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Projections of surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$ and the geometry of their singular images

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Abstract. We study the geometry of germs of singular surfaces in $\mathbb{R}^3$ whose parametrisations have an $\mathcal{A}$-singularity of $\mathcal{A}_e$-codimension less than or equal to 3, via their contact with planes. These singular surfaces occur as projections of smooth surfaces in $\mathbb{R}^4$ to $\mathbb{R}^3$. We recover some aspects of the extrinsic geometry of these surfaces in $\mathbb{R}^4$ from those of the images of their projections.

1. Introduction

Our investigation of singular surfaces is motivated by the study of the geometry of smooth surfaces in $\mathbb{R}^4$. Let P_v be the orthogonal projection in $\mathbb{R}^4$ along the non zero vector $v \in \mathbb{R}^4$ to the 3-space $v^\perp$. Given an embedded surface M in $\mathbb{R}^4$, the surface $P_v(M)$ can be regular or, at any given point, can generically have one of the local singularities in Table 1. We seek to extract geometric information about M from $P_v(M)$. We consider the geometric properties of $P_v(M)$, as a surface in the 3-space $v^\perp$, obtained via its contact with planes in $v^\perp$.

We take $\mathbb{R}^3$ as a model for $v^\perp$. Parametrised surfaces in $\mathbb{R}^3$ can have stable singularities of cross-cap (also called Whitney umbrella) type. The differential geometry of the cross-cap is studied, for instance, in [6], [8], [9], [18], [21], and [24]. We study in this paper the geometry of singular surfaces $S \subset \mathbb{R}^3$ derived from the contact of S with planes. We shall suppose that S is parametrised by $\phi\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, where ϕ is $\mathcal{A}$-equivalent to one of the normal forms in Table 1. (Two germs f and g are said to be $\mathcal{A}$-equivalent, written $f \sim_{\mathcal{A}} g$, if $g = k \circ f \circ h^{-1}$ for some germs of diffeomorphisms h and k of the source and target respectively.) Of course we cannot take ϕ as one of the normal forms in Table 1 as diffeomorphisms in the target do not preserve the geometry of the image of ϕ.

The singularities in Table 1 are of corank 1, so one can write ϕ in the form $(x, p(x, y), q(x, y))$, with p and q having no constant or linear parts. We can then

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Table 1. Classes of $\mathcal{A}$-map-germs of $\mathcal{A}_e$-codimension less than or equal to 3, [16].

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	$(x, y, 0)$	0
Crosscap	(x, y^2, xy)	0
$S_k^\pm$	$(x, y^2, y^3 \pm x^{k+1}y),\ k = 1, 2, 3$	k
$B_k^\pm$	$(x, y^2, x^2y \pm y^{2k+1}),\ k = 2, 3$	k
$C_3^\pm$	$(x, y^2, xy^3 \pm x^3y)$	3
H_k	$(x, xy + y^{3k-1}, y^3),\ k = 2, 3$	k
P_3 *	$(x, xy + y^3, xy^2 + ay^4),\ a \neq 0, \frac{1}{2}, 1, \frac{3}{2}$	3

* The codimension of P_3 is that of its stratum.

associate to ϕ a pair of quadratic forms (j^2p, j^2q), given by the second degree Taylor expansions of p and q at the origin. As the contact of a surface with a plane is invariant under affine transformations, we classify the singular points of S according to the $\mathcal{G} = GL(2,\mathbb{R}) \times GL(2,\mathbb{R})$-class of (j^2p, j^2q) (Definition 2.1). We obtain more geometric information about the cross-cap in Section 2. For instance, in Theorem 2.3 we relate the singularities of the height functions on the cross-cap to the torsion of the branches of its parabolic set. For the remaining singularities in Table 1, we identify in Theorem 2.7 the singularities of the parabolic set of S in the source (which we call the preparabolic set and denote by PPS) as well as those of the height functions on S (Theorem 2.8). We explain in Remark 2.10 and Table 4 the high degeneracy of the singularities of the PPS.

In Section 3 we apply the results in Section 2 to obtain geometric information about surfaces in $\mathbb{R}^4$. A point on a generic surface in $\mathbb{R}^4$ is called elliptic, hyperbolic, parabolic or an inflection point (see Section 3). One key observation we make here is that this classification is precisely that of the $\mathcal{G}$-classification of the singular points of $P_v(M)$ along any tangent direction v (Theorem 3.3). This explains a result in [18] comparing the type of the cross-cap of $P_v(M)$ at $P_v(p)$ and that of the point p.

It is worth observing that the results in this paper are independent of the metric as they are derived from the contact of the surfaces with planes and lines. They are valid, for instance, for projections of surfaces in projective 4-space to projective 3-space.

2. The geometry of singular surfaces

We consider the local geometry of a singular surface S parametrised locally by the germ of a smooth function $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, where ϕ is $\mathcal{A}$-equivalent to a singularity of $\mathcal{A}_e$-codimension less than or equal to 3 in Table 1. More specifically, we consider the contact of these singular surfaces with planes. This contact is measured by the $\mathcal{K}$-singularities of the members H_v of the family of height functions on S, $H : S \times \mathbb{S}^2 \to \mathbb{R}$, given by

$$H(x, y, v) = H_v(x, y) = \phi(x, y) \cdot v,$$

where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. (Two germs, at the origin, of functions f and g are $\mathcal{K}$-equivalent if $g(x, y) = k(x, y)f(h^{-1}(x, y))$, where h is the germ of a diffeomorphism and k is the germ of a function not vanishing at the origin.) The $\mathcal{K}$-singularities we shall use in this paper are the simple singularities (below left, [1]) and unimodal singularities (below right, [23]), having the following normal forms:

$A_k^{\pm} : x^2 \pm y^{k+1}, k \geq 0$	$J_{10} : x^3 + ax^2y^2 + y^6,\ 4a^3 + 27 \neq 0$
$D_k^{\pm} : x^2y \pm y^{k-1}, k \geq 4$	$X_{1,0} : x^4 + ax^2y^2 + y^4,\ a^2 - 4 \neq 0$
$E_6 : x^3 + y^4$	$X_{1,0} : xy(x^2 + axy + y^2),\ a^2 - 4 < 0$
$E_7 : x^3 + xy^3$	
$E_8 : x^3 + y^5$	

(In the complex case, the singularity $X_{1,0}$ has one normal form given by $x^4 + ax^2y^2 + y^4$, $a^2 - 4 \neq 0$, but this form does not include the case of two real roots.) The zero sets of the above singularities are drawn in Table 4. In this paper, the singularity type of the zero set of the germ of a function refers to the $\mathcal{K}$-singularity type of the germ of the function.

Contact with planes is affine invariant, therefore we can make affine changes of coordinates in the target (see [3]).

All the singularities in Table 1 are of corank 1, so we can make changes of coordinates in the source and rotations in the target and write ϕ in the form

$$\phi(x, y) = (x, p(x, y), q(x, y))$$

with $p, q \in \mathcal{M}^2(x, y)$ where $\mathcal{M}(x, y)$ denotes the maximal ideal in the ring of germs of functions in (x, y). We write $Q_1(x, y) = j^2p(x, y) = p_{20}x^2 + p_{21}xy + p_{22}y^2$ and $Q_2(x, y) = j^2q(x, y) = q_{20}x^2 + q_{21}xy + q_{22}y^2$, where the k-jet j^kf of a germ f at the origin is its Taylor polynomial of degree k at the origin.

