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Semiclassical estimates for measure potentials on the real line

Andrés Larraín-Hubach and Jacob Shapiro

Abstract. We prove an explicit weighted estimate for the semiclassical Schrödinger operator $P = -h^2\partial_x^2 + V(x;h)$ on $L^2(\mathbb{R})$, with $V(x;h)$ a finite signed measure, and where $h > 0$ is the semiclassical parameter. The proof is a one-dimensional instance of the spherical energy method, which has been used to prove Carleman estimates in higher dimensions and in more complicated geometries. The novelty of our result is that the potential need not be absolutely continuous with respect to Lebesgue measure. Two consequences of the weighted estimate are the absence of positive eigenvalues for P, and a limiting absorption resolvent estimate with sharp h-dependence. The resolvent estimate implies exponential time-decay of the local energy for solutions to the corresponding wave equation with a compactly supported measure potential, provided there are no negative eigenvalues and no zero resonance, and provided the initial data have compact support.

1. Introduction and statement of results

The goal of this note is to study the spectral and scattering theory for the one-dimensional semiclassical Schrödinger operator,

$$P = P(h) := -h^2\partial_x^2 + V(x;h) \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad h > 0, \tag{1}$$

with potential $V = V(x;h)$ a real, finite signed Borel measure on $\mathbb{R}$, which may depend on the semiclassical parameter h. Here and below, $L^2(\mathbb{R})$ is the usual Hilbert space of equivalence classes of functions $u \colon \mathbb{R} \to \mathbb{C}$ which are measurable with respect to the Lebesgue sigma algebra on $\mathbb{R}$, and for which $\int_{\mathbb{R}} |u|^2 dx < \infty$, where dx denotes Lebesgue measure.

Self-adjointness of singular Sturm–Liouville operators encompassing (1) was systematically addressed in earlier works [17, 27, 28]. With the objective of being self-contained, we proceed in elementary fashion to specify the domain $\mathcal{D}$ of P as a certain dense subspace of $L^2(\mathbb{R})$ which is contained in the Sobolev space $H^1(\mathbb{R})$. Recall each $u \in H^1(\mathbb{R})$ has a (unique) continuous, bounded representative, which we

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denote by u_c. Thus, for $u \in H^1(\mathbb{R})$, we prescribe the product of u and V to be the complex Borel measure $u_c V$, and define the expression

$$Pu := -h^2 \partial_x^2 u + u_c V \tag{2}$$

in the sense of distributions on $\mathbb{R}$. Using the calculus of functions of bounded variation (which we review in Section 2), we show in Section 3 that (for all $h > 0$) P is self-adjoint with respect to

$$\mathcal{D} := \{u \in H^1(\mathbb{R}) : u' \in L^\infty(\mathbb{R}) \text{ and } Pu \in L^2(\mathbb{R})\}. \tag{3}$$

The main result of this note, whose proof appears in Section 4, is the following weighted estimate on $L^2(\mathbb{R})$.

Theorem 1.1. *Fix $\delta > 0$. For all $E = E(h) > 0$ (which may depend on h), $\varepsilon \in [0, 1]$, $h > 0$, and $u \in \mathcal{D}$ with $(|x| + 1)^{(1+\delta)/2}(P - E \pm i\varepsilon)u \in L^2(\mathbb{R})$,*

$$\begin{aligned} &\int_{\mathbb{R}} (|x| + 1)^{-1-\delta}(E|u|^2 + |hu'|^2)dx \\ &\qquad \le C(V, E, h, \delta) \int_{\mathbb{R}} (|x| + 1)^{1+\delta}|(P - E \pm i\varepsilon)u|^2 dx. \end{aligned} \tag{4}$$

