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Some sharp inequalities of Mizohata–Takeuchi-type

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Abstract. Let Σ be a strictly convex, compact patch of a C^2 hypersurface in $\mathbb{R}^n$, with non-vanishing Gaussian curvature and surface measure $d\sigma$ induced by the Lebesgue measure in $\mathbb{R}^n$. The Mizohata–Takeuchi conjecture states that

$$\int |\widehat{g\,d\sigma}|^2\, w \leq C\, \|Xw\|_\infty \int |g|^2$$

for all $g \in L^2(\Sigma)$ and all weights $w\colon \mathbb{R}^n \to [0,+\infty)$, where X denotes the X-ray transform. As partial progress towards the conjecture, we show, as a straightforward consequence of recently-established decoupling inequalities, that for every $\varepsilon > 0$, there exists a positive constant C_ε, which depends only on Σ and ε, such that for all $R \geq 1$ and all weights $w\colon \mathbb{R}^n \to [0,+\infty)$, we have

$$\int_{B_R} |\widehat{g\,d\sigma}|^2\, w \leq C_\varepsilon\, R^\varepsilon \sup_T \Big(\int_T w^{(n+1)/2}\Big)^{2/(n+1)} \int |g|^2,$$

where T ranges over the family of tubes in $\mathbb{R}^n$ of dimensions $R^{1/2} \times \cdots \times R^{1/2} \times R$. From this we deduce the Mizohata–Takeuchi conjecture with an $R^{(n-1)/(n+1)}$-loss; i.e., that

$$\int_{B_R} |\widehat{g\,d\sigma}|^2\, w \leq C_\varepsilon\, R^{\frac{n-1}{n+1}+\varepsilon}\, \|Xw\|_\infty \int |g|^2$$

for any ball B_R of radius R and any $\varepsilon > 0$. The power $(n-1)/(n+1)$ here cannot be replaced by anything smaller unless properties of $\widehat{g\,d\sigma}$ beyond 'decoupling axioms' are exploited. We also provide estimates which improve this inequality under various conditions on the weight, and discuss some new cases where the conjecture holds.

1. Introduction

Let $n \geq 2$, and henceforth fix Σ to be a strictly convex, compact patch of a C^2 hypersurface in $\mathbb{R}^n$ with non-vanishing Gaussian curvature; a prototypical example is the sphere $\mathbb{S}^{n-1}$. Let $d\sigma$ be the surface measure on Σ, induced by the Lebesgue measure in $\mathbb{R}^n$. The *Fourier extension operator* associated to Σ is defined by

$$g \mapsto \widehat{g\,d\sigma}$$

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where

$$\widehat{g\,d\sigma}(x) := \int e^{2\pi i \langle x,\xi\rangle} g(\xi)\, d\sigma(\xi) \quad \text{for } x \in \mathbb{R}^n.$$

The Fourier restriction or extension conjecture [29], which lies at the heart of harmonic analysis, aims to understand the extension operator by determining its $L^p \to L^q$ mapping properties. However, while Fourier extension estimates provide information on the size of the level sets of $|\widehat{g\,d\sigma}|$, they do not reveal much about their shape. The Mizohata–Takeuchi conjecture aims to shed light in this direction, specifically regarding the clustering of level sets along lines. The conjecture arose in the study of dispersive PDE; see [25] for some background. In that setting, hypersurfaces such as the paraboloid and the cone are particularly relevant. Although the conjecture stated below arose first in the context of hypersurfaces with non-vanishing Gaussian curvature, it is nevertheless expected that it should hold for arbitrary sufficiently smooth hypersurfaces.

Conjecture 1.1 (Mizohata–Takeuchi). *For any C^2 compact convex hypersurface Σ in $\mathbb{R}^n$, the inequality*

$$\int |\widehat{g\,d\sigma}|^2\, w \le C\, \|Xw\|_\infty \int |g|^2$$

holds for all $g \in L^2(\Sigma)$ and all weights $w\colon \mathbb{R}^n \to [0,+\infty)$, for some $C > 0$ that only depends on Σ.

Here, X denotes the X-ray transform, so that

$$\|Xw\|_\infty = \sup_\ell \int_\ell w,$$

where the supremum is taken over all lines ℓ in $\mathbb{R}^n$. By the compactness of Σ and uncertainty principle considerations, the Mizohata–Takeuchi conjecture is equivalent to

$$\int |\widehat{g\,d\sigma}|^2\, w \le C \sup_T w(T) \int |g|^2$$

where the supremum is taken over all 1-neighbourhoods T of doubly-infinite lines in $\mathbb{R}^n$. In particular, we may – and indeed we shall – assume that w is roughly constant at scale 1.

The Mizohata–Takeuchi conjecture is open in all dimensions, including $n = 2$ (where the Fourier extension conjecture has been resolved).[1] It would directly follow from the truth of the stronger conjecture

$$\int |\widehat{g\,d\sigma}|^2\, w \le C \int |g(\xi)|^2 \sup_{\ell \| N(\xi)} Xw(\ell)\, d\sigma(\xi), \tag{1.1}$$

a formulation of which in the related context of the disc multipliers is due to Stein [29]; here, $N(\xi)$ denotes the normal to Σ at ξ.

When $\Sigma = \mathbb{S}^{n-1}$ and the weight is radial, the Mizohata–Takeuchi conjecture is known to hold (see [2, 10, 12–14]), and the Stein-like conjecture in the same setting is a trivial

[1] It is a nice observation of Bennett and Nakamura, see [4], p. 129, that when $n = 2$, the Mizohata–Takeuchi conjecture implies the Fourier extension conjecture.

consequence of this. When the weight is constant on parallel hyperplanes and the hypersurface is arbitrary, both conjectures are true. This can be seen by using an affine change of variables to reduce to the case of horizontal hyperplanes and a hypersurface parametrised as $(t, \gamma(t))$ for $t \in \mathbb{R}^{n-1}$, and in this case Plancherel's theorem in $\mathbb{R}^{n-1}$ gives the result directly. When $\Sigma = \mathbb{S}^1$ and the weight is a measure supported on $\mathbb{S}^1$, both conjectures are also known [3]. Little is known beyond these three cases.

One way to measure partial progress on the Mizohata–Takeuchi conjecture is to consider inequalities of the form

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \le C R^\alpha \|Xw\|_\infty \int |g|^2,$$

where B_R is the ball of radius R centred at 0, and to attempt to establish such inequalities with the exponent α as small as possible. By the Agmon–Hörmander trace inequality and the local constancy of w at scale 1, we have

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \le C R \|w\|_\infty \int |g|^2 \le C R \|Xw\|_\infty \int |g|^2 \tag{1.2}$$

in all dimensions $n \ge 2$, and it is known that

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \le C R^{1/2} \|Xw\|_\infty \int |g|^2 \quad \text{for } n = 2. \tag{1.3}$$

The latter inequality can be traced back to works of Bourgain [7], Erdoğan [17] and also Carbery and Seeger [11] – see Section 4 in [1] for further details of inequalities which can be found in the literature and which have (1.3) as a consequence. We give a more direct proof of this in Section 3 below. In more recent developments, it is a consequence of the main result in Du and Zhang [16] that one may take any $\alpha > (n-1)/n$ (in fact, with the significantly smaller functional $\sup_{x,\, 1 \le r \le R} w(B(x,r))/r^{n-1}$ in place of $\|Xw\|_\infty$) for arbitrary n. (See also Shayya [27] and Du et al [15], who gave alternative arguments when $n = 3$ for $\alpha > 6/7$ and $\alpha > 2/3$, respectively.) In Theorem 1.2 below, we show that one may take any $\alpha > (n-1)/(n+1)$ in all dimensions.

See also [4, 5] for a tomographic approach to the Mizohata–Takeuchi conjecture, [28] for related weighted $L^2 \to L^4$ estimates on the extension operator, and [18] for variants of the conjecture when the supports of g and w are respectively contained in and equal to neighbourhoods of algebraic varieties.

Notation

The control we shall obtain on $\int_{B_R} |\widehat{g d\sigma}|^2 w$ will be accompanied by multiplicative losses of the form $C_\varepsilon R^\varepsilon$ for any $\varepsilon > 0$. In order to facilitate expression of this, we adopt the following notation.

For any non-negative quantities A and B (which may depend on R), $A \lesssim B$ means that $A \le cB$ for some constant c that depends only on Σ and the ambient dimension. Likewise, $A \gtrsim B$ means that $B \lesssim A$, while $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$. With $R \ge 1$ fixed, $A \lessapprox B$ means that, for every $\varepsilon > 0$, there exists a constant C_ε, depending only on ε, Σ and the ambient dimension, such that $A \le C_\varepsilon R^\varepsilon B$. Similarly, $A \gtrapprox B$ means that $B \lessapprox A$, while $A \approx B$ means that $A \lessapprox B$ and $A \gtrapprox B$.

For a weight w on $\mathbb{R}^n$ and $A \subset \mathbb{R}^n$, we denote by $w(A)$ the integral $\int_A w$ with respect to Lebesgue measure on $\mathbb{R}^n$.

For $n \geq 2$, an n-dimensional ball of radius r will be referred to as an *r-ball*. A tube of length r and cross section an $(n-1)$-dimensional ball of radius $r^{1/2}$ will be referred to as an *$r^{1/2}$-tube*. With $R \geq 1$ fixed and $1 \leq r \leq R$, we let $\mathbb{T}_r$ be the set of $r^{1/2}$-tubes intersecting B_R.

For a line ℓ in $\mathbb{R}^n$ and $g \in L^2(\Sigma)$, we write $\ell \perp \operatorname{supp} g$ if the direction of ℓ is parallel to one of the normals to $\operatorname{supp} g \subset \Sigma$.

For a tube T in $\mathbb{R}^n$, we write $T \perp \operatorname{supp} g$ if the central line of T is parallel to one of the normals to $\operatorname{supp} g \subset \Sigma$.

Statement of results

In this paper, we present several L^2-weighted inequalities for the Fourier extension operator which are related to the Mizohata–Takeuchi conjecture. To place our results in context, we first observe that the Stein–Tomas inequality,

$$\|\widehat{g\,d\sigma}\|_{L^{2(n+1)/(n-1)}(\mathbb{R}^n)} \lesssim \|g\|_2,$$

together with Hölder's inequality, implies that

$$\int_{B_R} |\widehat{g\,d\sigma}|^2\, w \lesssim \Big(\int_{B_R} w^{(n+1)/2}\Big)^{2/(n+1)} \int |g|^2$$

for all g and all non-negative w. The first Mizohata–Takeuchi-type estimates that we present give a significant improvement over this inequality, and follow from the refined Stein–Tomas-type estimate in [21]. They are given in Theorem 1.2 below. The main inequality of this result, (1.4), is closely related to, but logically independent from, the Mizohata–Takeuchi conjecture, and it is sharp in the sense we discuss below the statement. Its consequence (1.5) is also sharp given the techniques that we employ; see [20], the remarks at the end of this section and Section 7. Estimates which improve on Theorem 1.2 appear in Lemma 1.4 (for g with small support), as well as in Theorems 1.6 and 1.8 (for weights that are constant on slabs), and arise as consequences of Theorem 1.2.

Theorem 1.2. *Let $n \geq 2$. For every $\varepsilon > 0$, there exists a positive constant C_ε, which depends only on Σ and ε, such that*

$$\int_{B_R} |\widehat{g\,d\sigma}|^2\, w \leq C_\varepsilon R^\varepsilon \sup_{T \in \mathbb{T}_R :\, T \perp \operatorname{supp} g} \Big(\int_T w^{(n+1)/2}\Big)^{2/(n+1)} \int |g|^2, \tag{1.4}$$

and in particular,

$$\int_{B_R} |\widehat{g\,d\sigma}|^2\, w \lessapprox R^{\frac{n-1}{n+1}} \Big(\sup_{\ell \perp \operatorname{supp} g} Xw(\ell)\Big) \int |g|^2 \tag{1.5}$$

for all $R \geq 1$, $g \in L^2(\Sigma)$ and weights $w\colon \mathbb{R}^n \to [0, +\infty)$.

The second statement follows from the first upon noting that

$$\sup_{T\in\mathbb{T}_R:\,T\perp\operatorname{supp} g}\Big(\int_T w^{(n+1)/2}\Big)^{2/(n+1)} \leq \|w\|_\infty^{\frac{n-1}{n+1}}\Big(\sup_{T\in\mathbb{T}_R:\,T\perp\operatorname{supp} g} w(T)\Big)^{2/(n+1)}$$
$$\lesssim R^{\frac{n-1}{n+1}}\,\|w\|_\infty^{\frac{n-1}{n+1}}\Big(\sup_{\ell\perp\operatorname{supp} g} Xw(\ell)\Big)^{2/(n+1)}$$

and using the approximate constancy of w at scale 1.

Notice that Theorem 1.2, unlike the Mizohata–Takeuchi conjecture itself, requires non-vanishing curvature of Σ.

Remark 1.3. Inequality (1.4) of Theorem 1.2 is sharp in the following senses. Firstly, if the exponent r is such that

$$\int_{B_R} |\widehat{g d\sigma}|^q\, w \lesssim \Big(\int_{B_R} w^r\Big)^{1/r}\Big(\int |g|^p\Big)^{q/p}$$

(which, by duality, is equivalent to an L^p-$L^{qr'}$ Fourier extension estimate) holds, then necessarily $1/qr' \leq (n-1)/(n+1)p'$; so the exponent $(n+1)/2$ appearing in (1.4) (in which $p = q = 2$) cannot be increased, irrespective of the size of the tubes $T \subset B_R$. Secondly, fixing $r = (n+1)/2$ in (1.4), we cannot reduce the width of the tubes appearing to be significantly smaller than $R^{1/2}$. These two assertions can both be seen by testing as usual on g the indicator function of an $R^{-1/2}$-cap and w the indicator of the dual $R^{1/2}$-tube. On the other hand, we do not know whether one may take $\varepsilon = 0$ in (1.4) and (1.5). It is likely that when $n = 2$, we may be able to replace the R^ε term by a power of $\log R$; see Remark 4.3 below.

Theorem 1.2 will follow from the more precise Theorem 4.1, in which $\mathbb{T}_R$ is replaced by the set of tubes featuring in the wave packet decomposition of g at scale R.

