{t^{-\theta}K(t,a)\} & \text{if } q=\infty \end{cases}$$

(see [4] and [30]). It is well known that the equivalence theorem still holds in the quasi-Banach setting (see [4, Theorem 3.11.3]), so $A_{\theta,q}$ can be equivalently realized as a J-space.

Using that $A_0 \hookrightarrow A_1$, it is not hard to check that $\|a\|_{A_{\theta,q}}$ is equivalent to any of the following quasi-norms:

$$\Big(\sum_{m=1}^{\infty} 2^{-\theta mq} K^q(2^m,a)\Big)^{\frac{1}{q}}, \quad \inf\Big\{\Big(\sum_{m=1}^{\infty} 2^{-\theta mq} J^q(2^m,a_m)\Big)^{\frac{1}{q}} : a=\sum_{m=1}^{\infty} a_m\Big\}$$

(with the usual modification if $q=\infty$), where the infimum is extended over all representations $a=\sum_{m=1}^{\infty} a_m$ (convergence in A_1), with $a_m \in A_0$ and $\big(\sum_{m=1}^{\infty} 2^{-\theta mq} J^q(2^m,a_m)\big)^{\frac{1}{q}} < \infty$. Constants in equivalences depend on θ and q, but if θ runs on a compact subset of $(0,1)$, say

$$\theta \in \{\eta + 2^{-j} : j \geq j_0\} \cup \{\eta - 2^{-j} : j \geq j_0\} \cup \{\eta\}$$

as it is the case in Definition 2.1, then it is possible to choose uniform constants for all those values of θ. Subsequently, we denote any of these three quasi-norms by the symbol $\|\cdot\|_{A_{\theta,q}}$. This will cause no confusion.

Replacing in the definition of $A_{\theta,q}$ the function t^θ by a more general function parameter $\varrho(t)$ we obtain the spaces $A_{\varrho;q}=(A_0,A_1)_{\varrho;q}$ that have been studied in [24, 16, 17] or [25]. We will mainly work here with the special function parameters

$$\varrho(t)=\varrho_{\theta,b}(t)=t^\theta(1+|\log t|)^{-b}, \quad t>0,$$

where $0<\theta<1$ and $b \in \mathbb{R}$. Again, we have

$$\begin{aligned}\|a\|_{A_{\varrho;q}} &= \Big(\int_0^\infty \Big(\frac{K(t,a)}{\varrho(t)}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\ &\sim \Big(\sum_{m=1}^{\infty} \frac{K^q(2^m,a)}{\varrho^q(2^m)}\Big)^{\frac{1}{q}} \\ &\sim \inf\left\{\Big(\sum_{m=1}^{\infty} \frac{J^q(2^m,a_m)}{\varrho^q(2^m)}\Big)^{\frac{1}{q}} : \begin{array}{c} a=\sum_{m=1}^{\infty} a_m \text{ with } \{a_m\}\subseteq A_0 \text{ and} \\ \Big(\sum_{m=1}^{\infty} \frac{J^q(2^m,a_m)}{\varrho^q(2^m)}\Big)^{\frac{1}{q}} < \infty \end{array}\right\}.\end{aligned}$$

Here $\sim$ means equivalence of quasi-norms.

Since $A_0 \hookrightarrow A_1$, we have for $0<p,q\leq\infty$

$$(A_0,A_1)_{\mu,p} \hookrightarrow (A_0,A_1)_{\theta,q} \quad \text{if } 0<\mu<\theta<1 \qquad (1)$$

(see [4, Theorem 3.4.1]). Let $A_{\theta+} = \bigcap_{\theta<\eta<1} A_{\eta,q}$, where $0<q\leq\infty$ and $0<\theta<1$. By (1), the space $A_{\theta+}$ is independent of q.

We shall now introduce *logarithmic interpolation spaces* in the quasi-Banach case by extending the definition of [7]. We denote by $\mathbb{N}$ the collection of all natural numbers.

Definition 2.1. Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < \theta < 1$ and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$,

$$\sigma_j = \theta + 2^{-j} < 1 \quad \text{and} \quad \lambda_j = \theta - 2^{-j} > 0.$$

Let $0 < q \leq \infty$.

(i) Assume $b < 0$. We let $A_{\theta,q}(\log A)_b$ denote the space of all $a \in A_{\theta+}$ which have a finite quasi-norm

$$\|a\|_{A_{\theta,q}(\log A)_b} = \left(\sum_{j=j_0}^{\infty} 2^{jbq} \, \|a\|_{A_{\sigma_j,q}}^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$).

(ii) Let $b > 0$. The space $A_{\theta,q}(\log A)_b$ consists of all $a \in A_1$ which can be represented as

$$a = \sum_{j=j_0}^{\infty} a_j, \quad \text{convergence in } A_1, \text{ with } a_j \in A_{\lambda_j,q} \tag{2}$$

such that $\left(\sum_{j=j_0}^{\infty} 2^{jbq} \, \|a_j\|_{A_{\lambda_j,q}}^q \right)^{\frac{1}{q}} < \infty$. We put

$$\|a\|_{A_{\theta,q}(\log A)_b} = \inf \left\{ \left(\sum_{j=j_0}^{\infty} 2^{jbq} \, \|a_j\|_{A_{\lambda_j,q}}^q \right)^{\frac{1}{q}} \right\}$$

where the infimum is taken over all sequences $\{a_j\}$ satisfying (2).

(iii) If $b = 0$, then $A_{\theta,q}(\log A)_b = A_{\theta,q}$.

Next we show that in the quasi-Banach setting the equality between logarithmic interpolation spaces and real interpolation spaces generated by function parameters $\varrho_{\theta,b}$ still holds. This result follows from [19, Theorems 4.2 and 4.6], because $A_{\theta,q}(\log A)_b$ can be realized as a $\delta^{(p)-}$ extrapolation space for $b < 0$ and as a $\Sigma^{(p)-}$ space for $b > 0$. However we prefer to give a direct and simpler proof, following the main lines of [7, Theorem 1].