We consider the action of $\mathcal{G} = GL(2, \mathbb{R}) \times GL(2, \mathbb{R})$ on the pairs (Q_1, Q_2), of binary forms given by linear changes of coordinates in the source and target. The $\mathcal{G}$-orbits (see for example [12]) are listed in Table 2.

Definition 2.1. A singular point of S is called hyperbolic, elliptic, or parabolic or is said to be an inflection point if the $\mathcal{G}$-class of (Q_1, Q_2) is as in Table 2.

At a singular point of S, $d\phi_0(T_0\mathbb{R}^2)$ is a line, which we call the tangent line to S. There is a plane of directions orthogonal to this tangent line. These directions are called the normal directions to S at the singular point. The Gauss map of S is not defined at a singular point. However, we can still define the closure of the parabolic set of S as the image under ϕ of the zero set of

$$\tilde{K}(x, y) = ((\phi_x \times \phi_y \cdot \phi_{xx})(\phi_x \times \phi_y \cdot \phi_{yy}) - (\phi_x \times \phi_y \cdot \phi_{xy})^2)(x, y). \tag{2.1}$$

Note that away from the singular point, $\tilde{K}$ vanishes if and only if the Gaussian curvature of S vanishes. We call the zero set of $\tilde{K}$ the *preparabolic set* (abbreviated PPS) of S.

Table 2. The $\mathcal{G}$-classes of pairs of quadratic forms.

$\mathcal{G}$-class	Name
(x^2, y^2)	hyperbolic point
$(xy, x^2 - y^2)$	elliptic point
(x^2, xy)	parabolic point
$(x^2 \pm y^2, 0)$	inflection point
$(x^2, 0)$	degenerate inflection
$(0, 0)$	degenerate inflection

Let X be one of the normal forms in Table 1. We define the subset

$$T_X := \{\phi \in \mathcal{E}(2,3) : \phi \sim_{\mathcal{A}} X\}.$$

of the set $\mathcal{E}(2,3)$ of all smooth map-germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$.

We give T_X the induced Whitney topology and say that a property (P) is generic if it is satisfied in a residual subset of T_X. Map-germs in such a residual subset are referred to as *generic* map-germs.

Let W be a codimension k subset of T_X. We can proceed as above and give W the induced Whitney topology. Then $\phi \in W$ is said to be a generic codimension k germ if it belongs to a residual subset of W.

2.1. The cross-cap

The study of the differential geometry of the cross-cap from the singularity theory point of view was initiated in [6], [24]; see also [8], [9], [18], and [21] for other studies of the geometry of the cross-cap. It is shown in [24] that, by a suitable choice of a coordinate system in the source and an affine coordinate change in the target, a parametrisation of a cross-cap can be taken to the form

$$\phi(x,y) = (x, xy + p(y), y^2 + ax^2 + q(x,y)), \tag{2.2}$$

where $p \in \mathcal{M}^4(y)$ and $q \in \mathcal{M}^3(x,y)$. The following is also shown in [24]. When $a<0$, the height function along any normal direction at a cross-cap point has an A_1-singularity. Such a cross-cap is called *hyperbolic cross-caps* as all its points other than the origin, have negative Gaussian curvature (Figure 1, left). When $a > 0$, there are two normal directions $(0, \pm 2\sqrt{a}, 1)$ at the cross-cap point along which the height function has a singularity more degenerate than A_1 (that is, of type $A_{\geq 2}$). Such a cross-cap is called an *elliptic cross-cap* (Figure 1, right). The singularity of the height function along the degenerate normal direction is precisely of type A_2 if and only if $q(\mp\frac{1}{\sqrt{a}}, 1) \neq 0$. When $a = 0$, there is a unique normal direction at the cross-cap point where the height function has a singularity more degenerate than A_1. The singularity of its corresponding height function is of type A_2 if and only if $\frac{\partial^3 q}{\partial x^3}(0,0) \neq 0$. Such a cross-cap is called a *parabolic cross-cap*.

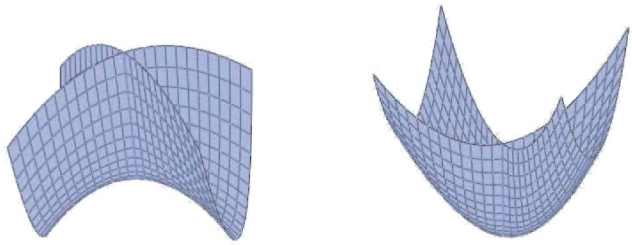


FIGURE 1. Hyperbolic and elliptic cross-caps.

We start with this simple but important observation.

Theorem 2.2. *A cross-cap is hyperbolic, elliptic, or parabolic if and only if its singular point is elliptic, hyperbolic, or parabolic (as in Table 2) respectively.*

Proof. The pair of quadratic forms associated to ϕ in (2.2) is $(xy, y^2 + ax^2)$. This is $\mathcal{G}$-equivalent to $(xy, x^2 - y^2)$, (x^2, y^2), or (x^2, xy) in Table 2 if and only if $a < 0$, $a > 0$, or $a = 0$, and the result follows from the discussion above. $\square$

We introduce a new notation and call an elliptic cross-cap where the height function has an A_i-singularity along one degenerate direction and an A_j-singularity along the other degenerate direction an *elliptic cross-cap of type* A_iA_j or an A_iA_j-*elliptic cross-cap*. Likewise, we call an A_k-*parabolic cross-cap* one where the height function has a degenerate singularity (of type A_k) along the unique degenerate normal direction.

When $a \neq 0$ above, the PPS has an A_1^+-singularity if $a < 0$ and A_1^--singularity if $a > 0$. The closure of the parabolic set on the cross-cap consists of two tangential curves, and each branch of the parabolic set is linked to one of the two degenerate normal directions at the cross-cap point.

Theorem 2.3. *Let $P_i(t), i = 1, 2$, be parametrisations of the branches of the parabolic set on an elliptic cross-cap (with $P_i(0)$ being the cross-cap point) and denote by $\tau_i(t)$ the torsions of these space curves. Then the height function along the degenerate normal direction associated to the branch P_i has a singularity at the cross-cap point of type*

$$\begin{aligned} A_2 &\iff \tau_i(0) \neq 0, \\ A_3 &\iff \tau_i(0) = 0, \tau_i'(0) \neq 0, \\ A_4 &\iff \tau_i(0) = \tau_i'(0) = 0, \tau_i''(0) \neq 0. \end{aligned}$$

Proof. The proof follows from direct calculations (using Maple). We parametrise the cross-cap as in (2.2) and set $a = 1$ with further affine changes of coordinates. We write $j^5p = p_{44}y^4 + p_{55}y^5$ and $j^5q = q_3 + q_4 + q_5$ with $q_i = \sum_{j=0}^{i} q_{ij}x^{i-j}y^j$. The PPS is given by the zero set of $\tilde{K}$ in (2.1). The 2-jet of $\tilde{K}$ is $4(x - y)(x + y)$.