We now turn to our other results. Theorems 1.6 and 1.8 below are improvements of Theorem 1.2 for weights that exhibit a level of local constancy along slabs. In the extreme case where there is no such local constancy beyond on unit scale, both theorems reduce to Theorem 1.2. Theorem 1.6 involves slabs that are ‘roughly parallel’ to caps of Σ, while Theorem 1.8 addresses the general case.

Both theorems (and, in fact, the more precise Theorems 6.1 and 6.2) will follow from a strengthened version of Theorem 1.2 for functions g with small support (Lemma 1.4 below), which we will prove for all weights.

In order to state Theorems 1.6 and 1.8, we first establish some further notation, and introduce a quantity which is intermediate between the quantity

$$\sup_{T\in\mathbb{T}_R:\,T\perp\operatorname{supp} g}\Big(\int_T w^{(n+1)/2}\Big)^{2/(n+1)}$$

occurring in Theorem 1.2 and a quantity more directly geared towards that occuring in the Mizohata–Takeuchi conjecture itself. This will involve considering an amalgam of ‘running averages’ of w at certain scales related to the level of constancy that we are assuming, which is measured by a parameter $1 \leq \rho \leq R$ which we now fix. Let $E \subset \Sigma$.

For each $T_R \in \mathbb{T}_R$ such that $T_R \perp E$, we cover T_R by essentially disjoint tubes $S_\rho \in \mathbb{T}_\rho$ which are parallel to and contained in T_R. For $w\colon \mathbb{R}^n \to [0, +\infty)$ and $E \subset \Sigma$, we define

$$A_{\rho,R,E}(w) := \frac{1}{\rho^{(n-1)/2}} \sup_{T_R \in \mathbb{T}_R : T_R \perp E} \Big(\sum_{S_\rho \subset T_R} w(S_\rho)^{(n+1)/2} \Big)^{2/(n+1)},$$

a quantity which can be expressed more geometrically as

$$\sup_{T_R \in \mathbb{T}_R : T_R \perp E} \Big(\sum_{S_\rho \subset T_R} \Big(\frac{w(S_\rho)}{|S_\rho|} \Big)^{(n+1)/2} |S_\rho| \Big)^{2/(n+1)},$$

and thus is seen to increase as ρ gets smaller.[2] For $\rho = 1$,

$$A_{1,R,E}(w) \sim \sup_{T_R \in \mathbb{T}_R : T_R \perp E} \Big(\int_{T_R} w^{(n+1)/2} \Big)^{2/(n+1)}$$

is the quantity appearing on the right-hand side of Theorem 1.2, controlling the $L^2(E) \to L^2(w)$-norm of the extension operator. Theorem 1.2 fails in general for g supported on E if the above quantity is replaced by the smaller

$$A_{R,R,E}(w) = \sup_{T_R \in \mathbb{T}_R : T_R \perp E} \frac{w(T_R)}{R^{(n-1)/2}}$$

(and in fact by $A_{\rho,R,E}(w)$ for any $\rho \gg 1$, as can be seen by taking g to be the indicator function of a 1-cap and w the indicator function of the unit ball). In the results which follow, however, we shall show that under certain auxiliary conditions (g being supported on a small cap, or the weight being the indicator function of a union of small slabs), Theorem 1.2 nevertheless does hold for $g \in L^2(E)$ if we replace the quantity $A_{1,R,E}(w)$ with $A_{\rho,R,E}(w)$ for an appropriate choice of ρ. To further compare these two quantities, observe that

$$A_{\rho,R,E}(w) \le \sup_{S_\rho \in \mathbb{T}_\rho : S_\rho \perp E} \Big(\frac{w(S_\rho)}{|S_\rho|} \Big)^{\frac{n-1}{n+1}} \sup_{T_R \in \mathbb{T}_R, T_R \perp E} w(T_R)^{2/(n+1)}, \tag{1.6}$$

which becomes

$$A_{\rho,R,E}(w) \le \sup_{S_\rho \in \mathbb{T}_\rho : S_\rho \perp E} \Big(\frac{w(S_\rho)}{|S_\rho|} \Big)^{\frac{n-1}{n+1}} A_{1,R,E}(w)$$

when w is an indicator function (which we may well assume for our purposes).

[2] By Hölder's inequality we have, for $\lambda \ge 1$ and a tessellation of an $S_{\lambda\rho}$ by S_ρ's,

$$\Big(\frac{w(S_{\lambda\rho})}{|S_{\lambda\rho}|} \Big)^{(n+1)/2} |S_{\lambda\rho}| \lesssim \sum_{S_\rho \subset S_{\lambda\rho}} \Big(\frac{w(S_\rho)}{|S_\rho|} \Big)^{(n+1)/2} |S_\rho|.$$

In situations in which we are able to bound the $L^2(E) \to L^2(w)$-norm of the extension operator by $A_{\rho,R,E}(w)$, inequality (1.6) leads to improved bounds in terms of $\|Xw\|_\infty$; in particular, to a gain on Theorem 1.2 by a factor $\rho^{-(n-1)/(n+1)}$. Indeed, by (1.6),

$$A_{\rho,R,E}(w) \le \Big(\frac{\|Xw\|_\infty}{\rho}\Big)^{\frac{n-1}{n+1}} (R^{(n-1)/2}\|Xw\|_\infty)^{2/(n+1)} \lessapprox \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sup_{\ell \perp E} Xw(\ell).$$

A situation such as this arises when g is supported in a $\rho^{-1/2}$-cap of Σ (that is, the intersection of Σ with a $\rho^{-1/2}$-ball), and is summarised in Lemma 1.4 below. The lemma will in turn be used in conjunction with a decoupling argument to derive Theorems 1.6 and 1.8 for all functions g and restricted classes of weights. Note that, in Lemma 1.4 below, the subscript τ on g_τ is not strictly needed, but we retain it to emphasise its support.

Lemma 1.4 (Small caps). *For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all weights $w\colon \mathbb{R}^n \to [0, +\infty)$, whenever $1 \le \rho \le R$, τ is a $\rho^{-1/2}$-cap of Σ and $g_\tau \in L^2(B^{n-1})$ is supported in τ, we have*

$$\int_{B_R} |\widehat{g_\tau d\sigma}|^2\, w \le C_\varepsilon R^\varepsilon\ A_{\rho,R,\operatorname{supp} g_\tau}(w) \int |g_\tau|^2,$$

and therefore also

$$\int_{B_R} |\widehat{g_\tau d\sigma}|^2\, w \lessapprox \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sup_{\ell \perp \operatorname{supp} g_\tau} Xw(\ell) \int |g_\tau|^2. \tag{1.7}$$

In order to state Theorems 1.6 and 1.8, we need to make precise what we mean by a slab, and by a slab being 'roughly parallel' to caps of Σ.

Definition 1.5. Fix $R \ge 1$, $1 \le \rho \le R$ and $0 \le \nu \le \pi/2$. We define a $\rho^{1/2}$-*slab* to be any affine copy of the 1-neighbourhood of an $(n-1)$-dimensional $\rho^{1/2}$-ball in $\mathbb{R}^n$. We say that a slab is ν-*parallel to* Σ if all normals to Σ create angle at least ν with the slab (that is, they create angle at most $\pi/2 - \nu$ with the normal to the slab).

In this definition, ν is a measure of how large the angles are between the slab and the normals to Σ. The larger ν is, the larger these angles are, and the more 'parallel' Σ and the slab look.

With these preliminaries in hand, we are now ready to state our remaining results. In the first two results which follow, the implicit constant blows up as $\nu \downarrow 0$. Thus, the interesting cases of these two results are those in which ν is large, i.e., when the slabs create large angles with the normals to Σ. If for instance Σ is roughly horizontal (i.e., all normals to Σ are within angle $\le 1/100$ from the vertical direction), then Theorem 1.6 gives meaningful results for slabs that are also nearly horizontal (e.g., creating angle $\ge 2/100$ with the vertical direction).

Theorem 1.6 (Slabs ν-parallel to Σ). *For every $0 < \nu \le \pi/2$ and $\varepsilon > 0$, there exists $C_{\varepsilon,\nu} > 0$ such that the following hold. Let $g \in L^2(\Sigma)$. For $R \ge 1$ and $R^\varepsilon \lesssim_\varepsilon \rho \le R$, let $w\colon \mathbb{R}^n \to [0, +\infty)$ be a weight of the form $\sum_{s \in \mathcal{S}} c_s \chi_s$, where $\mathcal{S}$ is a set of disjoint $\rho^{1/2}$-slabs ν-parallel to Σ. Then the inequality*

$$\int_{B_R} |\widehat{g\, d\sigma}|^2\, w \le C_{\varepsilon,\nu} R^\varepsilon A_{\rho,R,\operatorname{supp} g}(w) \int |g|^2 \lessapprox_\nu \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sup_{\ell \perp \operatorname{supp} g} Xw(\ell) \int |g|^2$$

holds. In fact, if

$$g = \sum_{\tau \in \mathfrak{T}} g_\tau, \quad \textit{with } \operatorname{supp} g_\tau \subset \tau,$$

for some boundedly overlapping family $\mathfrak{T}$ *of* $\rho^{-1/2}$*-caps* τ *of* Σ*, then*

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \lessapprox_\nu \sum_{\tau \in \mathfrak{T}} A_{\rho, R, \operatorname{supp} g_\tau}(w) \int |g_\tau|^2 \lessapprox_\nu \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sum_{\tau \in \mathfrak{T}} \sup_{\ell \perp \operatorname{supp} g_\tau} Xw(\ell) \int |g_\tau|^2.$$

It follows that Stein's stronger conjecture (1.1) (and thus the Mizohata–Takeuchi conjecture) holds under the conditions of Theorem 1.6 when the slabs involved are $R^{1/2}$-slabs. We single this out explicitly as a corollary.

Corollary 1.7. *Let* $R \geq 1$ *and suppose that* w *is a weight of the form* $\sum_{s \in \mathcal{S}} c_s \chi_s$*, where* $\mathcal{S}$ *is a set of disjoint* $R^{1/2}$*-slabs which are* ν*-parallel to* Σ *for some* $0 < \nu \leq \pi/2$*. Then*

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \lessapprox_\nu \int |g(\xi)|^2 \sup_{\ell \parallel N(\xi)} Xw(\ell)\, d\sigma(\xi)$$

for all $g \in L^2(\Sigma)$*.*

Stein's conjecture continues to hold even when the slabs are curved. The precise formulation of this appears in Corollary 3.4, and it is proved using a direct method, which does not rely on Theorem 1.2, and which also featured in [20].

A substitute result for Theorem 1.6 in the case where there is no restriction on ν (i.e., when the slabs can create arbitrarily small angles with normals to Σ) is as follows.

Theorem 1.8 (All slabs)**.** *For every* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that the following hold. Let* $g \in L^2(\Sigma)$*. For* $R \geq 1$ *and* $R^\varepsilon \lesssim_\varepsilon \rho \leq R$*, let* $w\colon \mathbb{R}^n \to [0, +\infty)$ *be a weight of the form* $\sum_{s \in \mathcal{S}} c_s \chi_s$*, where* $\mathcal{S}$ *is a set of disjoint* $\rho^{1/2}$*-slabs with no conditions on their directions. Then the inequality*

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \leq C_\varepsilon R^\varepsilon A_{\rho^{1/2}, R, \operatorname{supp} g}(w) \int |g|^2 \lessapprox \Big(\frac{R}{\rho^{1/2}}\Big)^{\frac{n-1}{n+1}} \sup_{\ell \perp \operatorname{supp} g} Xw(\ell) \int |g|^2$$

holds. In fact, if

$$g = \sum_{\tau \in \mathfrak{T}} g_\tau, \quad \textit{with } \operatorname{supp} g_\tau \subset \tau,$$

for some boundedly overlapping family $\mathfrak{T}$ *of* $\rho^{-1/4}$*-caps* τ *of* Σ*, then*

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \lessapprox \sum_{\tau \in \mathfrak{T}} A_{\rho^{1/2}, R, \operatorname{supp} g_\tau}(w) \int |g_\tau|^2 \lessapprox \Big(\frac{R}{\rho^{1/2}}\Big)^{\frac{n-1}{n+1}} \sum_{\tau \in \mathfrak{T}} \sup_{\ell \perp \operatorname{supp} g_\tau} Xw(\ell) \int |g_\tau|^2.$$

Corollary 1.9 ($R^{1/2}$-slabs)**.** *Let* $R \geq 1$ *and assume* w *is a weight of the form* $\sum_{s \in \mathcal{S}} c_s \chi_s$*, where* $\mathcal{S}$ *is a set of disjoint* $R^{1/2}$*-slabs. Then*

$$\int_{B_R} |\widehat{g d\sigma}|^2 w \lessapprox R^{\frac{n-1}{2(n+1)}} \sup_{\ell \perp \operatorname{supp} g} Xw(\ell) \int |g|^2$$

for all $g \in L^2(\Sigma)$*.*

Sharpness of inequality (1.5) given the choice of technique

During the recent talk [20], which in fact partially inspired the work in this paper, Guth explained that, using only basic local constancy and local L^2-orthogonality properties of the functions $\widehat{g\,d\sigma}$ – which are indeed the only properties that we exploit in proving Theorem 1.2 –, one *cannot* prove the Mizohata–Takeuchi conjecture for B_R with a loss better than $\sim (\log R)^{-3} R^{(n-1)/(n+1)}$.

This means that inequality (1.5) of Theorem 1.2, which establishes the conjecture with a loss of $\lessapprox R^{(n-1)/(n+1)}$, is essentially sharp given the techniques used.

Guth's argument is discussed in Section 7 for purposes of self-containment.