Theorem 2.2. *Let $0 < q \leq \infty$, $0 < \theta < 1$ and $b \in \mathbb{R}$. Let $\varrho_{\theta,b}(t) = t^{\theta}(1 + |\log t|)^{-b}$, $t > 0$. Then we have, with equivalent quasi-norms,*

$$A_{\theta,q}(\log A)_b = A_{\varrho_{\theta,b};q}.$$

Proof. The proof of the case $b \leq 0$ goes through as in the Banach case (see [7, Theorem 1/Step 1]), because triangle inequality is not used there. To establish the case $b > 0$, however, we have to modify the argument given in [7]. Assume therefore that $b > 0$. Take any $a \in A_{\theta,q}(\log A)_b$ and suppose that $0 < q < \infty$. Given any $\varepsilon > 0$, we can find a representation $a = \sum_{j=j_0}^{\infty} a_j$ with $a_j \in A_{\lambda_j,q}$ and

$$\sum_{j=j_0}^{\infty} 2^{jbq} \, \|a_j\|_{A_{\lambda_j,q}}^q \leq (1+\varepsilon) \|a\|_{A_{\theta,q}(\log A)_b}^q.$$

Choose now decompositions $a_j = \sum_{m=1}^{\infty} a_j^m$, $j \ge j_0$, such that $\{a_j^m\} \subseteq A_0$ and

$$\sum_{m=1}^{\infty} 2^{-mq(\theta-2^{-j})} J^q(2^m, a_j^m) \le (1+\varepsilon)\|a_j\|^q_{A_{\lambda_j,q}}.$$

Let c_j be the constant in the triangle inequality of A_j ($j = 0, 1$), put $c = \max\{c_0, c_1\}$ and define r by the formula $(2c)^r = 2$. We can suppose that the c_j are large, so that $r < q$. Let $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. By Hölder's inequality, we have

$$\begin{aligned}
&\Big(\sum_{j=j_0}^{\infty} J(2^m, a_j^m)^r\Big)^{\frac{1}{r}} \\
&\quad \le \Big(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})+jbq} J^q(2^m, a_j^m)\Big)^{\frac{1}{q}} \Big(\sum_{j=j_0}^{\infty} 2^{ms(\theta-2^{-j})-jbs}\Big)^{\frac{1}{s}} \\
&\quad \sim 2^{m\theta} m^{-b} \Big(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})+jbq} J^q(2^m, a_j^m)\Big)^{\frac{1}{q}}, \qquad (3)
\end{aligned}$$

where the last equivalence follows by using that $b > 0$ (see (33) in [7]).

The sum in (3) is finite as the argument below shows. Since the quasi-norm $J(2^m, \cdot)$ is a c-norm, it follows from [4, Lemma 3.10.2] that $\sum_{j=j_0}^{\infty} a_j^m$ is convergent in A_0, say to a^m, with

$$J(2^m, a^m) \le C\, 2^{m\theta} m^{-b} \Big(\sum_{j=j_0}^{\infty} 2^{-mq(\theta-2^{-j})+jbq} J^q(2^m, a_j^m)\Big)^{\frac{1}{q}}.$$

Consequently, $a = \sum_{m=1}^{\infty} a^m$ with

$$\begin{aligned}
\|a\|^q_{A_{\varrho;q}} &\le \sum_{m=1}^{\infty} \frac{J^q(2^m, a^m)}{\varrho^q_{\theta,b}(2^m)} \\
&\sim \sum_{m=1}^{\infty} 2^{-m\theta q} m^{bq} J^q(2^m, a^m) \\
&\le C^q \sum_{j=j_0}^{\infty} 2^{jbq} \sum_{m=1}^{\infty} 2^{-mq(\theta-2^{-j})} J^q(2^m, a_j^m) \\
&\le C^q (1+\varepsilon) \sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,q}} \\
&\le C^q (1+\varepsilon)^2 \|a\|^q_{A_{\theta,q}(\log A)_b}.
\end{aligned}$$

This implies that $A_{\theta,q}(\log A)_b \hookrightarrow A_{\varrho_{\theta,b};q}$. The case $q = \infty$ can be treated analogously.

The converse embedding can be checked by using the same argument as in [7, Theorem 1/Step 2] . Suppose $q < \infty$. The proof when $q = \infty$ can be carried out in the same way. Let $a \in A_{\varrho_{\theta,b};q}$ and take any representation $a = \sum_{m=1}^{\infty} a_m$ with $\{a_m\} \subseteq A_0$ and $\sum_{m=1}^{\infty} \frac{J^q(2^m, a_m)}{\varrho^q_{\theta,b}(2^m)} < \infty$. Put

$$a^j = \sum_{m=2^{j-j_0}}^{2^{j-j_0+1}-1} a_m \quad \text{for } j \geq j_0.$$

Then we have $a^j \in A_{\lambda_j,q}$, $a = \sum_{j=j_0}^{\infty} a^j$ and

$$\begin{aligned} \|a\|^q_{A_{\theta,q}(\log A)_b} &\leq \sum_{j=j_0}^{\infty} 2^{jbq} \|a^j\|^q_{A_{\lambda_j,q}} \\ &\leq \sum_{j=j_0}^{\infty} 2^{jbq} \sum_{m=2^{j-j_0}}^{2^{j-j_0+1}-1} 2^{-mq(\theta-2^{-j})} J^q(2^m, a_m) \\ &\sim \sum_{m=1}^{\infty} 2^{-mq\theta} m^{bq} J^q(2^m, a_m). \end{aligned}$$

This yields that $A_{\varrho_{\theta,b};q} \hookrightarrow A_{\theta,q}(\log A)_b$ and finishes the proof. □

In the proof of Theorem 2.2, it has been a help that summation over j in Definition 2.1 is taken to the same power q as in the spaces $A_{\sigma_j,q}$ and $A_{\lambda_j,q}$. However, in applications to concrete couples we shall need that q changes with j. Next we show that the choices of q that we shall take later generate the same logarithmic interpolation spaces. More general results valid for $\Sigma^{(p)}$ and $\Delta^{(p)}$ extrapolation spaces can be found in [19, Theorems 2.13 and 3.4].

Theorem 2.3. *Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $0 < \theta < 1$ and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$, $j \geq j_0$, $\sigma_j = \theta + 2^{-j} < 1$. Let $b < 0$, $0 < q \leq \infty$ and $r > 0$. Put $\frac{1}{s_j} = \frac{1}{q} + \frac{1}{r2^j}$, $j \geq j_0$. Then $A_{\theta,q}(\log A)_b$ consists of all $a \in A_{\theta+}$ such that*

$$\|a\|_* = \left(\sum_{j=j_0}^{\infty} 2^{jbq} \|a\|^q_{A_{\sigma_j,s_j}} \right)^{\frac{1}{q}} < \infty.$$

Moreover, the quasi-norms $\|\cdot\|_{A_{\theta,q}(\log A)_b}$ and $\|\cdot\|_*$ are equivalent.