Consider for example the branch with tangent direction $(1,1)$, which is the graph of the function $y(x)=x+\alpha_2x^2+\alpha_3x^3+\alpha_4x^4+$h.o.t, where the abbreviation h.o.t. indicates higher order terms, with

$$\begin{aligned}
\alpha_2 &= q_{31}+\tfrac{1}{2}q_{32}+\tfrac{3}{2}q_{30},\\
\alpha_3 &= -\tfrac{3}{4}q_{31}q_{33}+\tfrac{3}{8}q_{31}^2+\tfrac{1}{2}q_{31}q_{32}-\tfrac{1}{8}q_{32}^2+\tfrac{3}{4}q_{30}q_{32}-\tfrac{9}{8}q_{30}^2+3q_{40}+2q_{42}\\
&\quad+\tfrac{3}{2}q_{43}+\tfrac{5}{2}q_{41}+q_{44}-\tfrac{9}{8}q_{33}^2-\tfrac{3}{2}q_{33}q_{32}-2p_{44},\\
\alpha_4 &= \tfrac{9}{2}q_{51}+\tfrac{7}{2}q_{53}+5q_{50}+\tfrac{5}{2}q_{55}-5p_{55}-\tfrac{9}{8}q_{33}q_{31}^2+4q_{52}-\tfrac{3}{2}q_{41}q_{33}+9q_{33}p_{44}\\
&\quad+\tfrac{3}{2}q_{40}q_{32}+3q_{30}q_{42}-7q_{31}p_{44}-\tfrac{27}{8}q_{33}q_{31}q_{32}-\tfrac{9}{2}q_{33}q_{30}q_{32}-\tfrac{9}{8}q_{30}q_{31}q_{33}\\
&\quad+3q_{54}+\tfrac{5}{16}q_{31}^2q_{32}+3q_{31}q_{42}-\tfrac{3}{16}q_{30}q_{32}^2-\tfrac{9}{16}q_{32}q_{30}^2-\tfrac{3}{2}q_{32}q_{44}-\tfrac{9}{16}q_{30}q_{31}^2\\
&\quad-\tfrac{9}{2}q_{30}q_{40}-\tfrac{81}{16}q_{30}q_{33}^2+\tfrac{1}{16}q_{32}^3+\tfrac{27}{16}q_{30}^3+\tfrac{45}{16}q_{33}^2q_{32}+\tfrac{9}{2}q_{30}q_{43}+\tfrac{3}{2}q_{41}q_{32}\\
&\quad-12p_{44}q_{30}-\tfrac{9}{2}q_{33}q_{43}+2q_{41}q_{31}+q_{42}q_{32}+3q_{31}q_{43}+\tfrac{9}{2}q_{30}q_{44}+2q_{31}q_{44}\\
&\quad-3q_{42}q_{33}+\tfrac{27}{8}q_{33}^3-\tfrac{9}{4}q_{31}q_{33}^2-6q_{33}q_{44}.
\end{aligned}$$

We calculate the torsion of the curve $\phi(x,y(x))$ and its first two derivatives at $x=0$ (using Maple). Observe that $\tau(0)$, $\tau'(0)$, and $\tau''(0)$ depend only on α_2, α_3, and α_4.

The height function along the degenerate normal direction $v_1=(0,-2,1)$, which corresponds to the branch $(x,y(x))$ of the parabolic set is given by $h_{v_1}=(y-x)^2+q(x,y)+2p(y)$ and has a singularity at the origin of type

$$\begin{aligned}
A_2 &\iff q_3(1,1)\neq 0,\\
A_3 &\iff q_3(1,1)=0 \text{ and } (3q_{33}+q_{31}+2q_{32})^2-4q_4(1,1)+8p_{44}\neq 0,\\
A_4 &\iff q_3(1,1)=(3q_{33}+q_{31}+2q_{32})^2-4q_4(1,1)+8p_{44}=0 \text{ and } O\neq 0
\end{aligned}$$

with

$$\begin{aligned}
O = {} & q_5(1,1)-2p_{55}+\tfrac{1}{4}(q_{32}+3q_{33})(q_{31}+3q_{33}+2q_{32})^2\\
&-\tfrac{1}{2}(q_{41}+2q_{42}+3q_{43}+4q_{44}-8p_{44})(q_{31}+3q_{33}+2q_{32}).
\end{aligned}$$

The result now follows by observing that the above conditions for the singularities of the height function h_{v_1} can be expressed in terms of $\tau(0)$, $\tau'(0)$, and $\tau''(0)$ and these are as in the statement of the theorem. □

Remark 2.4. Theorem 2.3 gives a geometric characterisation of A_iA_j-elliptic cross-caps when $i,j\leq 4$. The A_2A_2-cross-caps are generic, the A_2A_3-cross-caps are of codimension 1, and the A_2A_4 and A_3A_3-cross-caps are of codimension 2.

2.2. Singularities more degenerate than a cross-cap

We turn now to the remaining singularities in Table 1. We shall describe the singularities of the PPS and those of the height functions along normal directions.

When S has an S_k, B_k, or C_3 singularity, we can make changes of coordinates in the source and affine changes of coordinates in the target to give it a parametrisation of the form

$$\phi(x,y)=(x,y^2+p(x),q(x,y)), \tag{2.3}$$

where $p \in \mathcal{M}^2(x)$ and $q \in \mathcal{M}^2(x,y)$ ([7]; the result follows from the fact that $p(x,y)$ is an $\mathcal{R}$-versal unfolding of y^2, so is $\mathcal{R}_+$-equivalent to $y^2+p(x)$. The parametrisation (2.3) can also be used for the cross-cap). We set

$$\begin{aligned} p(x) &= p_{20}x^2 + p_{30}x^3 + p_{40}x^4 + \dots \\ q(x,y) &= q_{20}x^2 + q_{22}y^2 + \textstyle\sum_{j=0}^{3} q_{3j}x^{3-j}y^j + \sum_{j=0}^{4} q_{4j}x^{4-j}y^j + \dots \end{aligned}$$

Note that $q_{21} = 0$, because the singularity of ϕ at the origin is more degenerate than a cross-cap. The conditions for ϕ in (2.3) to have one the $\mathcal{A}$-types in Table 1 are as follows:

$$\begin{aligned} B_1 = S_1 :&\quad q_{31} \neq 0,\ q_{33} \neq 0;\\ B_2 :&\quad q_{31} \neq 0,\ q_{33} = 0,\ 4q_{31}q_{55} - q_{43}^2 \neq 0;\\ B_3 :&\quad q_{31} \neq 0,\ q_{33} = 0, 4q_{31}q_{55} - q_{43}^2 = 0,\\ &\quad 2q_{31}^3q_{77} - (2q_{53}q_{55} + q_{43}q_{44})q_{31}^2 + (q_{43}q_{53} - q_{41}q_{55})q_{43}q_{31} - q_{41}q_{43}^2 \neq 0;\\ S_2 :&\quad q_{31} = 0,\ q_{33} \neq 0,\ q_{41} \neq 0;\\ S_3 :&\quad q_{31} = 0,\ q_{33} \neq 0,\ q_{41} = 0,\ q_{51} \neq 0;\\ C_3 :&\quad q_{31} = 0,\ q_{33} = 0,\ q_{41} \neq 0,\ q_{43} \neq 0. \end{aligned}$$

At an H_k or P_3-singularity, we can give S a parametrisation of the form

$$\phi(x,y) = (x, xy + p(x,y), q_{20}x^2 + q(x,y)), \tag{2.4}$$

where $p, q \in \mathcal{M}^3(x,y)$. The singularities of ϕ are identified as follows:

$$\begin{aligned} H_2 :&\quad q_{33} \neq 0,\ 3p_{55}q_{33}^2 - (4p_{44}\,q_{44} + 3p_{33}\,q_{55})\,q_{33} + 4p_{33}\,q_{44}^2 \neq 0\\ H_3 :&\quad q_{33} \neq 0,\ 3p_{55}\,q_{33}^2 - (4p_{44}\,q_{44} + 3p_{33}\,q_{55})\,q_{33} + 4p_{33}\,q_{44}^2 = 0,\ \xi \neq 0\\ P_3 :&\quad q_{33} = 0, p_{33}\,q_{32}\,q_{44} \neq 0,\ q_{44} \neq 0, 1/2, 3/2. \end{aligned}$$

The expression ξ depends on the 7-jets of p and q.