2. Preliminaries

For our purposes, we may assume that all normals to Σ have angle at most $1/100$ from the vertical direction, and that the projection of Σ on the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ is contained in the unit ball B^{n-1} centred at 0. This convention allows us to assume that Σ has a parametrisation

$$\Sigma = \{\Sigma(\omega) := (\omega, h(\omega)), \text{ for } \omega \in B^{n-1}\}$$

for some $h\colon B^{n-1} \to \mathbb{R}$, and to work with the operator E instead of $\widehat{\cdot\, d\sigma}$, where

$$Eg(x) := \int_{B^{n-1}} e^{2\pi i \langle x, \Sigma(\omega)\rangle}\, g(\omega)\, d\omega, \quad \text{for } x \in \mathbb{R}^n.$$

From now on, for fixed Σ and $\varepsilon > 0$, we say that a quantity $C(R,\varepsilon)$ satisfies

$$C(R,\varepsilon) = \mathrm{RapDec}_\varepsilon(R)$$

if for every $N \in \mathbb{N}$ there exists a non-negative constant $C_{N,\varepsilon}$ such that uniformly in $R \geq 1$ we have

$$|C(R,\varepsilon)| \leq C_{N,\varepsilon}\, R^{-N}.$$

Wave packet decomposition adapted to B_R

Let $\varepsilon > 0$ and $0 < \delta \ll \varepsilon$. Fix $R \gg 1$, and cover B^{n-1} by boundedly overlapping balls θ of radius $R^{-1/2}$. The set of these balls will be denoted by Θ_R, and the balls will be referred to as $R^{-1/2}$-*caps*. Let $\{\psi_\theta\}_{\theta\in\Theta_R}$ be a smooth partition of unity adapted to this cover. Thus,

$$g = \sum_{\theta\in\Theta_R} \psi_\theta\, g$$

for any $g\colon \mathbb{R}^{n-1} \to \mathbb{C}$ supported in B^{n-1} (and belonging to some suitable class). Now, cover $\mathbb{R}^{n-1}$ by boundedly overlapping balls of radius $CR^{(1+\delta)/2}$ and centres on the lattice $V_R := R^{(1+\delta)/2}\mathbb{Z}^{n-1}$. There exists a bump function η, adapted to the ball $B(0, R^{(1+\delta)/2})$, so that the bump functions $\eta_v := \eta(\cdot - v)$, over $v \in V_R$, form a partition of unity for this cover. It follows that, with $\hat{\cdot}$ and $\check{\cdot}$ denoting the $(n-1)$-dimensional Fourier transform and its inverse, respectively,

$$\check{g} = \sum_{(\theta,v)} \eta_v (\psi_\theta g)\check{}$$

and thus

$$g = \sum_{(\theta,v)} \widehat{\eta}_v * (\psi_\theta g)$$

for all g as above. Finally, restrict each of the above summands to the corresponding cap θ. In particular, let

$$g_{\theta,v} := \widetilde{\psi}_\theta \cdot (\widehat{\eta}_v * (\psi_\theta\, g)),$$

where $\widetilde{\psi}_\theta := \widetilde{\psi}(R^{1/2}(\cdot - \omega_\theta))$ for some fixed smooth bump function $\widetilde{\psi}$ (where ω_θ is the centre of the cap θ), chosen so that $\widetilde{\psi}_\theta$ is supported in θ and equals 1 on the $cR^{1/2}$-neighbourhood of $\operatorname{supp}\psi_\theta$, for some small $c > 0$.

The $g_{\theta,v}$ are the *wave packets of* g *at scale* R, while $\{g_{\theta,v}\}_{(\theta,v)\in\Theta_R\times V_R}$ constitutes the *wave packet decomposition of* g at this scale. Note that the decomposition is ε-dependent.

The function g is roughly the sum of its wave packets, all of which are roughly orthogonal. More precisely, note that the function $\widehat{\eta}_v$ is rapidly decaying when $|\omega| \gg R^{-(1+\delta)/2}$, so

$$\|g_{\theta,v} - \widehat{\eta}_v * (\psi_\theta g)\|_\infty \le \mathrm{RapDec}_\varepsilon(R)\, \|g\|_2, \quad \text{for each } (\theta, v),$$

hence

$$\text{(wp1)} \qquad \Big\| g - \sum_{(\theta,v)\in\Theta_R\times V_R} g_{\theta,v} \Big\|_\infty \le \mathrm{RapDec}_\varepsilon(R)\, \|g\|_2.$$

The functions $g_{\theta,v}$ are almost orthogonal, in the sense that

$$\text{(wp2)} \qquad \Big\| \sum_{(\theta,v)\in\mathbb{W}} g_{\theta,v} \Big\|_2^2 \sim \sum_{(\theta,v)\in\mathbb{W}} \|g_{\theta,v}\|_2^2$$

for every subset $\mathbb{W}$ of $\Theta_R \times V_R$.

It turns out that, for every (θ, v), $Eg_{\theta,v}$ is essentially supported in

$$T_{\theta,v} := \big\{x \in B_R : |x' + x_n \partial_\omega h(\omega_\theta) - v| \le R^{1/2+\delta}\big\},$$

the $R^{1/2+\delta}$-tube in B_R whose central line passes through $(v, 0)$ and has direction the normal $N(\theta) := (\partial_\omega h(\omega_\theta), -1)$ to the cap $\Sigma(\theta)$. Indeed, it follows by a non-stationary phase argument that

$$\text{(wp3)} \qquad |Eg_{\theta,v}(x)| \le (1 + R^{-1/2}|x' + x_n \partial_\omega h(\omega_\theta) - v|)^{-(n+1)}\, \mathrm{RapDec}_\varepsilon(R)\, \|g\|_2, \quad \text{for all } x \in B_R \setminus T_{\theta,v};$$

a detailed analysis can be found in [19].

Due to the curvature of Σ, different surface caps $\Sigma(\theta)$ have different normals, so there is a one-to-one correspondence between the pairs (θ, v) and the tubes $T_{\theta,v}$. We may thus denote each wave packet $g_{\theta,v}$ by g_T, for the tube $T = T_{\theta,v}$.

Henceforth, denote

$$\mathbb{T}_\varepsilon(B_R) := \{T_{\theta,v} : (\theta, v) \in \Theta_R \times V_R \text{ and } T_{\theta,v} \cap B_R \ne \emptyset\}$$

and

$$\mathbb{T}_\varepsilon^\theta(B_R) := \{T_{\bar{\theta},v} : |\bar{\theta} - \theta| \lesssim R^{-1/2},\ v \in V_R \text{ and } T_{\theta,v} \cap B_R \ne \emptyset\}$$

for each $\theta \in \Theta_R$, where the implicit multiplicative constant is sufficiently large. The above analysis ensures that

$$\text{(wp4)} \qquad Eg(x) = \sum_{T \in \mathbb{T}_\varepsilon(B_R)} Eg_T(x) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2, \quad \text{for all } x \in B_R,$$

while also that any function g_θ supported on $\theta \in \Theta_R$ satisfies

$$\text{(wp5)} \qquad Eg_\theta(x) = \sum_{T \in \mathbb{T}^\theta_\varepsilon(B_R)} Eg_T(x) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2, \quad \text{for all } x \in B_R.$$

We will refer to $\{g_T\}_{T \in \mathbb{T}_\varepsilon(B_R)}$ as the *wave packet decomposition* of g *adapted to* B_R.

Wave packet decompositions adapted to other balls

Let $R^\varepsilon \lesssim_\varepsilon \rho \le R$, and fix a ball $B = B(y,\rho)$. For $x \in \mathbb{R}^n$, set $\widetilde{x} := x - y$. It holds that

$$Eg(x) = \int e^{2\pi i \langle x, \Sigma(\omega)\rangle} g(\omega)\, d\omega = \int e^{2\pi i \langle \widetilde{x}, \Sigma(\omega)\rangle} e^{2\pi i \langle y, \Sigma(\omega)\rangle} g(\omega)\, d\omega = E\widetilde{g}(\widetilde{x}),$$

where $\widetilde{g}(\omega) = e^{2\pi i \langle y, \Sigma(\omega)\rangle} g(\omega)$. For every $x \in B$, $\widetilde{x}$ lives in B_ρ; therefore, by the earlier discussion,

$$\text{(wp6)} \qquad \begin{aligned} Eg(x) &= \sum_{T \in \mathbb{T}_\varepsilon(B_\rho)} E\widetilde{g}_T(\widetilde{x}) + \mathrm{RapDec}_\varepsilon(\rho) \int |\widetilde{g}|^2 \\ &= \sum_{T \in \mathbb{T}_\varepsilon(B_\rho)} E\widetilde{g}_T(x-y) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2, \quad \text{for all } x \in B. \end{aligned}$$

From now on, we will be referring to $\{\widetilde{g}_T\}_{T \in \mathbb{T}_\varepsilon(B_\rho)}$ as the *wave packet decomposition* of g *adapted to* B. Note that this decomposition is y-dependent.

By the above analysis, for every $\rho^{-1/2}$-cap τ, we have

$$\text{(wp7)} \qquad Eg_\tau(x) = \sum_{T \in \mathbb{T}^\tau_\varepsilon(B_\rho)} E\widetilde{g}_T(x-y) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2, \quad \text{for all } x \in B.$$

Each of the wave packets in the above summand is essentially constant in magnitude; this is made rigorous in the subsection below.

Fourier localisation and local constancy

Let $\varepsilon > 0$ and $R^\varepsilon \lesssim_\varepsilon \rho \le R$. Fix $g \in L^2(B^{n-1})$ and a $\rho^{-1/2}$-cap τ.

Roughly speaking, since g_τ is supported in τ, the Fourier transform of Eg_τ is supported in the ρ^{-1}-neighbourghood of $\Sigma(\tau)$. The uncertainty principle then dictates that $|Eg_\tau|$ is essentially constant on each dual object, i.e., on each $\rho^{1/2}$-tube pointing in the direction the normal to $\Sigma(\tau)$.

The above heuristic is made rigorous as follows. Let $\omega(\tau)$ be the centre of τ. The patch of the tangent space to Σ at $\Sigma(\omega_\tau)$ that lives over τ is the set

$$T_\tau\Sigma := \{\Sigma(\omega_\tau) + M_\tau \cdot (\omega - \omega_\tau) : \ \omega \in \tau\}, \quad \text{where } M_\tau := \begin{bmatrix} I_{n-1} & 0 \\ \partial_\omega h(\omega_\tau) & 1 \end{bmatrix}.$$

The convex set

$$S(\tau) := \Big\{\Sigma(\omega_\tau) + M_\tau \cdot (\omega - \omega_\tau) + t \cdot \tfrac{N(\omega_\tau)}{\|N(\omega_\tau)\|} \, : \, \omega \in \tau, \; t \in [-\rho^{-1}, \rho^{-1}]\Big\}$$

is a ‘thickening’ of the above tangent patch by ρ^{-1} in the direction normal to $\Sigma(\tau)$. The Fourier transform of $Eg_{\tau|_{B_R}}$ is essentially supported in a dilation of $S(\tau)$. We are interested in a precise version of this for appropriate cut-offs of Eg_τ.

In particular, let $\zeta\colon \mathbb{R}^n \to \mathbb{R}$ with $\zeta = 1$ on B_1 and $\zeta = 0$ outside B_2. For every ball $B = B(\bar{x}, \rho)$ in $\mathbb{R}^n$, define

$$\zeta_B(x) := \zeta\Big(\frac{x - \bar{x}}{\rho}\Big).$$

There exists a constant C, depending only on the dimension n, such that the following holds.

Proposition 2.1 (Fourier localisation). *Let $R^\varepsilon \lesssim_\varepsilon \rho \le R$, and let g_τ be supported in a $\rho^{-1/2}$-cap τ. Then, for every ρ-ball B in $\mathbb{R}^n$,*

$$Eg_\tau \cdot \zeta_B = G_\tau + \mathrm{RapDec}_\varepsilon(\rho)\, \|g_\tau\|_2,$$

for some $G_\tau\colon \mathbb{R}^n \to \mathbb{C}$ with the property that $\widehat{G}_\tau$ is supported in $S(C \cdot \tau)$.

The set $C \cdot \tau$ is the $C\rho^{-1/2}$-cap with the same centre as τ. The proof of Proposition 2.1 is exposed in full detail in [24].

When a function f is Fourier localised on a convex set (such as the slab $S(\tau)$), then to some extent it can be treated as a constant function on objects dual to that convex set. The precise statement appears in Lemmas 6.1 and 6.2 in [23]. For our purposes, we only need the following corollary.

Proposition 2.2 (Local constancy). *Let $R^\varepsilon \lesssim_\varepsilon \rho \le R$. Let τ be a $\rho^{-1/2}$-cap, and consider a function $f\colon \mathbb{R}^n \to \mathbb{C}$ with $\widehat{f} \subset S(\tau)$. Then, every tube T in $\mathbb{R}^n$ with direction $N(\tau)$, radius $\rho^{1/2}$ and length ρ satisfies*

$$\sup_{x \in T} |f(x)|^2 \lesssim \frac{1}{|T|} \int |f|^2\, w_T,$$

for some non-negative function $\omega_T\colon \mathbb{R}^n \to \mathbb{R}$, with $\omega_T = 1$ on T and $\omega(x) \sim C_N(1 + n(x,T))^{-N}$ for all $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$, where $n(x,T)$ is the smallest $n \in \mathbb{N}$ such that $x \in nT$. In particular, if $g \in L^2(B^{n-1})$ and B is a ρ-ball intersecting T, then

$$\sup_{x \in T} |Eg_\tau(x)|^2 \lesssim \rho^\delta \frac{1}{|2\widetilde{T}|} \int_{2\widetilde{T}} |Eg_\tau|^2 + \mathrm{RapDec}_\varepsilon(R) \int |g_\tau|^2.$$

for all $\widetilde{T}$ in $\mathbb{T}^\tau_\varepsilon(B)$ intersecting T.