Proof. Since $s_j < q$ for any $j \geq j_0$, we have

$$\|a\|_{A_{\sigma_j,q}} = \left(\sum_{m=1}^{\infty} 2^{-mq\sigma_j} K^q(2^m, a) \right)^{\frac{1}{q}} \leq \left(\sum_{m=1}^{\infty} 2^{-ms_j\sigma_j} K^{s_j}(2^m, a) \right)^{\frac{1}{s_j}} = \|a\|_{A_{\sigma_j,s_j}}.$$

Hence, given any $a \in A_{\theta+}$, we get

$$\|a\|_{A_{\theta,q}(\log A)_b} \leq \left(\sum_{j=j_0}^{\infty} 2^{jbq} \|a\|^q_{A_{\sigma_j,s_j}} \right)^{\frac{1}{q}} = \|a\|_*.$$

On the other hand, we claim that there exists $M > 0$ such that

$$\|a\|_{A_{\sigma_j,s_j}} \leq M \|a\|_{A_{\sigma_{j+1},q}} \quad \text{for all } j \geq j_0.$$

Indeed, using Hölder's inequality, we get

$$\begin{aligned} \|a\|_{A_{\sigma_j,s_j}} &= \left(\sum_{m=1}^{\infty} 2^{-ms_j\sigma_j} K^{s_j}(2^m, a) \right)^{\frac{1}{s_j}} \\ &\leq \left(\sum_{m=1}^{\infty} 2^{-mq\sigma_{j+1}} K^q(2^m, a) \right)^{\frac{1}{q}} \left(\sum_{m=1}^{\infty} 2^{mr2^j(\sigma_{j+1}-\sigma_j)} \right)^{\frac{1}{r2^j}} \\ &= \|a\|_{A_{\sigma_{j+1},q}} \left(\sum_{m=1}^{\infty} 2^{-mr/2} \right)^{\frac{1}{r2^j}} \\ &\leq M \|a\|_{A_{\sigma_{j+1},q}}. \end{aligned}$$

Consequently, for any $a \in A_{\theta+}$, we derive $\|a\|_* \leq M 2^{-b} \|a\|_{A_{\theta,q}(\log A)_b}$. □

The corresponding result for $b > 0$ reads as follows.

Theorem 2.4. *Let A_0, A_1 be quasi-Banach spaces with $A_0 \hookrightarrow A_1$. Let $b > 0$, $0 < \theta < 1$, $0 < q < \infty$, $r > 0$ and let $j_0 = j_0(\theta) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$, $j \geq j_0$, $\lambda_j = \theta - 2^{-j} > 0$ and $\frac{1}{r_j} = \frac{1}{q} - \frac{1}{r2^j} > 0$. Then $A_{\theta,q}(\log A)_b$ is formed by all those $a \in A_1$ which can be represented as $a = \sum_{j=j_0}^{\infty} a_j$, convergence in A_1, with $a_j \in A_{\lambda_j,r_j}$ and $\left(\sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,r_j}}\right)^{\frac{1}{q}} < \infty$. Moreover,*

$$\|a\|_{**} = \inf \left\{ \left(\sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,r_j}} \right)^{\frac{1}{q}} \right\}$$

is an equivalent quasi-norm in the space $A_{\theta,q}(\log A)_b$. Here the infimum is taken over all representations of the described type.

Proof. First we show that if $\sum_{j=j_0}^{\infty} 2^{jbq} \|a_j\|^q_{A_{\lambda_j,r_j}} < \infty$, then $\sum_{j=j_0}^{\infty} a_j$ is convergent in A_1. Let c be the constant in the triangle inequality of A_1 and define s by the formula $(2c)^s = 2$. We may assume that $s < q$. Put $\frac{1}{p} = \frac{1}{s} - \frac{1}{q}$. If $a \in A_{\lambda_j,r_j}$, we have

$$\|a\|_{A_1} \sim K(2, a) \leq 2^{\theta} \|a\|_{A_{\lambda_j,r_j}}.$$

Whence, applying Hölder's inequality, we obtain

$$\left(\sum_{j=j_0}^{\infty}\|a_j\|_{A_1}^s\right)^{\frac{1}{s}} \le 2^{\theta}\left(\sum_{j=j_0}^{\infty}2^{jbq}\|a_j\|_{A_{\lambda_j,r_j}}^q\right)^{\frac{1}{q}}\left(\sum_{j=j_0}^{\infty}2^{-jbp}\right)^{\frac{1}{p}} < \infty.$$

This yields that $\sum_{j=j_0}^{\infty} a_j$ is convergent in A_1.

Since $q < r_j$, it follows that $\|a\|_{**} \le \|a\|_{A_{\theta,q}(\log A)_b}$. To establish the converse inequality, we proceed as in Theorem 2.3. For any $j \ge j_0$ and any $a \in A_{\lambda_j,r_j}$, we get

$$\begin{aligned}\|a\|_{A_{\lambda_{j+1},q}} &= \left(\sum_{m=1}^{\infty}2^{-mq\lambda_{j+1}}K^q(2^m,a)\right)^{\frac{1}{q}}\\ &\le \left(\sum_{m=1}^{\infty}2^{-mr_j\lambda_j}K^{r_j}(2^m,a)\right)^{\frac{1}{r_j}}\left(\sum_{m=1}^{\infty}2^{m(\lambda_j-\lambda_{j+1})r2^j}\right)^{\frac{1}{r2^j}}\\ &= \|a\|_{A_{\lambda_j,r_j}}\left(\sum_{m=1}^{\infty}2^{-mr/2}\right)^{\frac{1}{r2^j}}\\ &\le M\|a\|_{A_{\lambda_j,r_j}}.\end{aligned}$$

This implies that $\|a\|_{A_{\theta,q}(\log A)_b} \le M2^b\|a\|_{**}$. □

Remark 2.5. Let $\frac{1}{s_j^*} = \frac{1}{q} - \frac{1}{r2^j}$ and $\frac{1}{r_j^*} = \frac{1}{q} + \frac{1}{r2^j}$. It is easy to check that Theorem 2.3 still holds for $q < \infty$ if we replace s_j by s_j^*. Similarly, Theorem 2.4 is also valid if we replace r_j by r_j^*.

Logarithmic spaces in the case $\theta = 0$ and $q = \infty$ will be also useful later. If A_0, A_1 are quasi-Banach spaces with $A_0 \hookrightarrow A_1$ and $b < 0$, we let $A_{0,\infty}(\log A)_b$ denote the space of all $a \in A_{0+}$ which have a finite quasi-norm

$$\|a\|_{A_{0,\infty}(\log A)_b} = \sup_{j\ge 1}\left\{2^{jb}\|a\|_{A_{2^{-j},\infty}}\right\}.$$

Here

$$\|a\|_{A_{2^{-j},\infty}} = \sup_{m\ge 1}\left\{2^{-2^{-j}m}K(2^m,a)\right\}.$$

We put $\varrho_{0,b}(t) = (1 + |\log t|)^{-b}$, $t > 0$, and we denote by $(A_0, A_1)_{\varrho_{0,b};\infty}$ the collection of all those $a \in A_1$ which have a finite quasi-norm

$$\|a\|_{A_{\varrho_{0,b};\infty}} = \sup_{t\ge 1}\left\{\frac{K(t,a)}{\varrho_{0,b}(t)}\right\}.$$

Theorem 2.6. *Let $b < 0$. Then we have, with equivalent quasi-norms,*

$$A_{0,\infty}(\log A)_b = A_{\varrho_{0,b};\infty}.$$

Proof. The result follows by using the same arguments as in Theorem 2.2. □

Remark 2.7. Spaces $A_{\theta,q}(\log A)_b$ might be considered as a quantitative counterpart to the notion of *inclusion indices* relative to an interpolation scale (see [12]).

3. Applications to function spaces

In this section we specialize the abstract results of Section 2 to Lorentz-Zygmund function spaces.