We start with the identification of the type of the singular point of S.

Theorem 2.5. (1) *Let ϕ be as in* (2.3). *Then the origin is either a hyperbolic point* (*if and only if $q_{20} - p_{20}q_{22} \neq 0$*) *or an inflection point* (*if and only if $q_{20} - p_{20}q_{22} = 0$*).

(2) *Let ϕ be as in* (2.4). *Then the origin is either a parabolic point* (*if and only if $q_{20} \neq 0$*) *or an inflection point* (*if and only if $q_{20} = 0$*).

Proof. For part (1), we make the affine change of coordinates $k(X,Y,Z) = (X, Y, Z - q_{22}Y)$ in the target, so that $j^2(k \circ \phi) = (x, y^2 + p_{20}x^2, (q_{20} - p_{20}q_{22})x^2)$. The result follows by comparing $(y^2 + p_{20}x^2, (q_{20} - p_{20}q_{22})x^2)$ with the normal forms in Table 2. Part (2) is immediate as $j^2\phi = (x, xy, q_{20}x^2)$. □

Remark 2.6. It is worth observing that it follows from Theorem 2.5 that a singular point of a surface with a singularity of type S_k, B_k, or C_3 is never an elliptic or a parabolic point. Similarly, for a surface with a singularity of type H_k and P_3, the singular point is never an elliptic or a hyperbolic point.

In the following, the singularity type of the PPS refers to the $\mathcal{K}$-singularity type of the germ of the function $\tilde{K}$.

Theorem 2.7. *If the singular point of S is not an inflection point, the generic singularities of the PPS are as shown in* Table 3. *If the singular point of S is an inflection point, the PPS has generically an $X_{1,0}$-singularity.*

TABLE 3. The singularities of ϕ and of the PPS of $\phi(\mathbb{R}^2, 0)$.

ϕ	$B_1^{\pm}$	B_2	B_3	S_2	S_3	C_3	H_2	H_3	P_3
PPS	$D_4^{\mp}$	D_5	D_5	E_7	J_{10}	$X_{1,0}$	D_5	D_5	J_{10}

Proof. The PPS is the zero set of the function $\tilde{K}$ in (2.1). For the S_k, B_k and C_3-singularities we take ϕ as in (2.3). Then,

$$\begin{aligned} j^4\tilde{K} = {} & 8(q_{20}-p_{20}q_{22})(-q_{31}x^2y+3q_{33}y^3)-4p_{20}q_{31}^2x^4\\ & -8(q_{31}(3q_{30}-p_{20}q_{32}-3p_{30}q_{22})+q_{41}(q_{20}-p_{20}q_{22}))x^3y\\ & +8q_{31}(2p_{20}q_{33}-3q_{31})x^2y^2+8(3q_{33}(3q_{30}-p_{20}q_{32}-3p_{30}q_{22})\\ & +3q_{43}(q_{20}-p_{20}q_{22})-4q_{31}q_{32})xy^3+16(4q_{44}(q_{20}-p_{20}q_{22})-q_{32}^2)y^4. \end{aligned}$$

The proof is an exercise in the recognition of singularities of functions. If $q_{20}-p_{20}q_{22}=0$ (that is, the origin is an inflection point, see Theorem 2.5), the 4-jet of $\tilde{K}$ is generically a nondegenerate quartic, so the singularity is of type $X_{1,0}$.

Suppose that $q_{20}-p_{20}q_{22}\neq 0$.

The map-germ ϕ has an $S_1^{\pm}(=B_1^{\pm})$-singularity if and only if $q_{31}q_{33}\neq 0$, so the PPS has a $D_4^{\mp}$-singularity.

At an S_2 singularity of ϕ, $q_{31}=0$ and $q_{41}q_{33}\neq 0$. Then the coefficient of x^3y in $\tilde{K}$ becomes $8(q_{20}-p_{20}q_{22})q_{41}$, so the PPS has an E_7-singularity.

At an S_3-singularity of ϕ, $q_{31}=q_{41}=0$ and $q_{51}q_{33}\neq 0$. Working with the 6-jet of $\tilde{K}$ we find that the PPS has a singularity of type J_{10}.

If ϕ has a B_2-singularity, then $q_{33}=0$, $q_{31}\neq 0$, and $4q_{31}q_{55}-q_{43}^2\neq 0$. The coefficient of y^4 in $\tilde{K}$ is not zero if and only if $4(q_{20}-p_{20}q_{22})q_{44}-q_{32}^2\neq 0$. Therefore, the PPS has generically a D_5-singularity. (When $4(q_{20}-p_{20}q_{22})q_{44}-q_{32}^2=0$, we get a D_6-singularity.) Observe that the condition to have a D_5-singularity is distinct from the condition $4q_{31}q_{55}-q_{43}^2=0$ for the map-germ ϕ to have a $B_{\geq 3}$-singularity. Therefore, at a B_3-singularity the PPS also has generically a D_5-singularity.

At a C_3-singularity, $q_{31}=q_{33}=0$ and $q_{41}q_{43}\neq 0$. The 3-jet of $\tilde{K}$ is identically zero and its 4-jet is generically a nondegenerate quartic. Therefore the singularity of the PPS is of type $X_{1,0}$.

At an H_k-singularity of S, we can take ϕ as in (2.4). Then the singularity is of type $H_{\geq 2}$ if and only if $q_{33}\neq 0$. The 4-jet of $\tilde{K}$ is given by

$$\begin{aligned} & 12q_{20}q_{33}yx^2+4q_{20}q_{32}x^3-9q_{33}^2y^4+36p_{33}q_{20}q_{33}y^3x\\ & +4(3q_{33}(p_{31}q_{20}+3q_{30})+3q_{20}(q_{43}-q_{31}p_{33}+p_{31}q_{33})+q_{32}(q_{31}+2p_{32}q_{20}))yx^3\\ & +6(q_{33}q_{31}+2p_{33}q_{20}q_{32}+2q_{20}(2q_{44}-q_{32}p_{33}+q_{33}p_{32})+2q_{33}(q_{31}+2p_{32}q_{20}))y^2x^2\\ & +(-q_{31}^2+4(p_{31}q_{20}+3q_{30})q_{32}+4q_{20}(p_{31}q_{32}+q_{42}-q_{31}p_{32}))x^4. \end{aligned}$$

We have a D_5-singularity if $q_{20}q_{33}\neq 0$. Note that the condition $q_{20}=0$ is that for the origin to be an inflection point (Theorem 2.5), and if it holds, the singularity of the PPS is generically of type $X_{1,0}$. Suppose that $q_{20} \neq 0$. Then the PPS has a D_5-singularity at an $H_{\geq 2}$-singularity of ϕ. If $q_{33} = 0$, we have a P_3-singularity of ϕ and the PPS has generically a J_{10}-singularity. □

We consider now the height functions on $S = \phi(\mathbb{R}^2, 0)$.