Here,

$$\begin{aligned} &C(V, E, h, \delta) \\ &\quad := e^{C_1(V,E,h,\delta)}\Big(\frac{2}{h} + \Big(\frac{4}{h^2} + \frac{1}{Eh^2}\Big(2 + 2E + \frac{\|V\|^2}{h^2}\Big)^2\Big)e^{C_1(V,E,h,\delta)}\Big), \end{aligned} \tag{5}$$

$$C_1(V, E, h, \delta) := 2\delta^{-1} + E^{-1/2}h^{-1}\|V\|, \tag{6}$$

and $\|V\| := |V|(\mathbb{R})$, with $|V|$ the total variation of V, defined by

$$|V| = V^+ + V^-,$$

where $\{V^+, V^-\}$ is the Jordan decomposition of V (see, e.g., [20, Theorem 3.4]*).*

Remark 1.2. In the case that $E > 0$ is fixed independent of h, and we restrict $h \in (0, 1]$, the constant (5) is bounded from above by the more succinct expression

$$\exp(\widetilde{C}(E, \delta)(1 + \|V\|)/h), \quad \text{for some } \widetilde{C}(E, \delta) > 0 \text{ depending on } E \text{ and } \delta. \tag{7}$$

Two consequences of Theorem 1.1 are the absence of positive eigenvalues of P, and a weighted limiting absorption resolvent bound.

Corollary 1.3. *The operator P on $L^2(\mathbb{R})$, given by (2) and equipped with domain (3), has no positive eigenvalues.*

Corollary 1.4. *Fix $\delta > 0$. Then for all $E = E(h) > 0$ (which may depend on h), $\varepsilon \in (0, 1]$, and $h > 0$,*

$$\|(|x| + 1)^{-\frac{1+\delta}{2}}(P(h) - E \pm i\varepsilon)^{-1}(|x| + 1)^{-\frac{1+\delta}{2}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq \Big(\frac{C(V, E, h, \delta)}{E}\Big)^{1/2}, \tag{8}$$

where $C(V, E, h, \delta)$ is as given in (5).

Remark 1.5. It is well known that $V \in L^1(\mathbb{R}; \mathbb{R})$ implies absence of positive eigenvalues. Furthermore, absence of positive eigenvalues was proved, by different methods, for locally H^{-1} potentials with L^1-type decay, see [30, Theorem 1.9]. This class includes finite signed measures as a special case. Recall also the celebrated von Neumann–Wigner potential W [32, Section XIII.13], which obeys

$$W(x) = -\frac{8\sin(2|x|)}{|x|} + O(|x|^{-2}), \quad \text{as } |x| \to \infty,$$

and has an eigenvalue at $E = 1$.

Remark 1.6. For $E > 0$ fixed and $h \in (0, 1]$, the right side of (8) is bounded from above by an expression of the form (7). In higher dimensions, resolvent upper bounds like (7) are usually proved by first establishing a Carleman estimate, which is similar to (4) but involves an additional weight of the form $e^{\varphi/h}$, where φ is a suitable phase function (see e.g., [7, Theorem 2.2] and [8, Lemma 2.2]). However, our proof of Theorem 1.1 in Section 4 shows that in one dimension it is not necessary to use a phase.

When V is compactly supported, we prove a simpler weighted estimate away from the support of V, which yields an improvement to (8) for h small.

Theorem 1.7 (Exterior estimate). *Fix $\delta > 0$ and $E > 0$. Suppose V is supported in $[-R_0, R_0]$ (independent of h) for some $R_0 > 0$. Set $h_0 = 2^{-1}\delta^{-1}(1 + R_0)^{-\delta}$. There exist C, depending only on R_0, E, and δ (see (40) below), so that for all $\varepsilon \in (0, 1]$ and $h \in (0, h_0]$,*

$$\|(|x| + 1)^{-\frac{1+\delta}{2}}\mathbf{1}_{>R_0}(P - E \pm i\varepsilon)^{-1}\mathbf{1}_{>R_0}(|x| + 1)^{-\frac{1+\delta}{2}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq \frac{C}{h}. \tag{9}$$

Here, $\mathbf{1}_{>R_0}$ denotes the characteristic function of $\{|x| > R_0\}$.