Let Ω be a domain in $\mathbb{R}^n$ with finite Lebesgue measure $|\Omega|$. For $0 < p < \infty$, $0 < q \leq \infty$ and $b \in \mathbb{R}$, the Lorentz-Zygmund function space $L_{p,q}(\log L)_b(\Omega)$ is formed by all (equivalent classes of) Lebesgue-measurable functions f on Ω which have a finite quasi-norm

$$\|f\|_{L_{p,q}(\log L)_b(\Omega)} = \left(\int_0^{|\Omega|} \left[t^{\frac{1}{p}} (1 + |\log t|)^b f^*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

(with the obvious modification if $q = \infty$). Here f^* is the non-increasing rearrangement of f

$$f^*(t) = \inf \big\{ s > 0 : \ |\{ x \in \Omega : \ |f(x)| > s \big\}| \leq t \}.$$

We refer to [2, 3] and [23] for properties of Lorentz-Zygmund function spaces. Note that if $p = q$, we get the Zygmund spaces $L_p(\log L)_b(\Omega)$. In particular, for $b = 0$ we obtain the Lebesgue spaces $L_p(\Omega)$. The case $b = 0$ and $p \neq q$ gives the Lorentz function spaces $L_{p,q}(\Omega)$.

The following result extends [10, Theorem 2.6.2/2] to the range $0 < p < 1$.

Corollary 3.1. *Let Ω be a domain in $\mathbb{R}^n$ with finite Lebesgue measure. Let $0 < p < \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$, $\frac{1}{p^{\lambda_j}} = \frac{1}{p} - \frac{1}{n 2^j} > 0$. Put $\frac{1}{p^{\sigma_j}} = \frac{1}{p} + \frac{1}{n 2^j}$.*

(i) *Let $b < 0$. Then $L_p(\log L)_b(\Omega)$ is the set of all Lebesgue-measurable functions f on Ω such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \, \|f\|^p_{L_{p^{\sigma_j}}(\Omega)} \right)^{\frac{1}{p}} < \infty. \tag{4}$$

Moreover, (4) defines an equivalent quasi-norm on $L_p(\log L)_b(\Omega)$.

(ii) *Let $b > 0$. Then $L_p(\log L)_b(\Omega)$ is the set of all Lebesgue-measurable functions f on Ω which can be represented as*

$$f = \sum_{j=j_0}^{\infty} f_j \,, \quad f_j \in L_{p^{\lambda_j}}(\Omega) \tag{5}$$

such that

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|f_j\|^p_{L_{p^{\lambda_j}}(\Omega)}\right)^{\frac{1}{p}} < \infty. \tag{6}$$

Moreover, the infimum over expression (6) *taken over all representations* (5), (6) *is an equivalent quasi-norm on* $L_p(\log L)_b(\Omega)$.

Proof. Take $0 < r < p$ and let $\theta = \frac{r}{p}$. Consider the spaces $L_\infty(\Omega)$ and $L_r(\Omega)$. Since $|\Omega| < \infty$, we have $L_\infty(\Omega) \hookrightarrow L_r(\Omega)$. According to [4, Theorem 5.2.1],

$$K(t, f; L_\infty(\Omega), L_r(\Omega)) \sim t\left(\int_0^{t^{-r}} (f^*(s))^r ds\right)^{\frac{1}{r}}. \tag{7}$$

Hence, interpolating with $\varrho_{\theta,b}(t) = t^\theta(1 + |\log t|)^{-b}, t > 0$, we get

$$(L_\infty(\Omega), L_r(\Omega))_{\varrho_{\theta,b};p} = L_p(\log L)_b(\Omega)$$

with equivalent quasi-norms.

Put $\frac{1}{p_*^{\sigma_j}} = \frac{1}{p} + \frac{1}{r2^j}$ and $\frac{1}{p_*^{\lambda_j}} = \frac{1}{p} - \frac{1}{r2^j}$. Then

$$(L_\infty(\Omega), L_r(\Omega))_{\theta+2^{-j}, p_*^{\sigma_j}} = L_{p_*^{\sigma_j}}(\Omega)$$

and

$$(L_\infty(\Omega), L_r(\Omega))_{\theta-2^{-j}, p_*^{\lambda_j}} = L_{p_*^{\lambda_j}}(\Omega)$$

with equivalence of quasi-norms where the constants do not depend on j. Whence, for $b < 0$, it follows from Theorems 2.2 and 2.3 that $L_p(\log L)_b(\Omega)$ is the set of all measurable functions f on Ω such that

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|f\|^p_{L_{p_*^{\sigma_j}}(\Omega)}\right)^{\frac{1}{p}} < \infty. \tag{8}$$

On the other hand, for $b > 0$, Theorems 2.2 and 2.4 imply that $L_p(\log L)_b(\Omega)$ consists of all measurable functions f on Ω which can be represented as

$$f = \sum_{j=j_0}^{\infty} f_j\,, \quad f_j \in L_{p_*^{\lambda_j}}(\Omega)$$

such that

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|f_j\|^p_{L_{p_*^{\lambda_j}}(\Omega)}\right)^{\frac{1}{p}} < \infty. \tag{9}$$

Finally, using Hölder's inequality and the fact that $|\Omega| < \infty$, it is not difficult to show that (8) and (4) are equivalent, and that the infimum over expression (9) is an equivalent quasi-norm to the one defined by (6). □

Part (i) in Corollary 3.1 was proved by Edmunds and Triebel in [10, Theorem 2.6.2/1] by direct calculations. Then, using duality, they derived part (ii) in Corollary 3.1 for $1 \leq p < \infty$ (see [10, Theorem 2.6.2/2]). Note that their technique does not allow to cover the case $0 < p < 1$, because in this range $(L_p(\Omega))' = \{0\}$.

If $p = \infty$, we can recover from the outcome for $\theta = 0$ the following result due to Triebel [31].

Corollary 3.2. *Let Ω be a domain in $\mathbb{R}^n$ with finite Lebesgue measure, let $b < 0$ and let $p^{\sigma_j} = n2^j$. Then $L_\infty(\log L)_b(\Omega)$ is the set of all Lebesgue-measurable functions f on Ω such that*

$$\sup_{j \geq 1} \left\{ 2^{jb} \|f\|_{L_{p^{\sigma_j}}(\Omega)} \right\} < \infty. \tag{10}$$

Moreover the expression in (10) *defines an equivalent norm on $L_\infty(\log L)_b(\Omega)$.*