Theorem 2.8. (1) *Suppose that the origin is not an inflection point of S. When S has an S_k, B_k, or C_3-singularity, there are two distinct normal directions v_i, $i = 1, 2$, at its singular point along which the height function H_{v_i} has a singularity of type $A_{\geq 2}$. We say that the surface is of type $A_k A_l$ if H_{v_1} has an A_k-singularity and H_{v_2} has an A_l-singularity.*

The S_k-surfaces are always of type $A_2 A_{\geq 2}$; the generic ones are of type $A_2 A_2$ and the type $A_2 A_3$ is of codimension 1.

The B_k and C_3 surfaces are always of type $A_{\geq 2} A_3$. The generic ones are of type $A_2 A_3$ and the type $A_3 A_3$ is of codimension 1.

If S has an H_k-singularity (respectively P_3-singularity), there is a unique degenerate normal direction at its singular point along which the height function has a singularity of type A_2 (respectively generically of type A_3).

(2) *If the singular point of S is an inflection point, there is a unique degenerate normal direction at this point along which the height function has generically a D_4-singularity.*

Proof. (1) We take ϕ as in (2.3). If we set $v = (\alpha, \beta, \gamma)$, we get

$$H_v(x, y) = \alpha x + \beta(y^2 + p(x)) + \gamma q(x, y).$$

This height function is singular at the origin if and only if $\alpha = 0$, that is, if and only if v is in the normal plane to S at the origin. For such v, the 2-jet of H_v is

$$(p_{20}\beta + q_{20}\gamma)x^2 + (\beta + q_{22}\gamma)y^2.$$

The singularity of H_v is of type A_1 if and only if $(p_{20}\beta + q_{20}\gamma)(\beta + q_{22}\gamma) \neq 0$. It is of type $A_{k\geq 2}$ if $p_{20}\beta + q_{20}\gamma = 0$ and $\beta + q_{22}\gamma \neq 0$ or vice-versa. Therefore, there are two distinct directions in the normal plane where the height function has a degenerate singularity of type $A_{k\geq 2}$ unless $p_{20}\beta + q_{20}\gamma = \beta + q_{22}\gamma = 0$. The last two equations are satisfied if and only if $q_{20} - p_{20}q_{22} = 0$, i.e., if and only if the origin is an inflection point. We suppose in this part of the proof that the origin is not an inflection point and deal with each degenerate direction separately.

(i) Suppose that $\beta + q_{22}\gamma \neq 0$ and $p_{20}\beta + q_{20}\gamma = 0$. Then v is parallel to $v_1 = (0, -q_{22}, 1)$ and the 3-jet of H_{v_1} is given by

$$(q_{20} - p_{20}q_{22})x^2 + (q_{30} - q_{22}p_{30})x^3 + q_{31}x^2y + q_{32}xy^2 + q_{33}y^3.$$

At an S_k-singularity of ϕ, $q_{33} \neq 0$, so the height function H_{v_1} has a singularity of type A_2.

Suppose now that $q_{33} = 0$, i.e., ϕ has a B_k or a C_3-singularity. The relevant part of the 4-jet of H_{v_1} is

$$(q_{20} - q_{22}p_{20})x^2 + q_{32}xy^2 + q_{44}y^4$$

and the singularity is of type A_3 if and only if the above expression is not a perfect square, that is, if and only if $4(q_{20}-q_{22}p_{20})q_{44}-q_{32}^2 \neq 0$. This is precisely the condition in the proof of Theorem 2.7 for the PPS to have a D_5-singularity when ϕ has a B_k-singularity, and is distinct from the conditions determining k in the B_k series or the C_3-singularity. When $4(q_{20}-q_{22}p_{20})q_{44}-q_{32}^2=0$, H_{v_1} has a singularity of type $A_{\geq 4}$.

(ii) We suppose now that $p_{20}\beta+q_{20}\gamma \neq 0$ and $\beta+q_{22}\gamma=0$. We have a degenerate direction parallel to $v_2=(0,-q_{20},p_{20})$ and the 3-jet of H_{v_2} is given by

$$-(q_{20}-p_{20}q_{22})y^2+(p_{20}q_{30}-q_{20}p_{30})x^3+p_{20}q_{31}x^2y+p_{20}q_{32}xy^2+p_{20}q_{33}y^3.$$

Thus, H_{v_2} has an A_2-singularity if and only if $p_{20}q_{30}-q_{20}p_{30}\neq 0$.

If $p_{20}q_{30}-q_{20}p_{30}=0$, by analysing the 4-jet of H_{v_2}, we find that its singularity is of type A_3 if and only if $p_{20}^2q_{31}^2-4(q_{20}-q_{22}p_{20})(q_{20}p_{40}-p_{20}q_{40})\neq 0$.

We turn now to the H_k and P_3-singularities and take ϕ as in (2.4). Then, $j^2H_v(x,y)=v_2xy+v_3q_{20}x^2$, so there is a unique direction $v=(0,0,1)$ along which H_v has a singularity more degenerate than A_1. We have $H_v(x,y)=q_{20}x^2+q(x,y)$. As it is assumed that the origin is not an inflection point, $q_{20}\neq 0$, so the singularity of H_v is precisely of type A_2 when $q_{33}\neq 0$, i.e. when ϕ has a singularity of type H_k. It is generically of type A_3 at a P_3-singularity of ϕ.

(2) Suppose now that the origin is an inflection point, so $q_{20}-p_{20}q_{22}=0$, and denote by v $(=v_1=v_2)$ the unique degenerate normal direction. Then the 3-jet of H_v is given by

$$(-q_{22}p_{30}+q_{30})x^3+q_{33}y^3+q_{31}x^2y+q_{32}xy^2.$$

This is a singularity of type D_4 unless the above cubic has a repeated root. □

When the height function on S is degenerate along two distinct normal directions (Theorem 2.8), we can split the PPS of S into two components, with each component related to one of the degenerate normal directions. The following result clarifies the high degeneracy of the singularities of the PPS in Theorem 2.7.

We denote by $\mathcal{L}_i$ the component of the PPS associated to the height function H_{v_i}, $i=1,2$, on S, where the v_i are as in the proof of Theorem 2.8.

Theorem 2.9. *The component $\mathcal{L}_2$ of the PPS is always a smooth curve.*

The component $\mathcal{L}_1$ has a singularity of type A_k when S has an S_k-singularity, $k=1,2,3$. At a $B_{\geq 2}$-singularity of S, the singularity of $\mathcal{L}_1$ is of type A_2 (the singularity can be of type A_3 in a codimension 1 B_k-surface), and at a C_3-singularity of S it is generically of type D_4.