Proof. The first conclusion is a direct application of Lemmas 6.1 and 6.2 in [23]. We now in turn apply this conclusion to the function $Eg_\tau \cdot \zeta_B$, which is essentially Fourier supported in $S(C \cdot \tau)$ by Proposition 2.1. Respecting the notation of Proposition 2.1, denote by T_C the tube with the same central line as T, radius $(C^{-2}\rho)^{1/2}$ and length $C^{-2}\rho$. We obtain

$$\sup_{x\in T}|Eg_\tau(x)|^2=\sup_{x\in T}|Eg_\tau(x)\cdot\zeta_B(x)|^2\lesssim\frac{1}{|T_C|}\int|Eg_\tau\cdot\zeta_B|^2\,w_{T_C}+\mathrm{RapDec}_\varepsilon(\rho)\int|g_\tau|^2.$$

Since $w_T(x)\sim w_{T_C}(x)$ for all $x\in\mathbb{R}^n$, it holds that

$$\begin{aligned}\frac{1}{|T_C|}\int|Eg_\tau\cdot\zeta_B|^2\,w_{T_C}&\lesssim\frac{1}{|T|}\int|Eg_\tau\cdot\zeta_B|^2\,w_T\lesssim\frac{1}{|T|}\int_{2B}|Eg_\tau|^2\,w_T\\&\sim\frac{1}{|T|}\int_{2B\cap2\widetilde{T}}|Eg_\tau|^2\,w_T+\frac{1}{|T|}\int_{2B\setminus2\widetilde{T}}|Eg_\tau|^2\,w_T\\&\lessapprox\frac{\rho^\delta}{|2\widetilde{T}|}\int_{2\widetilde{T}}|Eg_\tau|^2\,w_T+\mathrm{RapDec}_\varepsilon(\rho)\frac{1}{|T|}\int_{2B\setminus2\widetilde{T}}|Eg_\tau|^2\,w_T.\end{aligned}$$

The result follows as, due to the decay properties of w_T,

$$\mathrm{RapDec}_\varepsilon(\rho)\,\frac{1}{|T|}\int_{2B\setminus2\widetilde{T}}|Eg_\tau|^2\,w_T=\mathrm{RapDec}_\varepsilon(\rho)=\mathrm{RapDec}_\varepsilon(R)\int|g_\tau|^2.\qquad\blacksquare$$

3. Some new cases where Mizohata–Takeuchi holds

In this section, $\Sigma:=\{(\omega,h(\omega)):\omega\in B^{n-1}\}$ is a fixed hypersurface in $\mathbb{R}^n$, all of whose normals point within angle $1/100$ from the vertical direction. There is no requirement that Σ have non-vanishing Gaussian curvature.

The truth of the Mizohata–Takeuchi conjecture for some simple weights (such as indicator functions of neighbourhoods of roughly horizontal hyperplanes or hypersurfaces) implies that the conjecture holds for more complicated weights (superpositions of appropriately large patches of such surfaces). For instance, the Mizohata–Takeuchi conjecture holds for nearly horizontal $R^{1/2}$-slabs (case $\rho=R$ of Theorem 1.6) because it holds for horizontal hyperplanes (Plancherel).

Definition 3.1. A *ρ-flake* (or simply a *flake*) in $\mathbb{R}^n$ is the 1-neighbourhood of any hypersurface of the form $\{(\omega,\Gamma(\omega)):\omega\in B_\rho^{n-1}\}$, where B_ρ^{n-1} is a ρ-ball in $\mathbb{R}^{n-1}$ and $\Gamma\colon B_\rho^{n-1}\to\mathbb{R}$. A flake is *nearly horizontal* if all its tangent spaces create angle larger than $2/100$ with the vertical direction.

Note that ρ-slabs are ρ-flakes. We will usually be taking $\rho\geq1$. We emphasise that Γ and h are unrelated.

Every line normal to Σ which intersects a nearly horizontal flake will do so along a line segment of length about 1. Therefore, the following lemma states that the Mizohata–Takeuchi conjecture holds when the weight is the indicator of a single nearly horizontal flake.

Lemma 3.2. *Let γ be a nearly horizontal flake in $\mathbb{R}^n$. For all $R\geq1$ and $g\in L^2(B^{n-1})$,*

$$\int_{B_R\cap\gamma}|Eg|^2\lessapprox\int|g|^2.$$

Proof. The proof easily follows by induction on scales, and only a sketch is provided here. In particular, the estimate trivially holds when $R \lesssim 1$. For arbitrary larger R, we cover the flake γ by finitely overlapping $R^{1/2}$-balls B. For every one of these balls B, we may assume that

$$\int_{B\cap\gamma} |Eg|^2 \lessapprox \int |g_B|^2,$$

where g_B is the sum of the wave packets g_T of g at scale R that intersect B. The functions g_B are essentially orthogonal, as each of the tubes T in question has width $R^{1/2+\delta}$ (where as in Section 2, $0 < \delta \ll \varepsilon$) and creates angle $\gtrsim 1$ with the flake, hence it intersects $R^{O(\delta)}$ of the balls B. Adding up the above estimate over all B completes the proof. ∎

Remark 3.3. We emphasise that when γ is specifically a horizontal hyperplane, then the stronger estimate

$$\int_\gamma |Eg|^2 = \int |g|^2$$

directly follows by Plancherel's theorem. Indeed, for every $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$Eg(x,t) = \int e^{2\pi i\langle x,\omega\rangle}\, e^{2\pi i t h(\omega)}\, g(\omega)\, d\omega = \widehat{g_t}(x),$$

where $g_t := e^{2\pi i t h(\cdot)} g$ and $\widehat{\cdot}$ denotes the standard Fourier transform on $\mathbb{R}^{n-1}$. Therefore,

$$\int |Eg(\cdot,t)|^2 = \int |\widehat{g_t}|^2 = \int |g_t|^2 = \int |g|^2$$

for all $t \in \mathbb{R}$. (Note that this directly yields (1.2).) After an appropriate change of variables, a similar argument resolves the Mizohata–Takeuchi conjecture when the weight is the indicator function of the 1-neighbourhood of any hyperplane (independently of orientation), and subsequently when the weight is a sum of indicator functions of such 1-neighbourhoods. See Corollary 3 in [5] for a stronger estimate (a certain identity) in this specific scenario.

Lemma 3.2 easily implies the Mizohata–Takeuchi conjecture for superpositions of appropriately large flakes, and in fact an estimate stronger than Stein's conjecture (1.1).

Corollary 3.4 (MT holds for $R^{1/2}$-flakes). *The inequality*

$$\int_{B_R} |Eg|^2\, w \lessapprox \|Xw\|_\infty \int |g|^2$$

holds for every $g \in L^2(B^{n-1})$ and any weight $w\colon \mathbb{R}^n \to [0,+\infty)$ of the form $\sum_{\gamma\in\mathcal{F}} c_\gamma \chi_\gamma$, where $\mathcal{F}$ is a family of $R^{1/2}$-flakes. In fact, the stronger estimate

$$\int_{B_R} |Eg|^2\, w \lessapprox \sum_{T\in\mathbb{T}} \sup_{\ell\subset T} Xw(\ell) \int |g_T|^2$$

holds, where $\{g_T\}_{T\in\mathbb{T}}$ is the wave packet decomposition of g at scale R.

Proof. Fix $g\colon L^2(B^{n-1})$ and $\gamma \in \mathcal{F}$, and denote by $\mathbb{T}_\gamma$ the set of tubes in $\mathbb{T}$ that intersect γ. For all $x \in \gamma$,

$$Eg(x) = Eg_\gamma(x) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2,$$

where $g_\gamma := \sum_{T \in \mathbb{T}_\gamma} g_T$. Hence, by Lemma 3.2,

$$\int_\gamma |Eg|^2 \lessapprox \int |g_\gamma|^2 \sim \sum_{T \in \mathbb{T}_\gamma} \int |g_T|^2$$

up to an error of $\mathrm{RapDec}_\varepsilon(R) \int |g|^2$. Adding up over all $\gamma \in \mathcal{F}$, we obtain

$$\begin{aligned}\int |Eg|^2 w &\lessapprox \sum_{\gamma \in \mathcal{F}} c_\gamma \sum_{T \in \mathbb{T}_\gamma} \int |g_T|^2 = \sum_{T \in \mathbb{T}} \Big(\sum_{\gamma \in \mathcal{F} :\, \gamma \cap T \neq \emptyset} c_\gamma \Big) \int |g_T|^2 \\ &\lessapprox \sum_{T \in \mathbb{T}} \sup_{\ell \subset T} Xw(\ell) \int |g_T|^2\end{aligned}$$

up to an error of $\mathrm{RapDec}_\varepsilon(R) \int |g|^2$ (the final ≈ 1-loss is due to the fact that the tubes in $\mathbb{T}$ have width $R^{1/2+\delta}$, rather than $R^{1/2}$). The last quantity is at most $\|Xw\|_\infty \int |g|^2$. ∎

Remark 3.5. The idea behind the proof of Corollary 3.4 also appeared in [20], where the same result was presented in the special case where the flakes are horizontal slabs. Moreover, it was there pointed out that the statement of the corollary also implies (1.3), i.e., that the Mizohata–Takeuchi conjecture holds with loss $\lessapprox R^{1/2}$ in $\mathbb{R}^2$, by replacing each point in $\operatorname{supp} w$ by a horizontal $R^{1/2}$-slab (a process which enlarges the maximal line occupancy of w by $\lesssim R^{1/2}$). Perhaps an easier way to derive (1.3) is to observe that, by Proposition 2.2, the Mizohata–Takeuchi conjecture holds with ≈ 1-loss for each function g_θ supported in an $R^{1/2}$-cap θ; so (1.3) follows by the Cauchy–Schwarz inequality, as B^1 consists of $\sim R^{1/2}$ such caps.

4. Mizohata–Takeuchi with $R^{(n-1)/(n+1)}$-loss: Theorem 1.2

Theorem 1.2 immediately follows from the stronger Theorem 4.1 below, which takes into account the directions in which the waves propagate. Fix $n \geq 2$. For $g \in L^2(B^{n-1})$ and $\mathbb{T} \subset \mathbb{T}_\varepsilon(B_R)$, define

$$g_{\mathbb{T}} := \sum_{T \in \mathbb{T}} g_T,$$

where $\{g_T\}_{T \in \mathbb{T}_\varepsilon(R)}$ is the wave-packet decomposition of g adapted to B_R (at scale R).

Theorem 4.1. *For every $\varepsilon > 0$, there exists a positive constant C_ε, which depends only on Σ and ε, such that*

$$\begin{aligned}\int_{B_R} |Eg_{\mathbb{T}}|^2 w \leq C_\varepsilon R^\varepsilon \Big(\sum_{T \in \mathbb{T}} \Big[\sum_{B \in \mathcal{B} :\, B \cap T \neq \emptyset} w^{(n+1)/2}(B) \Big] \|g_T\|_2^2 \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^{\frac{2(n-1)}{n+1}} \\ + \mathrm{RapDec}_\varepsilon(R) \|w\|_\infty \int |g_{\mathbb{T}}|^2 \qquad (4.1)\end{aligned}$$

for all $R \geq 1$, $g \in L^2(\Sigma)$, $\mathbb{T} \subset \mathbb{T}_\varepsilon(R)$ and weights $w\colon \mathbb{R}^n \to [0,+\infty)$ on $\mathbb{R}^n$, and for every family $\mathcal{B}$ of boundedly overlapping $R^{1/2}$-balls.

As an immediate consequence of this, we have the following.

Corollary 4.2. *For every $\varepsilon > 0$, there exists a positive constant C_ε, which depends only on Σ and ε, such that*

$$\int_{B_R} |Eg_{\mathbb{T}}|^2 w \leq C_\varepsilon R^\varepsilon \Big(\int |g_{\mathbb{T}}(s)|^2 \sup_{T \| N(s),\, T \in \mathbb{T}} w^{(n+1)/2}(2T)\, ds \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^{\frac{2(n-1)}{n+1}}$$
$$\leq C_\varepsilon R^\varepsilon \sup_{T \in \mathbb{T}} \Big(\int_{2T} w^{(n+1)/2} \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^2 \tag{4.2}$$

up to a $\mathrm{RapDec}_\varepsilon(R)\|w\|_\infty \int |g_{\mathbb{T}}|^2$ *error term, for all $R \geq 1$, $g \in L^2(\Sigma)$, $\mathbb{T} \subset \mathbb{T}_\varepsilon(R)$ and weights $w\colon \mathbb{R}^n \to [0,+\infty)$ on $\mathbb{R}^n$.*

Remark 4.3. We need the error term $\mathrm{RapDec}_\varepsilon(R)\|w\|_\infty \int |g_{\mathbb{T}}|^2$ in these results because w may be large at some points of supp $Eg_{\mathbb{T}}$ which are outside $\bigcup_{T \in \mathbb{T}} T$. Theorem 4.1 manifestly implies Theorem 1.2 directly, since the error term is easily absorbed into the right-hand side of the first inequality of Theorem 1.2. However, unlike in the case of Theorem 1.2, it is definitively not possible to take $\varepsilon = 0$ in Theorem 4.1. This is because of the example (see p. 104 in [30], [26], [6], or pp. 125–126 in [31]) demonstrating the necessity of a logarithmic term in the discrete ℓ^2-L^6 restriction theorem for the paraboloid. For the argument linking the two phenomena, see pp. 355–358 in [8]. As we observe below, Theorem 4.1 is essentially a reformulation of the refined decoupling theorem [21]. It is furthermore closely related to the improved decoupling theorem of [22]. More precisely, if one takes the natural weight $w = |Eg_{\mathbb{T}}|^{4/(n-1)}$ in Theorem 4.1, one obtains an inequality slightly stronger than the one considered in Theorem 1.2 of [22], but with R^ε loss rather than the logarithmic loss obtained there when $n = 2$. Notice the Stein-like nature of the middle term appearing in (4.2).

Theorem 4.1 is actually a reformulation of the following refined Stein–Tomas or decoupling estimate. Theorem 4.4 was also discovered independently by Xiumin Du and Ruixiang Zhang (personal communication).

Theorem 4.4. (Refined decoupling [21]) *Let $\varepsilon > 0$, $g \in L^2(B^{n-1})$, and let $\mathbb{T}$ be a subset of $\mathbb{T}_\varepsilon(B_R)$ with the property that $\|g_T\|_2$ is roughly constant over all $T \in \mathbb{T}$. For each $k \in \mathbb{N}$, denote by U_k an essentially disjoint union of $R^{1/2}$-balls in B_R each intersecting $\sim k$ tubes in $\mathbb{T}$. Then the function*

$$g_{\mathbb{T}} = \sum_{T \in \mathbb{T}} g_T$$

satisfies

$$\|Eg_{\mathbb{T}}\|_{L^{2(n+1)/(n-1)}(U_k)} \leq C_\varepsilon R^\varepsilon \Big(\frac{k}{\#\mathbb{T}}\Big)^{1/(n+1)} \Big(\sum_{T \in \mathbb{T}} \|Eg_T\|^2_{L^{2(n+1)/(n-1)}} \Big)^{1/2}$$
$$\sim C_\varepsilon R^\varepsilon \Big(\frac{k}{\#\mathbb{T}}\Big)^{1/(n+1)} \Big(\sum_{T \in \mathbb{T}} \|g_T\|_2^2 \Big)^{1/2} \sim C_\varepsilon R^\varepsilon \Big(\frac{k}{\#\mathbb{T}}\Big)^{1/(n+1)} \|g_{\mathbb{T}}\|_2. \tag{4.3}$$

Since $k \leq \#\mathbb{T}$, estimate (4.3) provides an improvement on the classical Stein–Tomas inequality

$$\|Eg_{\mathbb{T}}\|_{L^{2(n+1)/(n-1)}(\mathbb{R}^n)} \lesssim \|g_{\mathbb{T}}\|_2$$

on the 'k-rich' sets U_k in B_R, according to their level k of richness.