Proof. Using (7) and Hardy's inequality (see [3, p. 246] or [2, Theorem 6.4]), we get

$$\begin{aligned}
\|f\|_{(L_\infty(\Omega), L_1(\Omega))_{\varrho_{0,b};\infty}} &= \sup_{t \geq 1} \left\{ \frac{K(t, f; L_\infty(\Omega), L_1(\Omega))}{(1 + |\log t|)^{-b}} \right\} \\
&= \sup_{t \geq 1} \left\{ (1 + |\log t|)^b \, t \int_0^{\frac{1}{t}} f^*(s)\, ds \right\} \\
&= \sup_{0 < t \leq 1} \left\{ (1 + |\log t|)^b \frac{1}{t} \int_0^t f^*(s) ds \right\} \\
&\sim \sup_{0 < t \leq 1} \left\{ (1 + |\log t|)^b f^*(t) \right\} \\
&= \|f\|_{L_\infty(\log L)_b(\Omega)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|f\|_{(L_\infty(\Omega), L_1(\Omega))_{2^{-j},\infty}} &= \sup_{m \geq 1} \left\{ 2^{-2^{-j} m} K(2^m, f) \right\} \\
&\sim \sup_{t > 0} \left\{ t^{1 - 2^{-j}} \int_0^{\frac{1}{t}} f^*(s) ds \right\} \\
&\sim \sup_{t > 0} \left\{ t^{2^{-j}} f^*(t) \right\},
\end{aligned}$$

where the constants in the equivalences do not depend on j. Since $|\Omega| < \infty$, for all $j \in \mathbb{N}$ and any $f \in L_{p^{\sigma_j}}(\Omega)$, we have

$$\sup_{t > 0} \left\{ t^{2^{-j}} f^*(t) \right\} \leq \max \left\{ 1, |\Omega|^{\frac{n-1}{2n}} \right\} \|f\|_{L_{p^{\sigma_j}}(\Omega)}.$$

Moreover, if $n < 2^{j_0}$, then we can find $M > 0$ such that for all $j \geq j_0$ and any $f \in L_{2^{j+j_0},\infty}(\Omega)$, we get

$$\|f\|_{L_{p^{\sigma_j}}(\Omega)} \leq M \sup_{t>0} \left\{ t^{2^{-(j+j_0)}} f^*(t) \right\}.$$

Now the result follows from Theorem 2.6. □

Next we show that [7, Corollary 2], holds for the full range of parameters. This result is due to Karadzhov and Milman [19, Theorems 4.4 and 4.7].

Corollary 3.3. *Let* Ω *be a domain in* $\mathbb{R}^n$ *with finite Lebesgue measure. Let* $0 < p < \infty$, $0 < q \leq \infty$ *and let* $j_0 = j_0(p) \in \mathbb{N}$ *such that, for all* $j \in \mathbb{N}$ *with* $j \geq j_0$, $\frac{1}{p^{\nu_j}} = \frac{1}{p} - 2^{-j} > 0$. *Put* $\frac{1}{p^{\mu_j}} = \frac{1}{p} + 2^{-j}$.

(i) *Let* $b < 0$. *Then* $L_{p,q}(\log L)_b(\Omega)$ *is the set of all measurable functions* f *on* Ω *such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbq} \, \|f\|^q_{L_{p^{\mu_j},q}(\Omega)} \right)^{\frac{1}{q}} < \infty$$

(equivalent norms).

(ii) *Let* $b > 0$. *Then* $L_{p,q}(\log L)_b(\Omega)$ *is the set of all measurable functions* f *on* Ω *which can be represented as*

$$f = \sum_{j=j_0}^{\infty} f_j \,, \quad f_j \in L_{p^{\nu_j},q}(\Omega) \tag{11}$$

such that

$$\left(\sum_{j=j_0}^{\infty} 2^{jbq} \, \|f_j\|^q_{L_{p^{\nu_j},q}(\Omega)} \right)^{\frac{1}{q}} < \infty. \tag{12}$$

The infimum of the expression in (12) *taken over all admissible representations* (11), (12) *is an equivalent quasi-norm in* $L_{p,q}(\log L)_b(\Omega)$.

Proof. The argument is similar to the one in the proof of Corollary 3.1. Take $0 < r < \min\{1, p, q\}$ and let $\theta = \frac{r}{p}$. Using (7), we have

$$(L_\infty(\Omega), L_r(\Omega))_{\varrho_{\theta,b};q} = L_{p,q}(\log L)_b(\Omega).$$

Put $\sigma_j = \theta + 2^{-j}$, $\lambda_j = \theta - 2^{-j}$, $\frac{1}{p^{\eta_j}} = \frac{1}{p} + \frac{1}{r2^j}$ and $\frac{1}{p^{\tau_j}} = \frac{1}{p} - \frac{1}{r2^j}$. Then

$$(L_\infty(\Omega), L_r(\Omega))_{\sigma_j,q} = L_{p^{\eta_j},q}(\Omega) \qquad \text{and} \qquad (L_\infty(\Omega), L_r(\Omega))_{\lambda_j,q} = L_{p^{\tau_j},q}(\Omega)$$

with equivalence of quasi-norms where the constants are independent of j. Since $\frac{1}{p^{\eta_j}} - \frac{1}{p^{\mu_j}} = \frac{1-r}{r2^j} \to 0$ as $j \to \infty$, we have

$$\begin{aligned}
\|f\|_{L_{p^{\eta_j},q}(\Omega)} &= \left(\int_0^{|\Omega|} \left(t^{\frac{1}{p^{\eta_j}}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\le \sup_{0<t<|\Omega|} \left\{ t^{\frac{1-r}{r2^j}} \right\} \left(\int_0^{|\Omega|} \left(t^{\frac{1}{p^{\mu_j}}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\le |\Omega|^{\frac{1-r}{r2^j}} \|f\|_{L_{p^{\mu_j},q}(\Omega)} \\
&\le M \|f\|_{L_{p^{\mu_j},q}(\Omega)}.
\end{aligned}$$

Similarly $\|f\|_{L_{p^{\nu_j},q}(\Omega)} \le M\|f\|_{L_{p^{\tau_j},q}(\Omega)}$. Let $j_1 > j_0$ with $1 < r2^{j_1}$. Since

$$\frac{1}{p^{\mu_j}} - \frac{1}{p^{\eta_{j_1+j}}} = \frac{1}{p^{\tau_{j_1+j}}} - \frac{1}{p^{\nu_j}} = \frac{r2^{j_1} - 1}{r2^{j_1+j}} \to 0 \quad \text{as} \quad j \to \infty,$$

we derive

$$\begin{aligned}
\|f\|_{L_{p^{\mu_j},q}(\Omega)} &\le M_1 \|f\|_{L_{p^{\eta_{j_1+j}},q}(\Omega)} \\
\|f\|_{L_{p^{\tau_{j_1+j}},q}(\Omega)} &\le M_1 \|f\|_{L_{p^{\nu_j},q}(\Omega)}.
\end{aligned}$$

Applying Theorem 2.2 to the couple $(L_\infty(\Omega), L_r(\Omega))$, we obtain a representation of $L_{p,q}(\log L)_b(\Omega)$ in terms of the Lorentz spaces $L_{p^{\eta_j},q}(\Omega)$ if $b < 0$, and in terms of $L_{p^{\tau_j},q}(\Omega)$ if $b > 0$. Then the result follows with the aid of the relationships between spaces $L_{p^{\eta_j},q}(\Omega)$ and $L_{p^{\mu_j},q}(\Omega)$, and between spaces $L_{p^{\tau_j},q}(\Omega)$ and $L_{p^{\nu_j},q}(\Omega)$, that we have shown before. □

Remark 3.4. Other kind of decompositions for Lorentz-Zygmund function spaces can be found in [8, 3.4.4] and the references given there.