The smooth curve $\mathcal{L}_2$ is transverse to $\mathcal{L}_1$ at an S_1, B_k, or C_3 singularity. The transversality fails at a $S_{\geq 2}$-singularity.

Proof. We parametrise the directions near $v_1=(0,-q_{22},1)$ by $(\alpha,\beta-q_{22},1)$, so the (modified) family of height functions on S is given by

$$H^1(x,y,\alpha,\beta)=\alpha x+(-q_{22}+\beta)(y^2+p(x))+q(x,y).$$

The component $\mathcal{L}_1$ of the PPS is the set of points (x, y) for which there exists (α, β) such that

$$\begin{aligned} H^1_x &= \alpha + 2(q_{20} - q_{22}p_{20})x + \text{h.o.t} = 0, \\ H^1_y &= 2\beta y + q_{31}x^2 + 2q_{32}xy + 3q_{33}y^2 + \text{h.o.t} = 0, \\ (H^1_{xy})^2 - H^1_{xx}H^1_{yy} &= -4(q_{20} - q_{22}p_{20})(q_{32}x + 3q_{33}y + \beta) + \text{h.o.t} = 0. \end{aligned}$$

We are assuming here that the origin is not an inflection point (see Theorem 2.8). The first (respectively third) equation gives α (respectively β) as a function of x and y. Substituting these in the second equation gives an equation with the 2-jet $q_{31}x^2 - 3q_{33}y^2$.

If $q_{31}q_{33} \neq 0$, i.e. ϕ has an S_1-singularity, then $\mathcal{L}_1$ has an A_1-singularity.

If $q_{33} \neq 0$ and $q_{31} = 0$, i.e. ϕ has an S_k-singularity, the relevant part of the equation of $\mathcal{L}_1$ is given by $-3q_{33}y^2 + q_{41}x^3$. Thus, this component has an A_2-singularity at an S_2-singularity of ϕ and an A_3-singularity at an S_3-singularity of ϕ.

If $q_{33} = 0$ and $q_{31} \neq 0$, i.e., ϕ has an B_k-singularity, then a calculation similar to that above shows that $\mathcal{L}_1$ has an A_2-singularity unless $4(q_{20} - q_{22}p_{20})q_{44} - q_{32}^2 = 0$, in which case the singularity has type A_3 (or more degenerate).

When $q_{33} = q_{31} = 0$, ϕ has a C_3-singularity and $\mathcal{L}_1$ has generically a singularity of type D_4.

For the component $\mathcal{L}_2$ of the PPS, we assume, without loss of generality, that $p_{20} \neq 0$ and parametrise the directions near $v_2 = (0, -q_{20}, p_{20})$ by $(\alpha, \beta - q_{20}, p_{20})$. Thus, the (modified) family of height functions on S is given by

$$H^2(x, y, \alpha, \beta) = \alpha x + (-q_{20} + \beta)(y^2 + p(x)) + p_{20}q(x, y).$$

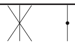
The component $\mathcal{L}_2$ of the PPS is the set of points (x, y) for which there exists (α, β) such that

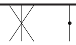
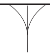
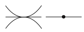
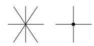
$$\begin{aligned} H^2_x &= \alpha + \text{h.o.t} = 0, \\ H^2_y &= -2(q_{20} - q_{22}p_{20})y + \text{h.o.t} = 0, \\ (H^2_{xy})^2 - H^2_{xx}H^2_{yy} &= -4(q_{20} - q_{22}p_{20}) \\ &\quad \cdot (3(p_{20}q_{30} - q_{20}p_{30})x + p_{20}q_{31}y + \beta p_{20}) + \text{h.o.t} = 0. \end{aligned}$$

The first (respectively third) equation gives α (respectively β) as functions in (x, y). Substituting these in the second equation gives $y = f(x)$, with $f(0) = f'(0) = 0$. Therefore the component $\mathcal{L}_2$ is always a smooth curve. Its tangent direction at the origin is in the direction of $(1, 0)$ and this is transverse to the tangent direction of $\mathcal{L}_1$ at an S_1, B_k, or C_3-singularity. The transversality fails at an $S_{\geq 2}$-singularity. $\square$

Remark 2.10. The results in Theorem 2.9 explain the high degeneracy of the singularities of the PPS when it has two components. Each component has a given singularity type and the two components are transverse except for the $S_{\geq 2}$-surfaces; see Table 4 where "tg" stands for tangency and "$\pitchfork$" stands for transversality between the components $\mathcal{L}_1$ and $\mathcal{L}_2$. Note that the case of an isolated point $X_{1,0}$-singularity does not occur on the PPS.

TABLE 4. The generic structure of the PPS and of its two components.

S	B_1	B_2	B_3
$\mathcal{L}_1$	$A_1^\pm$ 	A_2 	A_2 
$\mathcal{L}_2$	A_0 (⫛)	A_0 (⫛)	A_0 (⫛)
PPS	$D_4^\pm$ 	D_5 	D_5 

S	S_2	S_3	C_3
$\mathcal{L}_1$	A_2 	$A_3^\pm$ 	$D_4^\pm$ 
$\mathcal{L}_2$	A_0 (tg)	A_0 (tg)	A_0 (tg)
PPS	E_7 	J_{10} 	$X_{1,0}$ 

3. Projections of surfaces in $\mathbb{R}^4$ to 3-spaces

The geometry of surfaces in $\mathbb{R}^4$ is studied, for instance, in [4], [5], [10], [11], [13], [14], [15], [19], and [22]. Given a point $p \in M$ consider the unit circle in T_pM parametrised by $\theta \in [0, 2\pi]$. The curvature vectors $\eta(\theta)$ of the normal sections of M by the hyperplane $\langle\theta\rangle \oplus N_pM$ form an ellipse in the normal plane N_pM, called the curvature ellipse [14]. Points on the surface are classified according to the position of the point p with respect to the ellipse (N_pM is viewed as an affine plane through p). The point p is called *elliptic, parabolic, or hyperbolic* if it is inside, on, or outside the ellipse.

The curvature ellipse is the image of the unit circle in T_pM under a map formed by a pair of quadratic forms (Q_1, Q_2). This pair of quadratic forms is the 2-jet of the 1-flat map $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ (i.e. without constant or linear terms) whose graph, in orthogonal coordinates, is locally the surface M. As the contact of the surface with lines and planes is affine invariant [3], an alternative approach for studying the geometry of surfaces in $\mathbb{R}^4$ is given in [4]. It uses the pencil of the binary forms determined by the pair (Q_1, Q_2). Each point on the surface determines a pair of quadratics

$$(Q_1, Q_2) = (ax^2 + 2bxy + cy^2, lx^2 + 2mxy + ny^2).$$

A binary form $Ax^2+2Bxy+Cy^2$ is represented by its coefficients $(A, B, C) \in \mathbb{R}^3$, where the cone $B^2-AC = 0$ corresponds to perfect squares. If the forms Q_1 and Q_2 are independent, they determine a line in the projective plane $\mathbb{R}P^2$ and the cone determines a conic. This line meets the conic in 0, 1, or 2 points if $\delta(p)$ is negative, zero, or positive, where

$$\delta(p) = (an - cl)^2 - 4(am - bl)(bn - cm).$$

A point p is said to be *elliptic, parabolic, or hyperbolic* if $\delta(p)$ is negative, zero, or positive. The set of points (x, y) where $\delta = 0$ is called the *parabolic set* of M and is denoted by Δ. If Q_1 and Q_2 are dependent, the rank of the matrix $\left(\begin{smallmatrix} a & b & c \\ l & m & n \end{smallmatrix}\right)$ is 1 provided either of the forms is nonzero; the corresponding points on the surface are referred to as *inflection* points. (All the above notions agree with those defined using the curvature ellipse.)