The h-dependencies in (8) and (9) are sharp in general, and were proved previously for $V \in L^1(\mathbb{R}; \mathbb{R})$ [14]. Thus, the novelty of this work is that Theorem 1.1

implies optimal semiclassical resolvent bounds for potentials in one dimension which may not be absolutely continuous with respect to Lebesgue measure.

When V is smooth, the exponential bound (8) (with different constants) was first proved by Burq [4,5], who also considered higher dimensions and more general operators. Further proofs and generalizations can be found in [7–9, 22, 24, 31, 33, 37]. In dimension $n > 1$, current results require at least $V \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, with sufficient decay toward infinity, to obtain a semiclassical resolvent estimate. And often, only weaker versions of (8) are known [10, 23, 29, 38, 39, 45–50], with $e^{C/h}$ replaced by e^{C/h^ℓ} for some $\ell > 1$.

When V is smooth, the improvement (9) away from the support of V was first proved by Cardoso and Vodev [7], refining earlier work of Burq [5], and again analogous results hold in many settings [7–9, 22, 31, 33, 37]. When the dimension $n > 1$, Datchev and Jin [11] showed the cutoff $\mathbf{1}_{|x|>R_0}$ may need to be replaced by $\mathbf{1}_{|x|>R}$ with $R \gg R_0$, even when $V \in C_0^\infty(\mathbb{R}^n)$.

To prove Theorem 1.1, we employ a positive commutator-style argument in the context of the so-called spherical energy method. This strategy has long been used to prove Carleman and related estimates [7, 8, 14, 22, 29, 31]. In fact, as we work in one dimension, it suffices to use the pointwise energy

$$\begin{aligned} &F(x) = F_\pm[u](x) := |hu'(x)|^2 + E|u(x)|^2, \\ &u \in \mathcal{D} \text{ such that } (|x|+1)^{(1+\delta)/2}(P(h) - E \pm i\varepsilon)u \in L^2(\mathbb{R}). \end{aligned} \tag{10}$$

The goal is to construct a suitable weight $w(x)$ having locally bounded variation, so that, roughly speaking, the distributional derivative of wF is bounded from above by a term involving $2w\,\mathrm{Re}((P - E \pm i\varepsilon)u\bar{u}')$, and bounded from below by $w(E|u|^2 + |hu'|^2)$ (see (28) for the precise estimate). To attain the upper bound, V needs to be reintroduced after differentiation of wF, at the cost of a perturbation term (see (26), and note V does not appear in F in the first place because its distributional derivative may be irregular). This perturbation can be controlled, yielding the desired the lower bound, by designing w appropriately. In particular, since V may have discrete part V_d (i.e., countably many point masses which are absolutely summable), we use a family of weights $w_\eta(x)$ depending on a parameter $\eta > 0$. Each w_η controls a certain Gaussian approximation of $|V_d|$ (see (24)). We then show that the needed estimates hold uniformly as $\eta \to 0^+$.

When V is compactly supported, it is well known that Corollary 1.4 is related to the distribution of scattering resonances for the operator $-\partial_x^2 + V$. As in [40], we define the resonances of $-\partial_x^2 + V$ as the poles of the cutoff resolvent

$$\chi(-\partial_x^2 + V - \lambda^2)^{-1}\chi \colon L^2(\mathbb{R}) \to \mathcal{D}, \quad \chi \in C_0^\infty(\mathbb{R}; [0,1]), \chi \equiv 1 \text{ on } \operatorname{supp} V,$$

which continues meromorphically from $\operatorname{Im}\lambda \gg 1$ to the complex plane. In Section 6, we combine (8) with a resolvent identity argument of Vodev [44, Theorem 1.5] to show