Remark 3.5. As we have said in the Introduction, Corollary 3.1 is the basic tool for the estimates on entropy numbers derived in [9]. Another kind of applications can be found in the book by Edmunds and Triebel [10, Remark 5, pp. 74–75]. It refers to the Hardy-Littlewood maximal function

$$(Mf)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes Q containing x and with sides parallel to the coordinate axes. A classical result of Hardy and Littlewood says that if $f \in L_1(\log L)_1(\Omega)$, then $Mf \in L_1(\Omega)$. This assertion can be extended to $L_1(\log L)_{1+b}(\Omega)$ with $b \ge 0$. Indeed, see [28, p. 23] or [2, Theorem 3.4], it holds

$$\|Mf\|_{L_1(\log L)_b(\Omega)} \le c\, \|f\|_{L_1(\log L)_{b+1}(\Omega)}. \tag{13}$$

In [10, p. 75] one can find a simple proof of (13) by using Corollary 3.1.

Next consider the related operator

$$M_r f = \big[M(|f|^r)\big]^{\frac{1}{r}}$$

where $0 < r < 1$. This operator is useful in several situations (see, for example, [32, pp. 78, 108]). Inequality (13) yields that

$$M_r : L_r(\log L)_{b+1/r}(\Omega) \longrightarrow L_r(\log L)_b(\Omega)$$

is bounded. Indeed,

$$\begin{aligned}
\|M_r f\|_{L_r(\log L)_b(\Omega)} &= \left(\int_0^{|\Omega|} t\,(1+|\log t|)^{br} M(|f|^r)^*(t)\,\frac{dt}{t}\right)^{\frac{1}{r}}\\
&\le c\big\||f|^r\big\|^{\frac{1}{r}}_{L_1(\log L)_{br+1}(\Omega)}\\
&= c\bigg(\int_0^{|\Omega|}\Big(t^{\frac{1}{r}}(1+|\log t|)^{b+\frac{1}{r}} f^*(t)\Big)^r\,\frac{dt}{t}\bigg)^{\frac{1}{r}}\\
&= c\|f\|_{L_r(\log L)_{b+\frac{1}{r}}(\Omega)}.
\end{aligned}$$

4. Applications to operator spaces

In this final section we apply the abstract results of Section 2 to operator spaces defined in terms of approximation numbers.

Let E, F be quasi-Banach spaces and let $\mathcal{L}(E,F)$ be the quasi-Banach space of all bounded linear operators acting from E into F. For $k \in \mathbb{N}$, the k-th approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\big\{\,\|T-R\|\ :\ R \in \mathcal{L}(E,F)\ \text{ with }\ \operatorname{rank} R < k\big\}.$$

Let c be the constant in the triangle inequality of F and let s be defined by the equation $(2c)^s = 2$. It is easy to check that for $S, T \in \mathcal{L}(E,F)$ and $k, m \in \mathbb{N}$, it holds

$$a^s_{k+m-1}(S+T) \le a^s_k(S) + a^s_m(T). \tag{14}$$

For $0 < p < \infty$, $0 < q \le \infty$ and $b \in \mathbb{R}$, we define the Lorentz-Zygmund operator spaces $\mathcal{L}_{p,q,b}(E,F)$ as the collection of all those $T \in \mathcal{L}(E,F)$ having a finite quasi-norm

$$\|T\|_{p,q,b} = \left(\sum_{m=1}^{\infty}(m^{\frac{1}{p}}(1+\log m)^b a_m(T))^q m^{-1}\right)^{\frac{1}{q}} \tag{15}$$

(with the usual modification if $q = \infty$). In Banach spaces these operator spaces have been studied in [5] and [6]. For $b = 0$, we get the Lorentz operator spaces $(\mathcal{L}_{p,q}(E, F), \| \cdot \|_{p,q})$ (see [21] and [27]). The special case $b = 0$ and $p = q$ gives the spaces $(\mathcal{L}_p(E, F), \| \cdot \|_p)$, which are the analogues of the Schatten p-classes for approximation numbers (see [14] and [26]).

Approximation numbers coincide with singular numbers for operators in Hilbert space H. If $E = F = H$, $1 < p < \infty$, $1 \le q \le \infty$ and $b \in \mathbb{R}$, then the functional obtained from (15) replacing $a_m(T)$ by $m^{-1}\sum_{j=1}^{m} a_j(T)$ is a norm, equivalent to $\| \cdot \|_{p,q,b}$. The resulting Banach space was denoted by $\mathcal{L}_{p,q,b}(H)$ in [7], where it has been shown a representation theorem for $\mathcal{L}_{p,q,b}(H)$ in terms of spaces $\mathcal{L}_{p,q}(H)$. Next we establish the corresponding results for operators spaces in quasi-Banach spaces. We start with a result on the K-functional. In the Banach case, this was proved by König [20, Proposition 1].

Given two non-negative functions (or two sequences) $u(t)$, $v(t)$, we write $u(t) \lesssim v(t)$ if there is a positive constant c such that $u(t) \le cv(t)$ for all t. The equivalence $u(t) \sim v(t)$ holds if $u(t) \lesssim v(t)$ and $v(t) \lesssim u(t)$.

Lemma 4.1. *Let E, F be quasi-Banach spaces and let $0 < r < \infty$. Then*

$$K\left(t, T; \mathcal{L}_r(E, F), \mathcal{L}(E, F)\right) \sim K\left(t, \{a_m(T)\}; \ell_r, \ell_\infty\right).$$

Proof. It $t \le 1$, we have

$$K\left(t, T; \mathcal{L}_r(E, F), \mathcal{L}(E, F)\right) \sim t\,\|T\| = t\,\|\{a_m(T)\}\|_{\ell_\infty} = K\left(t, \{a_m(T)\}; \ell_r, \ell_\infty\right).$$

Suppose $t > 1$. Choose $T_t \in \mathcal{L}(E, F)$ such that $\operatorname{rank}(T_t) < [t^r]$ and $\|T - T_t\| \le 2a_{[t^r]}(T)$. Here $[\,\cdot\,]$ is the greatest integer function. If $m < [t^r]$, it follows from (14) that

$$a_m^s(T_t) \le a_m^s(T) + \|T - T_t\|^s \le (1 + 2^s)a_m^s(T).$$

If $m \ge [t^r]$, then $a_m(T_t) = 0$. Whence

$$\begin{aligned} K\left(t, T; \mathcal{L}_r(E, F), \mathcal{L}(E, F)\right) &\le \|T_t\|_r + t\,\|T - T_t\| \\ &\le c_1\Big(\sum_{m=1}^{[t^r]} a_m^r(T)\Big)^{\frac{1}{r}} + 2t\, a_{[t^r]}(T) \\ &\le c_1\Big(\sum_{m=1}^{[t^r]} a_m^r(T)\Big)^{\frac{1}{r}} + c_2\big([t^r]a_{[t^r]}^r(T)\big)^{\frac{1}{r}} \\ &\le c_3\Big(\sum_{m=1}^{[t^r]} a_m^r(T)\Big)^{\frac{1}{r}}. \end{aligned}$$