If we assume Theorem 4.1, we can immediately deduce Theorem 4.4 by testing on a weight $w \in L^{(n+1)/2}(U_k)$. Indeed, under the hypotheses of Theorem 4.4, we apply Theorem 4.1 and we have

$$\int_{B_R} |Eg_{\mathbb{T}}|^2\, w \leq C_\varepsilon R^\varepsilon \Big(\sum_{T\in\mathbb{T}} \Big[\sum_{B\in\mathcal{B}\,:\,B\cap T\neq\emptyset} w^{(n+1)/2}(B) \Big] \|g_T\|_2^2 \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^{\frac{2(n-1)}{n+1}} + \mathrm{RapDec}_\varepsilon(R)\|w\|_\infty \int |g_{\mathbb{T}}|^2$$

and, suppressing the error term (as we may) and letting $\lambda = \|g_{\mathbb{T}}\|_2^2/\#\mathbb{T}$ denote the common value of $\|g_T\|_2^2$, the right-hand side here equals

$$\begin{aligned} C_\varepsilon R^\varepsilon \lambda^{2/(n+1)} \Big(\sum_{T\in\mathbb{T}} \sum_{B\in\mathcal{B}\,:\,B\cap T\neq\emptyset} w^{(n+1)/2}(B) \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^{\frac{2(n-1)}{n+1}} \\ \sim C_\varepsilon R^\varepsilon (\lambda k)^{2/(n+1)} \Big(\sum_{B\in\mathcal{B}} w^{(n+1)/2}(B) \Big)^{2/(n+1)} \|g_{\mathbb{T}}\|_2^{\frac{2(n-1)}{n+1}} \\ \sim C_\varepsilon R^\varepsilon \Big(\frac{k}{\#\mathbb{T}} \Big)^{2/(n+1)} \|w\|_{(n+1)/2} \|g_{\mathbb{T}}\|_2^2, \end{aligned}$$

as needed to verify Theorem 4.4.

Likewise, Theorem 4.1 will in turn follow from (4.3), as the following simple argument shows.

Proof of Theorem 4.1. Let $\varepsilon > 0$, fix $g \in L^2(B^{n-1})$, $w\colon B_R \to [0, +\infty)$ and $\mathbb{T} \subset \mathbb{T}_\varepsilon(B_R)$. In order to prove (4.1), we may assume that

(a) w is supported in $\bigcup_{T\in\mathbb{T}} T$,

(b) $\|g_T\|_2 \sim 1$ for all $T \in \mathbb{T}$.

Indeed, assumption (a) is possible because, by (wp3), the part of the weight supported outside $\bigcup_{T\in\mathbb{T}} T$ contributes at most $\mathrm{RapDec}_\varepsilon(R)\|w\|_\infty \int |g_{\mathbb{T}}|^2$ to $\int_{B_R} |Eg_{\mathbb{T}}|^2 w$. For (b), observe that, in terms of our goal, it is trivial to control the contributions of the wave packets g_T with $\|g_T\|_2 < R^{-100n}\|g\|_2$. So, by dyadic pigeonholing, it suffices to prove (4.1) under the additional assumption that the g_T have roughly the same L^2 norms over all $T \in \mathbb{T}$. By scaling, we may assume this common value is 1.

We now fix a family $\mathcal{B}$ of boundedly overlapping $R^{1/2}$-balls covering B_R. By the above, it suffices to prove that

$$\int |Eg_{\mathbb{T}}|^2 w \lessapprox \Big(\frac{1}{\#\mathbb{T}} \sum_{T\in\mathbb{T}} \sum_{B\in\mathcal{B}\colon B\cap T\neq\emptyset} w^{(n+1)/2}(B) \Big)^{2/(n+1)} \int |g_{\mathbb{T}}|^2 \tag{4.4}$$

under assumptions (a) and (b).

Let U_k be the union of the balls in this family which meet $\sim k$ members of $\mathbb{T}$. Importantly, (a) ensures that there exists some dyadic $k \in \mathbb{N}$ for which

$$\int_{B_R} |Eg_{\mathbb{T}}|^2\, w \approx \int_{U_k} |Eg_{\mathbb{T}}|^2\, w,$$

So by Hölder's inequality and (4.3), we obtain

$$\int_{B_R} |Eg_{\mathbb{T}}|^2\, w \lessapprox \Big(\int_{U_k} |Eg_{\mathbb{T}}|^{\frac{2(n+1)}{n-1}}\Big)^{\frac{n-1}{n+1}} (w^{(n+1)/2}(U_k))^{2/(n+1)}$$
$$\leq C_\varepsilon R^\varepsilon \Big(\frac{k}{\#\mathbb{T}}\, w^{(n+1)/2}(U_k)\Big)^{2/(n+1)} \int |g_{\mathbb{T}}|^2 \sim C_\varepsilon R^\varepsilon \big(k\, w^{(n+1)/2}(U_k)\big)^{2/(n+1)} (\#\mathbb{T})^{\frac{n-1}{n+1}}.$$

We conclude with a simple counting argument. Indeed, let $\mathcal{B}_k$ be the set of $R^{1/2}$-balls comprising U_k. Then,

$$k\; w^{(n+1)/2}(U_k) \sim \sum_{B \in \mathcal{B}_k} w^{(n+1)/2}(B)\, k \sim \sum_{B \in \mathcal{B}_k} \sum_{T \in \mathbb{T}: T \cap B \neq \emptyset} w^{(n+1)/2}(B)$$
$$= \sum_{T \in \mathbb{T}} \sum_{B \in \mathcal{B}_k : B \cap T \neq \emptyset} w^{(n+1)/2}(B),$$

establishing (4.4) and thus (4.1). ∎

5. Improved Mizohata–Takeuchi estimates for small caps

In this section, we prove Lemma 1.4, which will be key to the proofs of Theorems 6.1 and 6.2. It is a Mizohata–Takeuchi-type estimate which holds for functions supported in small caps, and it represents an improvement over what we can obtain under no support hypothesis.

Towards proving the lemma, we may assume as in Section 2 that all normals to Σ have angle at most $1/100$ from the vertical direction, and that the projection of Σ on the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ is contained in the unit ball B^{n-1} centred at 0. It thus suffices to establish the analogous statement (Lemma 5.1 below) with Eg_τ in place of $\widehat{g_\tau\, d\sigma}$, where E is the extension operator associated to Σ and $g_\tau \in L^2(B^{n-1})$ is a function supported in a $\rho^{-1/2}$-cap τ in B^{n-1}.

To simplify notation, for $E \subset B^{n-1}$ (rather than $E \subset \Sigma$), and any line ℓ (or tube T in B_R), we write $\ell \perp E$ if $\ell \perp \Sigma(E)$ (similarly, we write $T \perp E$ if $T \perp \Sigma(E)$). We also define

$$A_{\rho,R,E}(w) := A_{\rho,R,\Sigma(E)}(w).$$

Lemma 5.1. *For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all weights $w\colon \mathbb{R}^n \to [0,+\infty)$, whenever $1 \leq \rho \leq R$, τ is a $\rho^{-1/2}$-cap in B^{n-1} and $g_\tau \in L^2(B^{n-1})$ is supported in τ, we have*

$$\int_{B_R} |Eg_\tau|^2\, w \leq C_\varepsilon R^\varepsilon\, A_{\rho,R,\operatorname{supp} g_\tau}(w) \int |g_\tau|^2,$$

and therefore also

$$\int_{B_R} |Eg_\tau|^2 \, w \le C_\varepsilon R^\varepsilon \left(\frac{R}{\rho}\right)^{\frac{n-1}{n+1}} \sup_{\ell \perp \operatorname{supp} g_\tau} Xw(\ell) \int |g_\tau|^2.$$

Notice that the tubes and lines featuring here have directions perpendicular to the support of g_τ.

Proof. Let $\varepsilon > 0$ and $R \ge 1$. For $\rho \lesssim R^\varepsilon$, the conclusion of the lemma follows directly from Theorem 1.2. We therefore consider $\rho \gtrsim R^\varepsilon$.

In order to prove the lemma for arbitrary weights, it suffices by dyadic pigeonholing to prove it for weights that are indicator functions. Indeed, first observe that we may assume that $w(x) \ge R^{-2n} \|w\|_\infty$ for all $x \in \operatorname{supp} w$. Therefore, after a dyadic pigeonholing causing losses of $\sim \log R$, we may assume that $w(x) \sim q$ for some fixed $q > 0$ over all $x \in \operatorname{supp} w$; and hence that w is an indicator function, due to the scaling properties of our desired estimate.

So, let w be an indicator function of a non-empty union of unit balls. Fix a $\rho^{-1/2}$-cap τ, and let g be a function supported in τ. Let $\mathbb{T}$ be a family of boundedly overlapping parallel $\rho^{1/2}$-tubes that cover $\operatorname{supp} w$, and point in some direction N normal to $\operatorname{supp} g$; observe that $\mathbb{T} \subset \mathbb{T}_\rho$. At a cost of a $\log R$-loss, it may be further assumed that

$$\frac{w(S_\rho)}{|S_\rho|} \sim \lambda, \quad \text{for all } S_\rho \in \mathbb{T},$$

for some $\lambda \le 1$, hence

$$\begin{aligned} A_{\rho,R,\operatorname{supp} g}(w) &= \sup_{T_R \in \mathbb{T}_R :\, T_R \perp \operatorname{supp} g} \Big(\sum_{S_\rho \subset T_R} \Big(\frac{w(S_\rho)}{|S_\rho|}\Big)^{(n+1)/2} |S_\rho| \Big)^{2/(n+1)} \\ &\sim \lambda \rho \sup_{T_R \in \mathbb{T}_R : T_R \perp \operatorname{supp} g} \#\big\{S_\rho \in \mathbb{T} : S_\rho \cap T_R \ne \emptyset\big\}^{2/(n+1)}. \end{aligned}$$

It therefore suffices to prove that

$$\int |Eg|^2 \, w \le C_\varepsilon R^\varepsilon \lambda \rho \sup_{T_R \in \mathbb{T}_R : T_R \perp \operatorname{supp} g} \#\big\{S_\rho \in \mathbb{T} : S_\rho \subset T_R\big\}^{2/(n+1)} \int |g|^2.$$

Proposition 2.2 ensures that, roughly speaking, $|Eg|$ is constant on each $S_\rho \in \mathbb{T}$. In particular, let $\mathbb{T}_N$ be a set of boundedly overlapping tubes in direction N, of width $\rho^{1/2+\delta}$ and length ρ, that cover B_R. For each $S_\rho \in \mathbb{T}$, fix $\widetilde{S}_\rho \in \mathbb{T}_N$ that intersects S_ρ. By Proposition 2.2,

$$\begin{aligned} \int_{S_\rho} |Eg|^2 w &\lesssim \frac{w(S_\rho)}{|S_\rho|} \int_{2\widetilde{S}_\rho} |Eg|^2 + \operatorname{RapDec}_\varepsilon(R) \int |g|^2 \\ &\sim \lambda \int_{2\widetilde{S}_\rho} |Eg|^2 + \operatorname{RapDec}_\varepsilon(R) \int |g|^2. \end{aligned}$$

By adding over all $S_\rho \in \mathbb{T}$, we obtain

$$\int |Eg|^2 \, w \lesssim \lambda \int |Eg|^2 \, \widetilde{w} + \operatorname{RapDec}_\varepsilon(R) \int |g|^2,$$

where

$$\widetilde{w} := \sum_{S_\rho \in \mathbb{T}} \chi_{2\widetilde{S}_\rho}.$$

Now by Theorem 1.2, we have

$$\int |Eg|^2 \,\widetilde{w} \lessapprox \sup_{T_R \in \mathbb{T}_R :\, T_R \perp \operatorname{supp} g} \widetilde{w}(T_R)^{2/(n+1)} \int |g|^2,$$

and for $T_R \in \mathbb{T}_R$ with $T_R \perp \operatorname{supp} g$, we have

$$\tilde{w}(T_R) \lesssim \rho^{(n+1)/2} \,\#\{S_\rho \in \mathbb{T} : 2S_\rho \cap T_R \neq \emptyset\}.$$

Therefore,

$$\int |Eg|^2 \,w \lessapprox \lambda\Big(\rho^{(n+1)/2} \sup_{T_R \in \mathbb{T}_R :\, T_R \perp \operatorname{supp} g} \#\{S_\rho \in \mathbb{T} : S_\rho \subset T_R\}\Big)^{2/(n+1)} \int |g|^2,$$

as required. ∎

6. Weights constant on slabs: Theorems 1.6 and 1.8

In this section, we will use the favourable estimates for functions g_τ supported in small caps which were established in Section 5 to obtain Mizohata–Takeuchi estimates which improve on Theorem 1.2 for general functions g and weights possessing a certain measure of local constancy. In particular, recall from (1.7) that if a function g_τ is supported in a $\rho^{-1/2}$-cap τ, then the Mizohata–Takeuchi conjecture holds for g_τ with an improved $(R/\rho)^{(n-1)(n+1)}$-loss. Therefore, for any fixed $g \in L^2(B^{n-1})$ and $w\colon \mathbb{R}^n \to [0, +\infty)$, a decoupling inequality of the form

$$\int_{B_R} |Eg|^2 \,w \lessapprox \sum_\tau \int_{B_R} |Eg_\tau|^2 \,w$$

for a boundedly overlapping collection of $\rho^{-1/2}$-caps τ (where $g = \sum_\tau g_\tau$ and $\operatorname{supp} g_\tau \subset \tau$) would directly imply that the Mizohata–Takeuchi conjecture holds for g with the inherited loss $(R/\rho)^{(n-1)(n+1)}$. The smaller the caps we manage to decouple into, the smaller the loss.