Theorem 1.8. *Suppose V is a finite signed Borel measure on $\mathbb{R}$, which is supported in $[-R_0, R_0]$ for some $R_0 > 0$. Fix $\chi \in C_0^\infty(\mathbb{R}; [0,1])$ such that $\chi = 1$ near $[-R_0, R_0]$, and fix $\lambda_0 > 0$. There exist $C, \varepsilon_0 > 0$ so that for all $|\operatorname{Re}\lambda| \geq \lambda_0$, and $|\operatorname{Im}\lambda| \leq \varepsilon_0$,*

$$\|\chi(-\partial_x^2 + V - \lambda^2)^{-1}\chi\|_{L^2 \to H^k} \leq C|\operatorname{Re}\lambda|^{k-1}, \quad k = 0, 1\, (H^0 := L^2(\mathbb{R})), \tag{11}$$

and

$$\|\chi(-\partial_x^2 + V - \lambda^2)^{-1}\chi\|_{L^2 \to \mathcal{D}} \leq C(|\operatorname{Re}\lambda| + 1), \tag{12}$$

where $\mathcal{D}$ is equipped with the graph norm $\|u\|_{\mathcal{D}} := (\|(-\partial_x^2 + V)u\|_{L^2}^2 + \|u\|_{L^2}^2)^{1/2}$.

The existence of resonance free regions below the real axis is a long-studied problem: [25, 26, 51] treat the cases $V \in L^\infty_{\mathrm{comp}}(\mathbb{R})$, $V \in L^1_{\mathrm{comp}}(\mathbb{R})$, and V exponentially decaying, respectively. More recent articles [12, 13, 34] describe the distribution of resonances for thin barriers in the semiclassical regime. To the authors' knowledge, Theorem 1.8 is the first demonstration of a resonance free strip for a class of potentials in one dimension that can have singularly continuous part.

Estimates such as (11) and (12) yield regularity and decay results for operators involving the measure V. As an illustration, in Section 7 we show how (11) implies an exponential local energy decay rate, modulo negative eigenvalues and a possible zero-resonance, for the associated wave equation,

$$\begin{cases} (\partial_t^2 - \partial_x^2 + V(x))w(x,t) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) = w_0(x), \\ \partial_t w(x,0) = w_1(x), \\ \operatorname{supp} w_0,\ \operatorname{supp} w_1 \subseteq (-R, R), & R > 0. \end{cases} \tag{13}$$

See Theorem 7.1 for a precise statement. Similar wave decay was previously established for $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{R})$ ([41] and [16, Theorem 2.9]). We also mention that exterior estimates like (9) have application to integrated wave decay [11, Lemma 5].

There is an extensive literature on second order operators whose coefficients are singular. Thus, we will not attempt to give a comprehensive review here. For the one-dimensional case, we point the reader to [2, 17–19, 27, 28, 35], which develop the Sturm–Liouville theory for such operators, and investigate topics including boundary conditions, self-adjoint extensions, and inverse spectral theory. The research monograph [1] gives a comprehensive treatment of point interactions in three and fewer dimensions. Higher-dimensional studies include [3, 21].

2. Review of BV

To keep the notation concise, for the rest of the article, we use "prime" notation to denote differentiation with respect to x, e.g., $u' := \partial_x u$.

In Section 2, we review the basics of functions of bounded variation (BV), and collect four well-known propositions concerning their calculus. This material is relied upon frequently in later Sections. We give the proof of Proposition 2.2, while proofs of Propositions 2.1, 2.3, and 2.4 may be found in [15, Appendix B].

Let $f : \mathbb{R} \to \mathbb{C}$ be a function of locally bounded variation. For all $x \in \mathbb{R}$, put

$$\begin{aligned} f^L(x) &:= \lim_{\delta \to 0+} f(x-\delta), \\ f^R(x) &:= \lim_{\delta \to 0+} f(x+\delta), \\ f^A(x) &:= (f^L(x) + f^R(x))/2, \end{aligned}$$

where the limits exist because both the real and imaginary parts of f are a difference of two increasing functions. Recall that f is differentiable Lebesgue almost everywhere, so $f(x) = f^L(x) = f^R(x) = f^A(x)$ for almost all $x \in \mathbb{R}$.