We claim that

$$K\left(t, T; \mathcal{L}_r(E,F), \mathcal{L}(E,F)\right) \sim \left(\sum_{m=1}^{[t^r]} a_m^r(T) \right)^{\frac{1}{r}}. \tag{16}$$

Indeed, take any $S \in \mathcal{L}_r(E,F)$ and let s be again the number appearing in (14). Without loss of generality we may assume that $s < r$. Using (14) and Minkowski's inequality we obtain

$$\begin{aligned} \left(\sum_{m=1}^{[t^r]} a_m^r(T) \right)^{\frac{1}{r}} &\sim \left(\sum_{m=1}^{[t^r]} a_{2m-1}^r(T) \right)^{\frac{1}{r}} \\ &\leq \left(\sum_{m=1}^{[t^r]} \left(a_m^s(S) + a_m^s(T-S) \right)^{\frac{r}{s}} \right)^{\frac{1}{r}} \\ &\leq \left(\left(\sum_{m=1}^{[t^r]} a_m^r(S) \right)^{\frac{s}{r}} + \left(\sum_{m=1}^{[t^r]} a_m^r(T-S) \right)^{\frac{s}{r}} \right)^{\frac{1}{s}} \\ &\leq c_4 \left(\left(\sum_{m=1}^{[t^r]} a_m^r(S) \right)^{\frac{1}{r}} + t \left\| T - S \right\| \right) \\ &\leq c_4 \left(\|S\|_r + t \left\| T - S \right\| \right). \end{aligned}$$

This yields (16). Now the result follows by using [4, Theorem 5.2.1]. □

Let $0 < p < \infty$, $0 < q \leq \infty$ and $b \in \mathbb{R}$. Take $0 < r < p$ and put $\theta = 1 - (\frac{r}{p})$. The quasi-Banach space $\mathcal{L}_r(E,F)$ is continuously embedded in $\mathcal{L}(E,F)$. By Lemma 4.1 and a similar argument as in [5, Theorem 5.2] we get

$$\left(\mathcal{L}_r(E,F), \mathcal{L}(E,F) \right)_{\varrho_{\theta,b};q} = \mathcal{L}_{p,q,b}(E,F).$$

Now we can establish

Corollary 4.2. *Let E and F be quasi-Banach spaces. Let $0 < p < \infty$, $0 < q \leq \infty$ and let $j_0 = j_0(p) \in \mathbb{N}$ such that, for all $j \in \mathbb{N}$ with $j \geq j_0$, $\frac{1}{p^{\nu_j}} = \frac{1}{p} - \frac{1}{2^j} > 0$. Put $\frac{1}{p^{\mu_j}} = \frac{1}{p} + \frac{1}{2^j}$.*

(i) *Let $b < 0$. Then $\mathcal{L}_{p,q,b}(E,F)$ is the set of all $T \in \mathcal{L}(E,F)$ such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbq} \left\| T \right\|_{p^{\nu_j},q}^q \right)^{\frac{1}{q}} < \infty$$

(equivalent quasi-norms).

(ii) *Let* $b > 0$. *Then* $\mathcal{L}_{p,q,b}(E,F)$ *consists of all* $T \in \mathcal{L}(E,F)$ *which can be represented as* $T = \sum_{j=j_0}^{\infty} T_j$ *with* $T_j \in \mathcal{L}_{p^{\mu_j},q}(E,F)$ *such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbq} \|T_j\|_{p^{\mu_j},q}^q\right)^{\frac{1}{q}} < \infty. \tag{17}$$

Furthermore, the infimum over expression (17) *is an equivalent quasi-norm in* $\mathcal{L}_{p,q,b}(E,F)$.

Proof. Take $0 < r < p$. Put $\theta = 1 - (\frac{r}{p}), \sigma_j = \theta + 2^{-j}, \lambda_j = \theta - 2^{-j}$, and let $p_*^{\nu_j}$, $p_*^{\mu_j}$ be the numbers defined by $\frac{1}{p_*^{\nu_j}} = \frac{1}{p} - \frac{1}{r2^j}$, $\frac{1}{p_*^{\mu_j}} = \frac{1}{p} + \frac{1}{r2^j}$. We have

$$(\mathcal{L}_r(E,F), \mathcal{L}(E,F))_{\sigma_j,q} = \mathcal{L}_{p_*^{\nu_j},q}\,,\ (\mathcal{L}_r(E,F), \mathcal{L}(E,F))_{\lambda_j,q} = \mathcal{L}_{p_*^{\mu_j},q}$$

with equivalence of quasi-norms where the constants do not depend on j. By Theorem 2.2, the space $\mathcal{L}_{p,q,b}(E,F)$ can be represented in terms of spaces $\mathcal{L}_{p_*^{\nu_j},q}(E,F)$ when $b < 0$, and in terms of spaces $\mathcal{L}_{p_*^{\mu_j},q}(E,F)$ if $b > 0$. Then the result follows by comparing $\mathcal{L}_{p_*^{\nu_j},q}(E,F)$ with $\mathcal{L}_{p^{\nu_j},q}(E,F)$ and $\mathcal{L}_{p_*^{\mu_j},q}(E,F)$ with $\mathcal{L}_{p^{\mu_j},q}(E,F)$. □

If $p = q$, we can derive representations in terms of the simpler spaces $\mathcal{L}_r(E,F)$:

Corollary 4.3. *Let* E, F *be quasi-Banach spaces. Let* $0 < p < \infty$ *and let* $j_0 = j_0(p) \in \mathbb{N}$ *such that, for all* $j \in \mathbb{N}$ *with* $j \geq j_0$, $\frac{1}{p^{\nu_j}} = \frac{1}{p} - 2^{-j} > 0$. *Put* $\frac{1}{p^{\mu_j}} = \frac{1}{p} + 2^{-j}$.

(i) *Let* $b < 0$. *Then* $\mathcal{L}_{p,p,b}(E,F)$ *is the set of all* $T \in \mathcal{L}(E,F)$ *such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|T\|_{p^{\nu_j}}^p\right)^{\frac{1}{p}} < \infty$$

(equivalent quasi-norms).