In general, it is not possible to decouple into small caps. However, we can indeed decouple into $\rho^{-1/2}$-caps when w is a weight of the form $\sum_{s\in\mathcal{S}} c_s \chi_s$, where $\mathcal{S}$ is a set of disjoint $\rho^{1/2}$-slabs that are ν-parallel to Σ; more precisely, we show that (6.1) below holds. This yields Mizohata–Takeuchi for such weights with an $(R/\rho)^{(n-1)(n+1)}$-loss. If the slabs in $\mathcal{S}$ are allowed to point in any direction, then we can decouple into larger $\rho^{-1/4}$-caps (6.3), inheriting Mizohata–Takeuchi with an $(R/\rho^{1/2})^{(n-1)(n+1)}$-loss.

These results are given in Theorems 6.1 and 6.2 below, which are more precise versions of Theorems 1.6 and 1.8, respectively. As per the above discussion, the new ingredients here are the decoupling inequalities (6.1) and (6.3) which follow. Note that, as in Section 5, we will be working with the extension operator E associated to Σ (rather than with $\cdot\,\widehat{d\sigma}$). When $E \subset B^{n-1}$, we will be using the simpler the notation $A_{\rho,R,E}(w)$ in place of $A_{\rho,R,\Sigma(E)}(w)$, and $\ell \perp E$ (or $T \perp E$) to mean $\ell \perp \Sigma(E)$ (similarly, $T \perp \Sigma(E)$) for any line ℓ and tube T in $\mathbb{R}^n$.

Theorem 6.1 (Roughly horizontal slabs). *Fix $\nu > 0$ and $\varepsilon > 0$. For $1 \le \rho \le R$, let $w\colon \mathbb{R}^n \to [0, +\infty)$ be a weight of the form $\sum_{s\in\mathcal{S}} c_s \chi_s$, where $\mathcal{S}$ is a set of disjoint $\rho^{1/2}$-slabs ν-parallel to Σ, and let $w^\star := \sum_{s\in\mathcal{S}} c_s \chi_{3s}$. For $g \in L^2(B^{n-1})$, write*

$$g = \sum_{\tau\in\mathfrak{T}} g_\tau, \quad \textit{with } \operatorname{supp} g_\tau \subset \tau,$$

where $\mathfrak{T}$ is a family of boundedly overlapping $\rho^{-1/2}$-caps τ in B^{n-1}. Then the decoupling inequality

$$\int_{B_R} |Eg|^2\, w \lessapprox_\nu \sum_{\tau\in\mathfrak{T}} \int_{B_R} |Eg_\tau|^2\, w^\star + \operatorname{RapDec}_\varepsilon(R) \int |g|^2 \tag{6.1}$$

holds. Consequently, we have

$$\begin{aligned}\int_{B_R} |Eg|^2\, w &\le C_{\varepsilon,\nu} R^\varepsilon \sum_{\tau\in\mathfrak{T}} A_{\rho,R,\operatorname{supp} g_\tau}(w) \int |g_\tau|^2 \\ &\lessapprox_\nu \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sum_{\tau\in\mathfrak{T}} \sup_{\ell\perp\operatorname{supp} g_\tau} Xw(\ell) \int |g_\tau|^2.\end{aligned} \tag{6.2}$$

Note that an immediate consequence of (6.2) is

$$\int_{B_R} |Eg|^2\, w \le C_{\varepsilon,\nu} R^\varepsilon A_{\rho,R,\operatorname{supp} g}(w) \int |g|^2 \lessapprox_\nu \Big(\frac{R}{\rho}\Big)^{\frac{n-1}{n+1}} \sup_{\ell\perp\operatorname{supp} g} Xw(\ell) \int |g|^2.$$

Theorem 6.2 (All slabs). *Fix $\varepsilon > 0$. For $1 \le \rho \le R$, let $w\colon \mathbb{R}^n \to [0, +\infty)$ be a weight of the form $\sum_{s\in\mathcal{S}} c_s \chi_s$, where $\mathcal{S}$ is a set of disjoint $\rho^{1/2}$-slabs. Let $w^\star := \sum_{s\in\mathcal{S}} c_s \chi_{3s}$. For $g \in L^2(B^{n-1})$, write*

$$g = \sum_{\tilde\tau\in\tilde{\mathfrak{T}}} g_{\tilde\tau}, \quad \textit{with } \operatorname{supp} g_{\tilde\tau} \subset \tilde\tau,$$

where $\tilde{\mathfrak{T}}$ is a family of finitely overlapping $\rho^{-1/4}$-caps $\tilde\tau$ in B^{n-1}. Then the decoupling inequality

$$\int_{B_R} |Eg|^2\, w \lessapprox \sum_{\tilde\tau\in\tilde{\mathfrak{T}}} \int_{B_R} |Eg_{\tilde\tau}|^2\, w^\star + \operatorname{RapDec}_\varepsilon(R) \int |g|^2 \tag{6.3}$$

holds. Consequently, we have

$$\begin{aligned}\int_{B_R} |Eg|^2\, w &\le C_\varepsilon R^\varepsilon \sum_{\tilde\tau\in\tilde{\mathfrak{T}}} A_{\rho^{1/2},R,\operatorname{supp} g_{\tilde\tau}}(w) \int |g_{\tilde\tau}|^2 \\ &\lessapprox \Big(\frac{R}{\rho^{1/2}}\Big)^{\frac{n-1}{n+1}} \sum_{\tilde\tau\in\tilde{\mathfrak{T}}} \sup_{\ell\perp\operatorname{supp} g_{\tilde\tau}} Xw(\ell) \int |g_{\tilde\tau}|^2.\end{aligned} \tag{6.4}$$

Note that an immediate consequence of (6.4) is

$$\int_{B_R} |Eg|^2\, w \le C_{\varepsilon,\nu} R^{\varepsilon} A_{\rho^{1/2},R,\,\operatorname{supp} g}(w) \int |g|^2 \lessapprox_{\nu} \Big(\frac{R}{\rho^{1/2}}\Big)^{\frac{n-1}{n+1}} \sup_{\ell \perp \operatorname{supp} g} Xw(\ell) \int |g|^2.$$

Proofs of (6.1) *and* (6.3). Fix $\varepsilon > 0$ and $R \ge 1$. Let s be a $\rho^{1/2}$ slab in B_R, and fix $g \in L^2(B^{n-1})$. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be collections of finitely overlapping $\rho^{-1/4}$ and $\rho^{-1/2}$-caps, respectively, that cover B^{n-1}. For $i = 1, 2$, write

$$g = \sum_{\tau \in \mathfrak{T}_i} g_\tau, \quad \text{with } \operatorname{supp} g_\tau \subset \tau.$$

We will show that

$$\int_s |Eg|^2 \le C_\varepsilon R^\varepsilon \sum_{\tau \in \mathfrak{T}_1} \int_{3s} |Eg_\tau|^2$$

and that, if additionally s is ν-parallel to Σ for some $\nu > 0$, then

$$\int_s |Eg|^2 \le C_{\nu,\varepsilon} R^\varepsilon \sum_{\tau \in \mathfrak{T}_2} \int_{3s} |Eg_\tau|^2.$$

Note that henceforth we may assume that $\rho \gtrsim_\varepsilon R^{\varepsilon/n}$ (as otherwise (6.1) and (6.3) follow trivially by the Cauchy–Schwarz inequality), and that $\nu \gtrsim_\varepsilon R^{-\varepsilon}$ (as otherwise $C_{\varepsilon,\nu}$ may be chosen to be an appropriately large power of R for (6.1) to follow).

For this proof, it will be useful to think of g as truly supported on Σ. And indeed, due to our assumption that the normals to Σ create angles at most $1/100$ with the vertical direction, it suffices instead to prove the above decoupling inequalities for $g \in L^2(\Sigma)$, for $\widehat{g\,d\sigma}$ in place of Eg and for $\mathfrak{T}_i$ collections of finitely overlapping $\rho^{-1/4}$-caps and $\rho^{-1/2}$-caps, respectively, of Σ.

Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a non-negative, smooth bump function with $\eta(x) = 1$ for all $x \in B_1$ and $\eta(x) = 0$ for all $x \in B_2$. Denote by η_s a smooth bump function adapted to s. In particular, if $s_0 = [0, \rho^{1/2}]^{n-1} \times [0, 1]$, define

$$\eta_{s_0}(x) := \eta\Big(\frac{x'}{\rho^{1/2}}, x_n\Big),$$

and let $\eta_s(x) := \eta_{s_0}(Mx)$, where M is a rigid motion mapping s to s_0. Let $s^\star$ be a 'dual' object to s, specifically the tube with centre 0, direction the normal to s, length 1 and cross section of radius $\rho^{-1/2+\delta}$. It is easy to see by stationary phase that $\widehat{\eta_s(x)}$ is essentially supported in $s^\star$; more precisely,

$$|\widehat{\eta_s}(y)| = \operatorname{RapDec}_\varepsilon(\rho)\, \|\eta_s\|_1 = \operatorname{RapDec}_\varepsilon(R) \quad \text{for all } y \in \mathbb{R}^n \setminus s^\star.$$

Therefore, for $i = 1, 2$,

$$\int_s |\widehat{g\,d\sigma}|^2 \le \int |\widehat{g\,d\sigma}|^2 \eta_s = \int \Big| \sum_{\tau \in \mathfrak{T}_i} \widehat{g_\tau\, d\sigma} \Big|^2 \eta_s = \int \Big(\sum_{\tau \in \mathfrak{T}_i} \widehat{g_\tau\, d\sigma} \Big) \Big(\sum_{\tau' \in \mathfrak{T}_i} \overline{\widehat{g_{\tau'}\, d\sigma}} \Big) \eta_s$$

$$= \sum_{\tau,\tau' \in \mathfrak{T}_i} \int \Big(\widehat{g_\tau\, d\sigma}\; \overline{\widehat{g_{\tau'}\, d\sigma}} \Big) \eta_s = \sum_{\tau,\tau' \in \mathfrak{T}_i} \int (g_\tau\, d\sigma) * \widetilde{(g_{\tau'}\, d\sigma)}\, \widehat{\eta_s},$$

where, for every $f : \mathbb{R}^n \to \mathbb{C}$, $\tilde{f}$ is defined by $\tilde{f}(y) := \overline{f(-y)}$.

For every $\tau, \tau' \in \mathfrak{T}_i$, the function $(g_\tau \, d\sigma) * (\widetilde{g_{\tau'} \, d\sigma})$ is supported in $\tau - \tau'$, and thus its contribution to the above sum is negligible unless $\tau - \tau'$ intersects $s^\star$. More precisely,

$$\int (g_\tau \, d\sigma) * (\widetilde{g_{\tau'} \, d\sigma}) \, \widehat{\eta_s} = \int_{\mathbb{R}^n \setminus s^\star} (g_\tau \, d\sigma) * (\widetilde{g_{\tau'} \, d\sigma}) \, \widehat{\eta_s} = \mathrm{RapDec}_\varepsilon(R) \, \|g_\tau\|_2 \, \|g_{\tau'}\|_2$$

whenever $\tau - \tau' \cap s^\star = \emptyset$, whence

$$\begin{aligned} \int_s |\widehat{g \, d\sigma}|^2 &= \sum_{\tau,\tau' \in \mathfrak{T}_i : (\tau - \tau') \cap s^\star \neq \emptyset} \int (g_\tau \, d\sigma) * (\widetilde{g_{\tau'} \, d\sigma}) \, \widehat{\eta_s} + \mathrm{RapDec}_\varepsilon(R) \int |g|^2 \\ &= \sum_{\tau,\tau' \in \mathfrak{T}_i : (\tau - \tau') \cap s^\star \neq \emptyset} \int \left(\widehat{g_\tau \, d\sigma} \; \overline{\widehat{g_{\tau'} \, d\sigma}} \right) \eta_s + \mathrm{RapDec}_\varepsilon(R) \int |g|^2 \\ &\le \sum_{\tau,\tau' \in \mathfrak{T}_i : (\tau - \tau') \cap s^\star \neq \emptyset} \left(\int_{3s} |\widehat{g_\tau \, d\sigma}|^2 + \int_{3s} |\widehat{g_{\tau'} d\sigma}|^2 \right) + \mathrm{RapDec}_\varepsilon(R) \int |g|^2 \\ &\le N_i \cdot \sum_{\tau \in \mathfrak{T}_i} \int_{3s} |\widehat{g_\tau \, d\sigma}|^2 + \mathrm{RapDec}_\varepsilon(R) \int |g|^2, \end{aligned} \tag{6.5}$$

where

$$N_i := \max_{\tau \in \mathfrak{T}_i} \#\{\tau' \in \mathfrak{T}_i : (\tau - \tau') \cap s^\star \neq \emptyset\}.$$

Note that for the last inequality in (6.5), we used that $s^\star$ is symmetric around 0.

It now suffices to show that

$$N_1 \le C_\varepsilon \, R^\varepsilon \tag{6.6}$$

and that, if additionally s is ν-parallel to Σ for some $\nu \gtrsim_\varepsilon R^\varepsilon$, then

$$N_2 \le C_{\varepsilon,\nu} R^\varepsilon. \tag{6.7}$$

We first focus on the case $i = 1$. Fix $\tau \in \mathfrak{T}_1$, and let $\omega(\tau)$ denote its centre. The family $\mathfrak{T}_1$ consists of $\rho^{-1/4}$-caps, so the $\tau' \in \mathfrak{T}_1$ with $(\tau - \tau') \cap s^\star \neq \emptyset$ cover the set

$$A(\tau) := \{\omega \in \Sigma : (\tau - \omega) \cap s^\star \neq \emptyset\}.$$

Let e denote the direction of the tube $s^\star$. For every $\omega \in A(\tau)$, there exists $\omega_0 \in \tau$ such that $\omega_0 - \omega \in s^\star$, which implies that

$$|\omega_0 - \omega| \lesssim \rho^{-1/4+\delta} \quad \text{or} \quad \mathrm{Angle}(\omega_0 - \omega, e) \lesssim \rho^{-1/4+\delta},$$

hence

$$|\omega - \omega(\tau)| \lesssim \rho^{-1/4+\delta} \quad \text{or} \quad \mathrm{Angle}(\omega - \omega(\tau), e) \lesssim \rho^{-1/4+\delta}.$$

It follows that $A(\tau)$ can be covered by two $\sim \rho^{-1/4+\delta}$-caps of Σ, and thus by $O(\rho^\delta) = O(R^\varepsilon)$ $\rho^{-1/4}$-caps of Σ. This immediately implies (6.6), which in turn establishes the desired estimate (6.3) when combined with (6.5).