We may decompose f as

$$f = f_{r,+} - f_{r,-} + i(f_{i,+} - f_{i,-}),$$

where the $f_{\sigma,\pm}$, $\sigma \in \{r, i\}$, are increasing functions on $\mathbb{R}$. Each $f^R_{\sigma,\pm}$ uniquely determines a regular Borel measure $\mu_{\sigma,\pm}$ on $\mathbb{R}$ satisfying $\mu_{\sigma,\pm}(x_1, x_2] = f^R_{\sigma,\pm}(x_2) - f^R_{\sigma,\pm}(x_1)$, see [20, Theorem 1.16]. We put

$$df := \mu_{r,+} - \mu_{r,-} + i(\mu_{i,+} - \mu_{i,-}),$$

which is a complex measure when restricted to any bounded Borel subset. For any $a < b$,

$$\int_{(a,b]} df = f^R(b) - f^R(a), \qquad \int_{(a,b)} df = f^L(b) - f^R(a). \tag{14}$$

Proposition 2.1 (Integration by parts). *Let $f : \mathbb{R} \to \mathbb{C}$ have locally BV. For any $a < b$, and any continuous φ with φ' piecewise continuous,*

$$\int_{(a,b]} \varphi df + \int_{(a,b]} \varphi' f dx = f^R(b)\varphi(b) - f^R(a)\varphi(a). \tag{15}$$

Proposition 2.2 (Fundamental theorem of calculus). *Let $\mu_{\sigma,\pm}$, $\sigma \in \{r, i\}$ be positive Borel measures on $\mathbb{R}$ which are finite on all bounded Borel subsets of $\mathbb{R}$. Suppose $u \in \mathcal{D}'(\mathbb{R})$ has distributional derivative equal to $\mu = \mu_{r,+} - \mu_{r,-} + i(\mu_{i,+} - \mu_{i,-})$.*

(For example, this will hold for $u \in \mathcal{D}$, with $\mu = u_c V + g dx$ for some $g \in L^2(\mathbb{R})$.) Then u is a function of locally BV. For any $a \in \mathbb{R}$, u differs by a constant from the right continuous, locally BV function

$$f_\mu(x) := \begin{cases} \int_{[a,x]} d\mu & x \geq a, \\ -\int_{(x,a)} d\mu & x < a. \end{cases} \tag{16}$$

Proof. We need to show that the function (16) has distributional derivative μ. First, it is straightforward to check that $f_\mu(x_2) - f_\mu(x_1) = \mu(x_1, x_2]$ for all $x_1 < x_2$. Hence, $df_\mu = \mu$. Then (15) implies

$$-\int_{\mathbb{R}} \varphi' f_\mu dx = \int_{\mathbb{R}} \varphi d\mu, \quad \varphi \in C_0^\infty(\mathbb{R}), \tag{17}$$

where all boundary terms vanish, and the right side of (17) is finite, due to the compact support of φ. ∎

Proposition 2.3 (Product rule). *Let $f, g\colon \mathbb{R} \to \mathbb{C}$ be functions of locally bounded variation. Then*

$$d(fg) = f^A dg + g^A df \tag{18}$$

as measures on a bounded Borel subset of $\mathbb{R}$.

Proposition 2.4 (Chain rule). *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and have locally bounded variation. Then, as measures on a bounded Borel set of $\mathbb{R}$,*

$$d(e^f) = e^f df. \tag{19}$$

3. Self-adjointness for V a measure

The goal of this section is to use the tools of Section 2 to show $(P, \mathcal{D})$ is self-adjoint on $L^2(\mathbb{R})$, where $\mathcal{D}$ is given by (3). This strategy sets the stage for several steps in the proof of Theorem 1.1 in Section 4. We demonstrate that $(P, \mathcal{D})$ is merely the self-adjoint operator naturally associated to the quadratic form

$$q(u, v) := \int h^2 \bar{u}' v' dx + \int \overline{u_c} v_c V, \quad u, v \in H^1(\mathbb{R}). \tag{20}$$

As mentioned in Section 1, self-adjointness was addressed in greater generality elsewhere [17, 27, 28].