We consider the action of $\mathcal{G}$ (see the introduction) on the pairs of binary forms (Q_1, Q_2). The $\mathcal{G}$-orbits and the characterisation of the corresponding point on the surface are as those given in Table 2.

The geometrical characterisation of points of M using singularity theory was first obtained in [15] using the family of height functions $H\colon M \times S^3 \to \mathbb{R}$, with $H(p, w) = p \cdot w$.

The height function $H_w(p) = H(p, w)$ is singular if and only if $w \in N_pM$. It is shown in [15] that an elliptic point is a nondegenerate critical point of H_w for any $w \in N_pM$. At a hyperbolic point, there are exactly two directions in N_pM, called *binormal directions*, such that p is a degenerate critical point of the corresponding height functions. The two binormal directions coincide at a parabolic point. A hyperplane orthogonal to a binormal direction is called an *osculating hyperplane*.

The direction of the kernel of the Hessian of a height function along a binormal direction is an *asymptotic direction* associated to the given binormal direction [15]. The asymptotic directions are called conjugate directions in [14], and are defined as the directions along θ such that the curvature vector $\eta(\theta)$ is tangent to the curvature ellipse (see also [10], [15]). Thus, if p is not an inflection point, there are 0, 1, or 2 asymptotic directions at p depending on p being an elliptic, parabolic, or hyperbolic point. If p is an inflection point, then every direction in T_pM is asymptotic [15]. The configurations of the asymptotic curves at inflection points of imaginary type (where the parabolic set Δ has an A_1^+-singularity) are given in [10], and the configurations at inflection points of real type (where Δ has an A_1^--singularity) and at other points on the curve Δ are given in [5].

Asymptotic directions can also be described as in [17] and [4] via the singularities of the members of the family of projections P of M on hyperplanes. The family of orthogonal projections in $\mathbb{R}^4$ is given by $P : \mathbb{R}^4 \times S^3 \to TS^3$ with

$$P(p, v) = (v, p - (p \cdot v)v).$$

We denote the second component of P by $P_v(p) = p - (p \cdot v)v$. For v fixed, the projection can be viewed locally at a point $p \in M$ as a map-germ $P_v : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. For a generic surface, the germ P_v has only local singularities of $\mathcal{A}_e$-codimension less than or equal to 3 in Table 1. (This is why in Section 2 we considered only surfaces with singularities as in Table 1.)

The projection P_v is singular at p if and only if $v \in T_pM$. The singularity is a cross-cap unless v is an asymptotic direction at p. The codimension 2 singularities occur generically on curves on the surface and the codimension 3 ones at special points on these curves (see Figure 2 for their configurations at noninflection points). The H_2-curve coincides with the Δ-set [4]. The B_2-curve of P_v, with v asymptotic, is also the A_3-set of the height function along the binormal direction associated to v [4]. The A_3-set is called a *flat ridge* in [20]. This curve meets the Δ-set tangentially at isolated points [5], [15]. At inflection points the Δ-set has a Morse singularity and the configuration of the B_2 and S_2-curves there is given in [4].

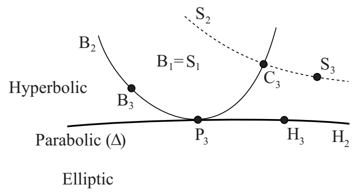


FIGURE 2. Special curves and points on generic surfaces in $\mathbb{R}^4$ away from inflection points.

Let M be a smooth surface in $\mathbb{R}^4$ and let $\psi : U \subset \mathbb{R}^2 \to \mathbb{R}^4$ be a local parametrisation of M. To simplify notation, we write $M = \psi(U)$ and denote also the restriction of P to M by P . Thus, the family of orthogonal projections $P : U \times S^3 \to TS^3$ on M is given by $P((x, y), v) = (v, P_v(\psi(x, y)))$.

Let w be a unit vector in T_vS^3, so $w \cdot v = 0$ and $w \cdot w = 1$. We write

$$\mathcal{D} = \{(v, w) \in S^3 \times S^3 \,|\, v \cdot w = 0\}.$$

Given $(v, w) \in \mathcal{D}$, the height function on the projected surface $P_v(M)$ along the vector w is given by

$$H_{(v,w)}(x, y) = P_v(x, y) \cdot w = (\psi(x, y) - (\psi(x, y) \cdot v)v) \cdot w = \psi(x, y) \cdot w.$$

This is precisely the height function on M along the direction w. It particular, it follows that:

Remark 3.1. The height function $H_{(v,w)}$ on $P_v(M)$ along the direction w has the same singularities as the height function H_w on M along w.

The family $H : U \times \mathcal{D} \to \mathbb{R}$ has parameters in $\mathcal{D}$ which is a 5-dimensional manifold. However, it is trivial along the parameter v. Thus, the generic singularities that can appear in $H_{(v,w)}$ are those of $\mathcal{K}_e$-codimension less than or equal to 3.

For v fixed, w varies in a 2-dimensional sphere, so for a generic M and for most directions v, the height function on $P_v(M)$ has $\mathcal{K}$-singularities of types $A_1^\pm$, A_2, and $A_3^\pm$, and these are versally unfolded by varying w. For isolated directions v, we expect the following singularities: A_4, $D_4^\pm$ and an A_2 or an A_3 singularity which is not versally unfolded by the family H_v. We denote the latter by NVA_2 or NVA_3.

We recover in this section geometric information about the surface M from the geometry of the surface $P_v(M)$. In [18] we considered the $\mathcal{K}$-singularities of the pre-image on M of the parabolic set of $P_v(M)$. We called this pre-image the v-PPS. The generic singularities that appear on the v-PPS can be of high codimension. The results in Section 2 explain the source of this high degeneracy (Theorem 2.9 and Table 4).

We take the point p of interest on M to be the origin in $\mathbb{R}^4$, and take the surface locally at p to have the Monge form $\psi(x, y) = (x, y, f^1(x, y), f^2(x, y))$, with

$$\begin{aligned} f^1(x, y) &= Q_1(x, y) + \textstyle\sum_{i=0}^{3} c_{3i}x^{3-i}y^i + \sum_{i=0}^{4} c_{4i}x^{4-i}y^i + \text{h.o.t.},\\ f^2(x, y) &= Q_2(x, y) + \textstyle\sum_{i=0}^{3} d_{3i}x^{3-i}y^i + \sum_{i=0}^{4} d_{4i}x^{4-i}y^i + \text{h.o.t.}, \end{aligned}$$

where the pair (Q_1, Q_2) of quadratics is one of the normal forms in Table 2.

3.1. Projecting along a nontangential direction

Suppose that $v \in S^3$ is not a tangent direction at $p \in M$. We write $v = v_T + v_N$ where v_T is the orthogonal projection of v on the tangent space T_pM and v_N is its orthogonal projection on the normal space N_pM. Since $v_N \neq 0$, the surface $P_v(M)$ is smooth at $P_v(p)$.