For the case $i = 2$, let $\nu \gtrsim_\varepsilon R^\varepsilon$. Fix $\tau \in \mathfrak{T}_2$ and denote by $\omega(\tau)$ its centre. Similarly to the previous case, the $\tau' \in \mathfrak{T}_2$ with $(\tau - \tau') \cap s^\star \neq \emptyset$ cover the set

$$A(\tau) := \{\omega \in \Sigma : (\tau - \omega) \cap s^\star \neq \emptyset\} = \Sigma \cap (\tau - s^\star).$$

Now however the family $\mathfrak{T}_2$ consists of $\rho^{-1/2}$-caps; moreover, s is ν-parallel to Σ, which implies that all tangents to τ create angles at least ν with the (roughly vertical) direction e of $s^\star$. Therefore,

$$\tau - s^\star \subset R_{s^\star},$$

for some vertical rectangle $R_{s^\star}$, with vertical side of length $\sim_\nu 1$ (roughly the length of $s^\star$) and all other sides of length $\sim_\nu \rho^{-1/2+\delta}$ (approximately the sum of the width of $s^\star$ and the radius of τ).

Due to our assumption that all tangents to Σ create angle at most $1/100$ with the vertical direction, it follows that $\Sigma \cap R_{s^\star}$ (and consequently $A(\tau)$) is contained in a single $\sim_\nu \rho^{-1/2+\delta}$-cap of Σ, and can thus be covered by $O(R^\varepsilon)$ $\rho^{-1/2}$-caps in $\mathfrak{T}_2$. This implies the desired estimate (6.7) and hence completes the proof of (6.1). ∎

Proof of Theorem 6.1. Let ν, ε, R, ρ, w and g be as in the statement of the theorem. Now that (6.1) has been established, it suffices to prove the first assertion in (6.2).

To that end, observe that $w^\star$ is the sum of 3^{n-1} weights: the weight $w_0 := w$ (supported in B_R), and weights w_j of the form $w(\cdot - t_j)$ (for appropriate $t_j \in \mathbb{R}^{n-1} \times \{0\}$, with $|t_j| \leq R$, for $j = 1, 2, \ldots$). It thus suffices to show that

$$\int |Eg|^2 w_j \leq C_{\varepsilon,\nu} R^\varepsilon \sum_{\tau \in \mathfrak{T}} A_{\rho,R,\operatorname{supp} g_\tau}(w) \int |g_\tau|^2$$

for all $j = 1, 2, \ldots$ For $j = 0$ the inequality follows by Lemma 5.1. For $j = 1, 2, \ldots$,

$$Eg = Eg_j(\cdot - t_j), \quad \text{where } g_j := e^{2\pi i \langle t_j, \Sigma(\cdot)\rangle} g.$$

Observe that, denoting $g_{j,\tau} := e^{2\pi i \langle t_j, \Sigma(\cdot)\rangle} g_\tau$, we can write

$$g_j = \sum_{\tau \in \mathfrak{T}} g_{j,\tau}, \quad \text{with } \operatorname{supp} g_{j,\tau} = \operatorname{supp} g_\tau \subset \tau.$$

Therefore, by Lemma 5.1,

$$\int |Eg|^2 w_j = \int |Eg_j(\cdot - t_j)|^2 w(\cdot - t_j) = \int |Eg_j|^2 w$$
$$\leq C_{\varepsilon,\nu} R^\varepsilon \sum_{\tau \in \mathfrak{T}} A_{\rho,R,\operatorname{supp} g_{j,\tau}}(w) \int |g_{j,\tau}|^2 = C_{\varepsilon,\nu} R^\varepsilon \sum_{\tau \in \mathfrak{T}} A_{\rho,R,\operatorname{supp} g_\tau}(w) \int |g_\tau|^2,$$

completing the proof. ∎

Proof of Theorem 6.2. The proof follows the same steps as that of Theorem 6.1, but with the family $\mathfrak{T}$ replaced by $\widetilde{\mathfrak{T}}$. ∎

7. Guth's argument: the $R^{(n-1)/(n+1)}$ barrier

In his recent talk [20],

(a) Guth identified two 'decoupling axioms' (appropriate local constancy and local L^2-orthogonality conditions) that are satisfied by all Eg, and are sufficient to ensure that the Bourgain–Demeter decoupling inequality [8] holds in B_R for every function F satisfying them.

(b) He then constructed a function $F\colon B_R \to \mathbb{C}$ which satisfies the decoupling axioms, but for which the Mizohata–Takeuchi conjecture fails by a factor of $\sim (\log R)^{-3} R^{\frac{n-1}{n+1}}$. Notably, $F_{|B_R}$ is not of the form $Eg_{|B_R}$ for any $g \in L^2(B^{n-1})$.

Guth's decoupling axioms for all Eg are also sufficient to imply the refined decoupling Theorem 4.4 (as a careful review of its proof reveals), and thus its corollary Theorem 1.2, which established the conjecture with a loss of $\lessapprox R^{(n-1)/(n+1)}$. Therefore, our main result is essentially sharp given the techniques used.

In this section, we outline Guth's axiomatic approach and argument demonstrating the existence of a counterexample [20], and briefly review our result within this context. We emphasise that these results are not ours, and we present them only for self-containment.

Fix $R \geq 1$ and $\varepsilon > 0$. In this section, for every $g \in L^2(B^{n-1})$ and every cap τ in B^{n-1}, we denote $g_\tau := g_{|\tau}$. In particular, $g_{B^{n-1}} = g$.

We call a cap τ in B^{n-1} *admissible* if its diameter $d(\tau)$ is a dyadic number that belongs to $[R^{-1/2}, R^{-\varepsilon}] \cup \{2\}$. In this analysis, B^{n-1} is the only admissible cap of diameter 2. Denote by $\mathcal{D}_R$ the set of all admissible caps.

For every $\tau \in \mathcal{D}_R$, let $F_\tau\colon \mathbb{R}^n \to \mathbb{C}$ be some function. Note that the caps τ are simply used for enumeration here, and may be entirely unrelated to properties of F_τ. This is in contrast to, say, functions of the form Eg_τ, which are Fourier-localised close to $\Sigma(\tau)$.

Axiomatic decoupling (Guth [20])

If the decoupling axioms (DA1) *and* (DA2) *below hold for the full sequence* $(F_\tau)_{\tau\in D_R}$, *then the function* $F := F_{B^{n-1}}$ *in* B_R *can be decoupled into the functions* F_θ *corresponding to the smallest possible scale, as follows*:

$$\|F\|_{L^p(B_R)} \leq C_\varepsilon R^{O(\varepsilon)} \Big(\sum_{\theta\in\mathcal{D}_R\colon d(\theta)\sim R^{-1/2}} \|F_\theta\|^2_{L^p(B_R)} \Big)^{1/2} \quad \text{for all } 2 \leq p \leq \frac{2(n+1)}{n-1}.$$

The *decoupling axioms* (DA1) and (DA2) for a sequence $(F_\tau)_{\tau\in\mathcal{D}_R}$ are the following statements.

(DA1) (Local constancy). For every $\tau \in \mathcal{D}_R$ with $d(\tau) \leq R^{-\varepsilon}$, the function $|F_\tau|$ is essentially constant on each translate of

$$\Sigma(\tau)^\star := \{x : |x \cdot (\xi - \xi_\tau)| \leq 1 \text{ for all } \xi \in \Sigma(\tau)\},$$

where ξ_τ denotes the centre of $\Sigma(\tau)$.[3]

[3] Formally, a function is essentially constant on translates of $\Sigma(\tau)^\star$ if it satisfies estimate (24) in the statement of Lemma 6.1 in [18], with θ replaced by the smallest rectangle containing $\Sigma(\tau)$.

(DA2) (Local L^2–orthogonality). Let $\gamma \in D_R$, and suppose that $\gamma = \sqcup_{\tau \in \mathfrak{T}} \tau$, where $\mathfrak{T}$ is a family of finitely overlapping caps in $\mathcal{D}_R$ with diameters smaller than $d(\gamma)$. Then, the estimate

$$\int_K |F_\gamma|^2 \sim \sum_{\tau \subset \gamma} \int_K |F_\tau|^2 + \mathrm{RapDec}_\varepsilon(R) \int |F_\gamma|^2$$

holds for every convex $K \subset \mathbb{R}^n$ such that the sets $\tau + K^\star$, over all $\tau \in \mathfrak{T}$, are finitely overlapping.[4]

It is not hard to see that, for all $g \in L^2(B^{n-1})$, the sequence $(Eg_\tau)_{\tau \in \mathcal{D}_R}$ satisfies (DA1) and (DA2). Guth's axiomatic decoupling statement above, together with a careful review of the proof [18] of the refined decoupling Theorem 4.4 (which directly led to our Theorem 1.2, or equivalently to (7.1) below), reveal the following.

Fact A. (DA1 & DA2 $\Rightarrow$ MT with $\lessapprox R^{(n-1)/(n+1)}$-loss for all Eg) *The fact that*

$$(Eg_\tau)_{\tau \in \mathcal{D}_R} \textit{ satisfies } (\text{DA1}) \textit{ and } (\text{DA2}) \textit{ for all } g \in L^2(B^{n-1})$$

implies the inequality

$$\int_{B_R} |Eg|^2\, w \leq C_\varepsilon R^{\frac{n-1}{n+1}+\varepsilon}\, \|Xw\|_\infty \frac{1}{R} \int_{B_R} |Eg|^2 \tag{7.1}$$

for all $g \in L^2(B^{n-1})$ and $w\colon \mathbb{R}^n \to [0, +\infty)$.

To improve on the Mizohata–Takeuchi conjecture, one needs to reduce the lossy factor $R^{(n-1)/(n+1)}$ in (7.1) (and ideally to remove it altogether). Up to ≈ 1 factors, this is impossible if one insists on only using that all $(Eg_\tau)_{\tau \in \mathcal{D}_R}$ satisfy (DA1) and (DA2). Indeed, Guth [20] proved the following.

Fact B. (DA1 & DA2 $\not\Rightarrow$ MT with $\ll R^{(n-1)/(n+1)}$-loss for general F) *There exists a function $F\colon \mathbb{R}^n \to \mathbb{C}$, with*

$$F = F_{B^{n-1}} \textit{ for some } (F_\tau)_{\tau \in \mathcal{D}_R} \textit{ satisfying } (\text{DA1}) \textit{ and } (\text{DA2}), \tag{7.2}$$

such that

$$\int_{B_R} |F|^2\, w \gtrsim (\log R)^{-3} R^{\frac{n-1}{n+1}}\, \|Xw\|_\infty \frac{1}{R} \int_{B_R} |F|^2 \tag{7.3}$$

for some $w\colon \mathbb{R}^n \to [0, +\infty)$.

Proof. Let Σ be as earlier. The scale $R^{-1/(n+1)}$ plays a key role in the upcoming argument; thus, denote by $\mathcal{D}$ the set of all $\tau \in \mathcal{D}_R$ with $d(\tau) = R^{-1/(n+1)}$ (or, precisely, with $d(\tau)$ equal to the smallest dyadic number that is at least $R^{-1/(n+1)}$). For each $\tau \in \mathcal{D}$, let $\mathbb{T}_\tau$ be a family of finitely-overlapping parallel tubes in $\mathbb{R}^n$ that intersect and cover B_R, of radius $R^{1/(n+1)}$, length $R^{2/(n+1)}$ and direction the normal to $\Sigma(\tau)$ (these tubes are essentially translates of $\Sigma(\tau)^\star$). Let

$$\mathbb{T} := \{T \in \mathbb{T}_\tau : \tau \in \mathcal{D}\}.$$

[4] Without (DA2), no relationship between the different F_τ would be imposed. Observe that, in contrast to the case where $(F_\tau)_{\tau \in \mathcal{D}_R} = (Eg_\tau)_{\tau \in \mathcal{D}_R}$, the equality $F_\gamma = \sum_{\tau \in \mathfrak{T}} F_\tau$ may not hold for a sequence $(F_\tau)_{\tau \in \mathcal{D}_R}$ satisfying the decoupling axioms.

There exists a weight $w\colon \mathbb{R}^n \to [0,+\infty)$ such that the following hold.

(1) w is the characteristic function of a union of $\sim R^{n-1}$ unit balls in B_R.

(2) Each tube L of radius 1 satisfies $w(L) \lesssim \log R$.

(3) Each tube $T \in \mathbb{T}$ satisfies $w(T) \lesssim \log R$, and fully contains every 1-ball in $\operatorname{supp} w$ that it intersects.

This is the weight that will feature in (7.3), and its existence is guaranteed by prior work of the first author (see Theorem 3 in [9]) on aspects of the Mizohata–Takeuchi conjecture. The details are omitted.

The function F will be carefully defined as a sum of wave packets, so that it is large on a big proportion of $\operatorname{supp} w$; more precisely, on a large set $\mathcal{B}$ of unit balls in $\operatorname{supp} w$. The set $\mathcal{B}$ is the one appearing in the claim below. The proof is postponed to the end of the section. (Note that the claim would be trivial if each tube in $\mathbb{T}$ intersected and fully contained at most one 1-ball in $\operatorname{supp} w$.)

Claim 7.1. *There exist*

(i) *a set* $\mathcal{B} = \{B_1, \dots, B_m\}$ *of* $\gtrsim (\log R)^{-2} R^{n-1}$ *disjoint unit balls in* $\operatorname{supp} w$*, and*

(ii) *sets* $\mathbb{T}_j \subset \mathbb{T}$ *with* $\#\mathbb{T}_j \gtrsim \#\mathcal{D}$ *for every* $j = 1, \dots, m$*,*

such that the following hold.