Lemma 3.1. *Let $V = V(x; h)$ be a real, finite signed Borel measure on $\mathbb{R}$. Then $\mathcal{D}$ specified by (3) is dense in $L^2(\mathbb{R})$. The operator $P\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by (2) with domain $\mathcal{D}$ is self-adjoint.*

Proof. Throughout the proof, we work with $u \in L^2(\mathbb{R})$ that have locally integrable distributional derivative u'. Such u have a (unique, locally absolutely) continuous representative u_c, hence we can define the product $u_c V$ as a distribution on $\mathbb{R}$, since it is a complex measure when restricted to bounded Borel subsets of $\mathbb{R}$.

Let $\mathcal{D}_{\max} \supseteq \mathcal{D}$ be the set of all $u \in L^2(\mathbb{R})$ such that $u' \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $Pu := -h^2 u'' + u_c V \in L^2(\mathbb{R})$. By Proposition 2.2, for any $u \in \mathcal{D}_{\max}$, we may fix a representative u'_{bv} for u' that has locally bounded variation. If necessary, we redefine u'_{bv} on a set of Lebesgue measure zero so that $u'_{\mathrm{bv}}(x) = (u'_{\mathrm{bv}})^A(x)$ for all $x \in \mathbb{R}$ (this updated u'_{bv} still has locally BV).

In the computations to follow, we always work with the representatives u_c and u'_{bv}, but we drop subscripts to keep notation concise. This convention ensures that expressions like $u'V$ are well defined as complex Borel measures (on bounded Borel subsets), since locally BV functions are Borel measurable. It also simplifies some calculations that involve (14) or (18).

Our first step is to prove $\mathcal{D}_{\max} \subseteq \mathcal{D}$. Since the reverse containment is trivial, we will conclude $\mathcal{D}_{\max} = \mathcal{D}$. Indeed, for $u \in \mathcal{D}_{\max}$ and any $a > 0$,

$$\begin{aligned}
\int\limits_{(-a,a)} |u'|^2 dx &= \int\limits_{(-a,a)} u' d(\bar{u}) \\
&= \int\limits_{(-a,a)} d(u'\bar{u}) - \bar{u}\, d(u') \\
&= (u'\bar{u})(a) - (u'\bar{u})(-a) + h^{-2} \int\limits_{(-a,a)} \bar{u} P u\, dx - h^{-2} \int\limits_{(-a,a)} \bar{u} u V \\
&\le 2 \sup_{(-a,a)} |u'| \sup_{(-a,a)} |u| + h^{-2} \|V\| \sup_{(-a,a)} |u|^2 + h^{-2} \|Pu\|_{L^2} \|u\|_{L^2},
\end{aligned} \tag{21}$$

where $\|V\| := |V|(\mathbb{R})$, with $|V|$ the total variation of V. The second line of (21) follows from (18) and $u = u^A$, $u' = (u')^A$; the third line follows from the fact that $-h^2 d(u') = Pu - uV$ as Borel measures, which is a consequence of (17).

Since u is locally absolutely continuous,

$$\begin{aligned}
\sup_{(-a,a)} |u|^2 &= \sup_{x \in (-a,a)} \Big(|u(0)|^2 + 2 \operatorname{Re} \int_0^x u' \bar{u}\, dx \Big) \\
&\le |u(0)|^2 + 2 \Big(\int\limits_{(-a,a)} |u'|^2 dx \Big)^{1/2} \|u\|_{L^2}.
\end{aligned}$$