Proposition 3.2. *The height function $H_{(v,w)}$ on $P_v(M)$ is singular at $P_v(p)$ if and only if $w \in N_pM$. For a generic surface, the singularity of $H_{(v,w)}$ at $P_v(p)$ has the type*

- *A_2 if p is a hyperbolic or parabolic point and $w = v_N^\perp$ is a binormal direction, where $v_N^\perp$ is the orthogonal direction to v_N in N_pM.*
- *A_3 if $w = v_N^\perp$ is a binormal direction, p is on the B_2-curve, and v does not lie on a circle of directions C in the sphere $w^\perp \in \mathcal{D}$. Then the v-PPS is a regular curve.*
- *NVA_3 if $w = v_N^\perp$ is a binormal direction, p is on the B_2-curve and $v \in C$. For generic $v \in C$ the singularity of the v-PPS is an A_1-singularity. For isolated directions in C the singularity becomes an A_2-singularity, and for special points on the B_2-curve it becomes an A_3-singularity.*
- *A_4 if $w = v_N^\perp$ is a binormal direction and p is an A_4-point on the B_2-curve.*
- *D_4 if $w = v_N^\perp$ is a binormal direction and p is an inflection point.*

Proof. The identification of the singularities of $H_{(v,w)}$ follows from Remark 3.1. To analyse the structure of the v-PPS, we follow the method in [2] (see also [3]) and consider (locally) the family of Monge–Taylor maps $\theta : M \times S^3 \to V_k$, where V_k denotes the vector space of polynomials in x and y of degree at least 2 and at most k. The family θ is constructed as follows. Given a point q on M near p, we choose an orthonormal coordinate system in $v^\perp \subset \mathbb{R}^4$ so that $\theta_v(M)$ is given locally at $P_v(q)$ in the Monge form $(x, y, f_v(x, y))$. We take $\theta(q, v)$ to be the degree k Taylor polynomial of f_v at the origin.

The singularities of interest are determined by the 3-jet of f_v, so we shall work in V_3. The set of functions in V_3 that have an $A_{\geq 2}$-singularity form a smooth variety of codimension 1, called the A_2-set. Following similar arguments in [2], there is a residual set of embeddings of M in $\mathbb{R}^4$ such that the map θ is transverse to the A_2-set. The intersection of the image of θ with the A_2-set is then a smooth manifold of dimension 4. Therefore, near (p, v_0) its preimage is a smooth manifold W of dimension 4 in $M \times S^3$. The v-PPS are the sections of this manifold by the sets where v is constant. By the Thom transversality theorem, for a generic set of embeddings of M in $\mathbb{R}^4$, the projection $\pi : W \subset (M \times S^3, (p, v_0)) \to (S^3, v_0)$ is $\mathcal{A}$-stable. Thus, the models of the v-PPS are obtained by considering the fibres of $\mathcal{A}$-stable map-germs $(\mathbb{R}^4, 0) \to (\mathbb{R}^3, 0)$. These are (x, y, z); $(x, y, z^2 \pm w^2)$; $(x, y, z^3 + xz + w^2)$; and $(x, y, z^4 + xz^2 + yz \pm w^2)$, where (x, y, z, w) denote the coordinates in $\mathbb{R}^4$. The fibres of these maps (which are models of curves on M, so are plane curves) have singularities of type A_0, A_1, A_2, and A_3 respectively. The specific conditions for these to occur can be found in [18]. □

3.2. Projecting along a tangent direction

Theorem 3.3. *Suppose that v is a tangent direction at $p \in M$. Then the point p on M is an elliptic, hyperbolic, parabolic, or an inflection point if and only if the singular point $P_v(p)$ of $P_v(M)$ is an elliptic, hyperbolic, parabolic, or an inflection point, respectively.*

Proof. Suppose that $v = a\psi_x + b\psi_y$, with $b \neq 0$. We make the affine change of coordinates $(X, Y, Z, W) \to (bX - aY, aX + bY, Z, W)$ in the target so that $P_v(x, y) = (bx - ay, 0, f^1(x, y), f^2(x, y))$, which we simplify to

$$P_v(x, y) = (bx - ay, f^1(x, y), f^2(x, y)).$$

The result follows by observing that $(j^2 f^1(\frac{1}{b}(x + ay), y), j^2 f^2(\frac{1}{b}(x + ay), y))$ is $\mathcal{G}$-equivalent to $(j^2 f^1(x, y), j^2 f^2(x, y))$. (The case $b = 0$ follows similarly.) $\square$

It follows from Theorems 2.2 and 3.3 that if v is a tangent but not an asymptotic direction at $p \in M$, the surface $P_v(M)$ has a hyperbolic, elliptic, or parabolic cross-cap at $P_v(p)$ if and only if p is an elliptic, hyperbolic, or parabolic point (see also [18] for an alternative proof). We have more information on such cross-caps.

Proposition 3.4. *Suppose that $v \in T_pM$ is not an asymptotic direction at p.*

(i) *If p is a hyperbolic point, then $P_v(M)$ is a surface with an elliptic cross-cap of type A_2A_2 if p is not on the B_2-curve. If p is on the B_2-curve, the elliptic cross-cap becomes of type A_2A_3 and at isolated points on this curve it can be of type A_2A_4 or A_3A_3.*

(ii) *If p is a parabolic point, then $P_v(M)$ is in general an A_2-parabolic cross-cap and becomes an A_3-parabolic cross-cap if p is the point of tangency of the B_2-curve with the parabolic set Δ.*

Proof. The type of the cross-cap is determined by the singularities of the height function $H_{(v,w_i)}$ on $P_v(M)$ at $P_v(p)$ along the binormal directions w_i, $i = 1, 2$. It follows from Remark 3.1 that these are the same as the singularities of the height function H_{w_i} on M at p.

In (i) the A_2A_4 cross-cap occurs at special points on the B_2-curve where the height function has an A_4-singularity and these are distinct in general from the B_3 and C_3-points. The A_3A_3 cross-cap occurs at the point of intersection of two B_2-curves associated to the two binormal directions. $\square$

Remark 3.5. With the conditions of Proposition 3.4, the v-PPS has a Morse singularity of type A_1^- when p is a hyperbolic point. When p is on the Δ-curve, the v-PPS has an A_2-singularity if p is not on the B_2-curve and has an A_3-singularity if it is. The v-PPS is studied in [18] by considering the singularities of the function $\tilde{K}$ in (2.1). We observe that the normal to the surface $P_v(M)$ does not have a limit as its singular point is approached. It is of interest to find a way of extending the Monge–Taylor map ([2]) in the proof of Proposition 3.2 to such cases.

When projecting along an asymptotic direction at p (so p is not an elliptic point), the generic singularities of P_v are as those in Table 1 which are more degenerate than a cross-cap. Suppose that p is not an inflection point. The generic singularities of the PPS in Table 3 also occur in the v-PPS. However, when p is on the B_2-curve, there are isolated points where a D_6-singularity occurs on the v-PPS (with v the binormal direction associated to the B_2-curve). These points are precisely those where the height function along v has an A_4-singularity. For the remaining singularities of $P_v(M)$ of a generic M, the singularities of the v-PPS are as in Table 3 (see also Table 4 for the components of the v-PPS).

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