(P1) *The tubes in* $\mathbb{T}_j$ *contain* B_j*, for all* $j = 1, \dots, m$*.*

(P2) *For* $j = 2, \dots, m$*, the tubes in* $\mathbb{T}_j$ *do not intersect any of the balls* $B_1, \dots, B_{j-1}$*.*

We now construct a sequence $(F_\tau)_{\tau \in \mathcal{D}_R}$ of functions $F_\tau\colon \mathbb{R}^n \to \mathbb{C}$ as follows.

- For each $\tau \in \mathcal{D}_R$ with $R^{-1/2} \lesssim d(\tau) < R^{-1/(n+1)}$, define $F_\tau := d(\tau)^{(n-1)/2} \chi_{B_R}$.
- For $\tau \in \mathcal{D}_R$ with $d(\tau) = R^{-1/(n+1)}$ (or, precisely, for each $\tau \in \mathcal{D}$), define

$$F_\tau := \sum_{T \in \mathbb{T}_\tau} c_T \, e^{-2\pi i \langle \cdot\,,\, \xi_\tau \rangle} \, d(\tau)^{(n-1)/2} \, \phi_T,$$

  where ϕ_T is a bump function on T and ξ_τ is the centre of $\Sigma(\tau)$. The coefficients $c_T \in \mathbb{C}$ are defined below.
- For $\gamma \in \mathcal{D}_R$ with $R^{-1/(n+1)} < d(\tau) \le 2$, define

$$F_\gamma := \sum_{\tau \in \mathcal{D}, \tau \subset \gamma} F_\tau.$$

Let $F := F_{B^{n-1}} = \sum_{\tau \in \mathcal{D}} F_\tau$. The coefficients c_T will all have modulus 1, and will be chosen below so that

$$|F| \gtrsim R^{\frac{n-1}{2(n+1)}} \quad \text{on } \bigcup_{B \in \mathcal{B}} B. \tag{7.4}$$

Verifying (7.2) *and* (7.3). For each $\tau \in \mathcal{D}$, F_τ is Fourier supported roughly in the smallest slab containing $\Sigma(\tau)$. It easily follows that $(F_\tau)_{\tau \in \mathcal{D}_R}$ satisfies the decoupling axioms (DA1) and (DA2).

On the other hand, (7.4) and the small line occupancy of w imply (7.3), so F and w do not respect the numerology of the Mizohata–Takeuchi conjecture. Indeed,

$$\int_{B_R} |F|^2 w \gtrsim R^{\frac{n-1}{n+1}} \#\mathcal{B} \gtrsim (\log R)^{-2} R^{\frac{n-1}{n+1}} R^{n-1}$$

by (7.4), while

$$\int |F|^2 \lesssim \sum_{\tau\in\mathcal{D}} |F_\tau|^2 \lesssim \sum_{\tau\in\mathcal{D}} \sum_{T\in\mathbb{T}_\tau} \int_T |c_T\, d(\tau)^{(n-1)/2}|^2 \sim \sum_{\tau\in\mathcal{D}} \sum_{T\in\mathbb{T}_\tau} |T|\cdot|\tau|$$
$$= |B^{n-1}|\cdot|B_R| \sim R^n$$

due to the essential disjointness of the Fourier supports of the F_τ, and therefore

$$\|Xw\|_\infty \frac{1}{R} \int_{B_R} |F|^2 \lesssim (\log R)\, R^{n-1} \lesssim (\log R)^3 R^{-\frac{n-1}{n+1}} \int_{B_R} |F|^2 w.$$

Defining the c_T. For $T \in \mathbb{T}$, let $\tau(T)$ be the cap $\tau \in \mathcal{D}$ with $T \in \mathbb{T}_\tau$. For $B \in \mathcal{B}$, let

$$\mathbb{T}_B := \{T \in \mathbb{T} : T \text{ intersects } B\},$$

and observe that, once the c_T are defined for all $T \in \mathbb{T}$, it will hold that

$$F_{|B} = R^{\frac{-(n-1)}{2(n+1)}} \sum_{T\in\mathbb{T}_B} c_T\, e^{-2\pi i\langle\,\cdot\,,\,\xi_{\tau(T)}\rangle}\, \phi_{T_{|B}}, \quad \text{for all } B \in \mathcal{B}.$$

The c_T are thus defined via an iteration, the j-th step of which ensures that the above sum has large magnitude for $B = B_j$. First, for all $T \in \mathbb{T}_{B_1}$ define

$$c_T := e^{2\pi i\langle x_1,\,\xi_{\tau(T)}\rangle},$$

where x_1 is the centre of B_1. Due to the small radius of B_1,

$$\mathrm{Re}\left(c_T\, e^{-2\pi i\langle x,\,\xi_{\tau(T)}\rangle}\right) = \mathrm{Re}\left(e^{2\pi i\langle x_1 - x,\,\xi_{\tau(T)}\rangle}\right) \gtrsim 1 \quad \text{for all } x \in B_1,$$

hence

$$\mathrm{Re}\Big(R^{\frac{-(n-1)}{2(n+1)}} \sum_{T\in\mathbb{T}_{B_1}} c_T\, e^{-2\pi i\langle\,\cdot\,,\,\xi_{\tau(T)}\rangle}\, \phi_T\Big) \gtrsim R^{\frac{-(n-1)}{2(n+1)}} \#\mathbb{T}_1 \gtrsim R^{\frac{n-1}{2(n+1)}}$$

on B_1. Therefore, once the remaining c_T have been defined, we will have that

$$|F| \geq \mathrm{Re}\, F \gtrsim R^{\frac{n-1}{2(n+1)}} \quad \text{on } B_1,$$

as desired.

Now, fix $j = 2, \ldots, m$. Suppose that, for each $i = 1, \ldots, j-1$, we have performed the i-th step of the iteration, by defining c_T for all $T \in \mathbb{T}_{B_1}$ (when $i = 1$) and for all $T \in \mathbb{T}_{B_i} \setminus (\mathbb{T}_{B_1} \cup \cdots \cup \mathbb{T}_{B_{i-1}})$ (when $i \geq 2$) so that

$$\Big|\mathrm{Re}\Big(R^{\frac{-(n-1)}{2(n+1)}} \sum_{T\in\mathbb{T}_{B_i}} c_T\, e^{-2\pi i\langle\,\cdot\,,\,\xi_{\tau(T)}\rangle}\, \phi_T\Big)\Big| \gtrsim R^{\frac{n-1}{2(n+1)}}$$

on B_i (which ensures that, once the remaining c_T have been defined, we will have that

$$|F| \gtrsim R^{\frac{n-1}{2(n+1)}} \quad \text{on } B_1, \ldots, B_{j-1}.$$

During the j-th step of the iteration, we will define c_T for $T \in \mathbb{T}_{B_j} \setminus (\mathbb{T}_{B_1} \cup \cdots \cup \mathbb{T}_{B_{j-1}})$ so that

$$\Big|\mathrm{Re}\Big(R^{\frac{-(n-1)}{2(n+1)}} \sum_{T \in \mathbb{T}_{B_j}} c_T\, e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle}\, \phi_T\Big)\Big| \gtrsim R^{\frac{n-1}{2(n+1)}}$$

on B_j (ensuring that eventually $|F| \gtrsim R^{\frac{n-1}{2(n+1)}}$ on B_j as well). Write

$$\mathbb{T}_{B_j} := \mathbb{T}^1_{B_j} \sqcup \mathbb{T}^2_{B_j},$$

where $\mathbb{T}^1_{B_j} := \mathbb{T}_{B_j} \setminus (\mathbb{T}_{B_1} \cup \cdots \cup \mathbb{T}_{B_{j-1}})$ (the set of tubes through B_j for which we still need to define the c_T), while $\mathbb{T}^2_{B_j}$ consists of the tubes through B_j for which the c_T have already been defined. Importantly, $\mathbb{T}^1_{B_j} \supset \mathbb{T}_j$.

Let σ_{B_j} be the sign of $F^2_j := \mathrm{Re}\big(\sum_{T \in \mathbb{T}^2_{B_j}} c_T\, e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle}\, \phi_T\big)$ on[5] B_j, and define

$$c_T := \sigma_{B_j}\, e^{2\pi i \langle x_j, \xi_{\tau(T)}\rangle} \quad \text{for all } T \in \mathbb{T}^1_{B_j},$$

where x_j is the centre of B_j. As earlier,

$$\big|\mathrm{Re}(c_T\, e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle})\big| \gtrsim 1 \quad \text{on } B_j;$$

and, crucially, $\mathrm{Re}(c_T\, e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle})$ also has sign σ_{B_j} on B_j, for all $T \in \mathbb{T}^1_{B_j}$. Therefore, the functions F^2_j and

$$F^1_j := \mathrm{Re}\Big(R^{\frac{-(n-1)}{2(n+1)}} \sum_{T \in \mathbb{T}^1_{B_j}} c_T\, e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle}\, \phi_T\Big)$$

have the same sign on B_j, so

$$\Big|\mathrm{Re}\Big(R^{\frac{-(n-1)}{2(n+1)}} \sum_{T \in \mathbb{T}_{B_j}} c_T e^{-2\pi i \langle\, \cdot\,,\, \xi_{\tau(T)}\rangle} \phi_T\Big)\Big| = |F^1_j + F^2_j| \geq |F^1_j| \gtrsim R^{\frac{-(n-1)}{2(n+1)}}\, \#\mathbb{T}_j \gtrsim R^{\frac{n-1}{2(n+1)}}$$

on B_j, as desired.

For all $T \in \mathbb{T}$ that do not contain any of the balls in $\mathcal{B}$, we define $c_T = 1$. By the end of the iteration, (7.4) holds. ∎

Proof of Claim 7.1. Let $\mathcal{P}$ be a family of disjoint unit balls inside $\operatorname{supp} w$, with

$$\#\mathcal{P} \sim |\operatorname{supp} w| \sim R^{n-1}.$$

For each $B \in \mathcal{P}$, denote by $\mathbb{T}_B$ the set of tubes in $\mathbb{T}$ through B; observe that $\#\mathbb{T}_B = \#\mathcal{D}$.

[5] Technically, this sign does not have to be uniform over all points of B_j; we can however choose the dominant sign over B_j, and eventually control the sum of the F_τ on a large subset of B_j. We omit this additional technicality from our exposition.

Write $\mathcal{P} = \{B_1, B_2, \dots, B_N\}$. To prove the claim, we will show that there exist indices $k_1 < k_2 < \cdots < k_m$ such that

- $m \gtrsim (\log R)^{-10} R^{n-1}$,
- $B_{k_1} = B_1$, and for each $j = 2, 3, \dots, m$, at least $\#\mathcal{D}/2$ tubes in $\mathbb{T}_{B_{k_j}}$ do not lie in $\mathbb{T}_{B_{k_1}} \cup \mathbb{T}_{B_{k_2}} \cup \cdots \cup \mathbb{T}_{B_{j-1}}$.

Indeed,

- let $k_1 := 1$,
- let k_2 be the smallest $j > k_1$ such that at most $\#\mathcal{D}/2$ tubes through B_j contain B_{k_1},
- let k_3 be the smallest $j > k_2$ such that at most $\#\mathcal{D}/2$ tubes through B_j contain B_{k_1} or B_{k_2},

and so on, until no further k_j as above exists. Let $\mathcal{P}^1$ be the set of balls B_{k_j}, over all the k_j selected via the above process. To complete the proof of the claim, it will now be shown that

$$\#\mathcal{P}^1 \gtrsim (\log R)^{-2} R^{n-1},$$

by studying the incidences between $\mathcal{P}$ and $\mathbb{T}$. For any $\mathcal{S} \subset \mathcal{P}$ and $\mathbb{L} \subset \mathbb{T}$, denote

$$I(\mathcal{S}, \mathbb{L}) := \#\{(B, T) \in S \times \mathbb{L} : B \text{ is contained in } T\},$$

the number of incidences between $\mathcal{S}$ and $\mathbb{L}$.

Assume for contradiction that

$$\#\mathcal{P}^1 \lesssim (\log R)^{-2} \#\mathcal{P} \tag{7.5}$$

for an appropriately small implicit constant. Then, the set $\mathbb{T}^1$ of tubes in $\mathbb{T}$ that pass through balls in $\mathcal{P}^1$ is not too large; in particular,

$$\#\mathbb{T}^1 \le I(\mathcal{P}^1, \mathbb{T}^1) \le \mathcal{P}^1 \#\mathcal{D} \lesssim (\log R)^{-2} \#\mathcal{P} \#\mathcal{D} \sim (\log R)^{-2} I(\mathcal{P}, \mathbb{T}) \lesssim (\log R)^{-1} \#\mathbb{T},$$

for a small implicit constant. Therefore, the tubes in $\mathbb{T}^1$ only contribute a small fraction of the total incidences between $\mathbb{T}$ and $\mathcal{P}$:

$$I(\mathcal{P}, \mathbb{T}^1) \lesssim \#\mathbb{T}^1 \log R \lesssim \mathbb{T} \sim (\log R)^{-1} \#\mathcal{D} R^{n-1} \sim \#\mathcal{D} \#\mathcal{P} \le \tfrac{1}{10} I(\mathcal{P}, \mathbb{T})$$

(the implicit constant in (7.5) is chosen so that this is true).

This is a contradiction, as $\mathcal{P}^1$ was selected so that $\mathbb{T}^1$ $(= \bigcup_{j=1}^m \mathbb{T}_{B_{k_j}})$ contributes at least half of the total incidences between $\mathbb{T}$ and $\mathcal{P}$. Indeed, each $B_i \in \mathcal{P} \setminus \mathcal{P}^1$ is incident to at least $\#\mathcal{D}/2$ tubes in $\bigcup_{k_j < i} \mathbb{T}_{B_{k_j}} \subset \mathbb{T}^1$; while each $B_i \in \mathcal{P}$ has all the $\#\mathcal{D}$ tubes in $\mathbb{T}$ through it in $\mathbb{T}^1$. Therefore,

$$I(\mathcal{P}, \mathbb{T}^1) \ge \#\mathcal{P} \#\mathcal{D}/2 = I(\mathcal{P}, \mathbb{T})/2,$$

contradicting (7.5). ∎

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