(ii) *Let* $b > 0$. *Then* $\mathcal{L}_{p,p,b}(E,F)$ *is the set of all operators* $T \in \mathcal{L}(E,F)$ *which can be represented as* $T = \sum_{j=j_0}^{\infty} T_j$ *with* $T_j \in \mathcal{L}_{p^{\mu_j}}(E,F)$ *such that*

$$\left(\sum_{j=j_0}^{\infty} 2^{jbp} \|T_j\|_{p^{\mu_j}}^p\right)^{\frac{1}{p}} < \infty. \tag{18}$$

Moreover, the infimum over expression (18) *is an equivalent quasi-norm in* $\mathcal{L}_{p,p,b}(E,F)$.

Proof. The result is a consequence of Theorems 2.2, 2.3, 2.4 and Remark 2.5. □

Next we deal with bounded linear maps between operator spaces. The corresponding result to (13) reads as follows:

Theorem 4.4. *Let E and F be quasi-Banach spaces and let $\mathcal{F}$ be a bounded linear operator from $\mathcal{L}_p(E,F)$ into $\mathcal{L}_p(E,F)$ for $1 < p \le 2$. If $b < 0$ and*

$$\|\mathcal{F}\|_{\mathcal{L}_p(E,F),\mathcal{L}_p(E,F)} \lesssim (p-1)^{-1} \quad \textit{as} \quad p \downarrow 1,$$

then $\mathcal{F}$ is bounded from $\mathcal{L}_{1,1,b}(E,F)$ into $\mathcal{L}_{1,1,b-1}(E,F)$.

Proof. According to Corollary 4.3/(i) with $p = 1$, we have

$$\frac{1}{p^{\nu_j}} = 1 - \frac{1}{2^j} \quad \text{or} \quad (p^{\nu_j} - 1) \sim 2^{-j}.$$

Whence, for any $T \in \mathcal{L}_{1,1,b}(E,F)$, using the information on $\mathcal{F}$ and Corollary 4.3/(i), we derive

$$\|\mathcal{F}T\|_{1,1,b-1} \sim \sum_{j=j_0}^{\infty} 2^{j(b-1)} \|\mathcal{F}T\|_{p^{\nu_j}} \lesssim \sum_{j=j_0}^{\infty} 2^{jb} \|T\|_{p^{\nu_j}} \sim \|T\|_{1,1,b}. \qquad \square$$

We finish the paper with a result for the case $b = 0$. We define the space $\mathcal{L}_{\mathcal{M}}(E,F)$ as the collection of all those $T \in \mathcal{L}(E,F)$ having a finite quasi-norm

$$\|T\|_{\mathcal{M}} = \sup_{m\ge 1} \left\{ \frac{\sum_{j=1}^{m} a_j(T)}{1 + \log m} \right\}.$$

Clearly, $\mathcal{L}_{1,1,-1}(E,F) \subseteq \mathcal{L}_{\mathcal{M}}(E,F) \subseteq \mathcal{L}_{1,\infty,-1}(E,F)$, and in general the inclusions are strict. For example, if $a_m(T) \sim 1/m$, then $T \in \mathcal{L}_{\mathcal{M}}(E,F)$ but $T \notin \mathcal{L}_{1,1,-1}(E,F)$. If $a_m(T) \sim \frac{1}{m}(1 + \log m)$, then $T \in \mathcal{L}_{1,\infty,-1}(E,F)$, but $T \notin \mathcal{L}_{\mathcal{M}}(E,F)$.

For operators in Hilbert space H, the space $\mathcal{L}_{\mathcal{M}}(H)$ is referred in the literature as one of Macaev ideals (see [15] or [1]).

Lemma 4.5. *Let E, F be quasi-Banach spaces. Then we have, with equivalent quasi-norms,*

$$(\mathcal{L}_1(E,F), \mathcal{L}(E,F))_{\varrho_{0,-1};\infty} = \mathcal{L}_{\mathcal{M}}(E,F).$$

Proof. Using Lemma 4.1, we get

$$\begin{aligned} \|T\|_{\varrho_{0,-1};\infty} &= \sup_{t\ge 1} \left\{ \frac{K(t,T)}{1 + |\log t|} \right\} \\ &\sim \sup_{m\ge 1} \left\{ \frac{K(m,T)}{1 + \log m} \right\} \\ &\sim \sup_{m\ge 1} \left\{ \frac{\sum_{j=1}^{m} a_j(T)}{1 + \log m} \right\} \\ &= \|T\|_{\mathcal{M}}. \qquad \square \end{aligned}$$

In the Hilbert case, interpolation properties of the Macaev ideal can be found in [1] and [22].

Theorem 4.6. *Let E and F be quasi-Banach spaces and let $\mathfrak{F}$ be a bounded linear operator from $\mathcal{L}_p(E,F)$ into $\mathcal{L}_p(E,F)$ for $1 < p \leq 2$. If*

$$\|\mathfrak{F}\|_{\mathcal{L}_p(E,F),\mathcal{L}_p(E,F)} \lesssim (p-1)^{-1} \quad as \quad p \downarrow 1,$$

then $\mathfrak{F}$ is bounded from $\mathcal{L}_1(E,F)$ into $\mathcal{L}_{\mathcal{M}}(E,F)$.

Proof. Let $p_j = (1-2^{-j})^{-1}$. According to Theorem 2.6 and Lemma 4.5, we have

$$\|T\|_{\mathcal{M}} \sim \sup_{j\geq 1} \left\{ 2^{-j} \|T\|^*_{p_j,\infty} \right\},$$

where $\|.\|^*_{p_j,\infty}$ is the norm in $\mathcal{L}_{p_j,\infty}(E,F)$ obtained by real interpolation on the couple $(\mathcal{L}_1(E,F),\mathcal{L}(E,F))$ with parameters 2^{-j} and ∞. That is

$$\|T\|^*_{p_j,\infty} = \sup_{m\geq 1} \left\{ 2^{-2^{-j}m} \sum_{k=1}^{2^m} a_k(T) \right\}.$$

This norm satisfies that $\|T\|^*_{p_j,\infty} \leq \|T\|_{p_j}$ for all $T \in \mathcal{L}_{p_j}(E,F)$. Indeed, using Hölder's inequality, we obtain

$$2^{-2^{-j}m} \sum_{k=1}^{2^m} a_k(T) \leq 2^{-2^{-j}m} \left(\sum_{k=1}^{2^m} a_k^{p_j}(T) \right)^{\frac{1}{p_j}} 2^{m\left(1-\frac{1}{p_j}\right)} \leq \|T\|_{p_j}.$$

Therefore, for any $T \in \mathcal{L}_1(E,F)$, we derive

$$\|\mathfrak{F}T\|_{\mathcal{M}} \sim \sup_{j\geq 1} \left\{ 2^{-j} \|\mathfrak{F}T\|^*_{p_j,\infty} \right\} \leq \sup_{j\geq 1} \left\{ 2^{-j} \|\mathfrak{F}T\|_{p_j} \right\} \lesssim \sup_{j\geq 1} \left\{ \|T\|_{p_j} \right\} \leq \|T\|_1. \qquad \square$$

In the Hilbert case, Theorem 4.6 can be found in [22].

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