Furthermore, if $x \in (0, a)$, then by (14), (18), and $u' = (u')^A$,

$$
\begin{aligned}
&(u'\bar{u}')(x) \\
&\quad = (u'\bar{u}')(0) + \int\limits_{(0,x]} d(u'\bar{u}') \\
&\quad = |u'|^2(0) - 2h^{-2}\operatorname{Re}\Bigg(\int\limits_{(0,x]} \bar{u}'Pu\,dx - \int\limits_{(0,x]} \bar{u}'uV\Bigg) \\
&\quad \le |u'|^2(0) + 2h^{-2}\|V\| \sup_{(-a,a)}|u| \sup_{(-a,a)}|u'| + 2h^{-2}\|Pu\|_{L^2}\Bigg(\int\limits_{(-a,a)} |u'|^2 dx\Bigg)^{1/2},
\end{aligned}
$$

while if $x \in (-a, 0)$, we similarly find

$$
\begin{aligned}
&(u'\bar{u}')(x) \\
&\quad = (u'\bar{u}')(0) - \int\limits_{(x,0]} d(u'\bar{u}') \\
&\quad \le |u'|^2(0) + 2h^{-2}\|V\| \sup_{(-a,a)}|u| \sup_{(-a,a)}|u'| + 2h^{-2}\|Pu\|_{L^2}\Bigg(\int\limits_{(-a,a)} |u'|^2 dx\Bigg)^{1/2}.
\end{aligned}
$$

We thus arrive at a system of inequalities of the form $x^2 \le 2yz + Ay^2 + B$, $y^2 \le C + Dx$, $z^2 \le E + Fyz + Gx$, where $x := (\int_{(-a,a)} |u'|^2 dx)^{1/2}$, $y := \sup_{(-a,a)} |u|$, and $z := \sup_{(-a,a)} |u'|$. After using the second inequality to eliminate y, we obtain a system in x and z with quadratic left-hand sides and subquadratic right-hand sides. Hence, x, y, and z are each bounded in terms of $A, B, \ldots, G$. Letting $a \to \infty$, we conclude that $u' \in L^2(\mathbb{R})$ and $u, u' \in L^\infty(\mathbb{R})$. Hence, $\mathcal{D}_{\max} \subseteq \mathcal{D}$ as desired.

Next, we equip P with the domain $\mathcal{D}_{\max} = \mathcal{D}$, and show that P is symmetric. Let $u, v \in \mathcal{D}$, and take $\{\varphi_k\}_{k=1}^\infty \subseteq C_0^\infty(\mathbb{R})$ converging to v in $H^1(\mathbb{R})$. Using the distributional definition of Pu,

$$
\begin{aligned}
\langle Pu, v\rangle_{L^2} &= \lim_{k\to\infty} \langle Pu, \varphi_k\rangle_{L^2} \\
&= \lim_{k\to\infty} \int \bar{u}(-h^2\varphi_k'')dx + \int \bar{u}\varphi_k V \\
&= \lim_{k\to\infty} \int h^2\bar{u}'\varphi_k' dx + \int \bar{u}\varphi_k V \\
&= \int h^2\bar{u}'v'dx + \int \bar{u}vV, \qquad (22)
\end{aligned}
$$

where the last equal sign follows since $|V|$ is finite and $\|w\|_{L^\infty}^2 \le \|w\|_{L^2}\|w'\|_{L^2}$ for any $w \in H^1(\mathbb{R})$. Approximating $u \in H^1(\mathbb{R})$ by $C_0^\infty(\mathbb{R})$-functions, we similarly have $\langle u, Pv\rangle_{L^2} = \int h^2\bar{u}'v'dx + \int \bar{u}vV$. Thus, P is symmetric.

The last step is to establish that $(P, \mathcal{D})$ is densely defined and $P^* \subseteq P$. For this, define on $H^1(\mathbb{R})$ the quadratic form (20). Since, for any $\gamma > 0$,

$$\begin{aligned} \left| \int V|u|^2 \right| &\le \|V\| \|u\|_{L^\infty}^2 \\ &\le \|V\| \|u\|_{L^2} \|u'\|_{L^2} \\ &\le \|V\| \Big(\frac{1}{2\gamma} \|u\|_{L^2}^2 + \frac{\gamma}{2} \|u'\|_{L^2}^2 \Big), \end{aligned}$$

setting $\gamma = h^2/\|V\|$ yields

$$-\frac{\|V\|^2}{2h^2} \|u\|_{L^2}^2 + \frac{h^2}{2} \|u'\|_{L^2}^2 \le q(u,u) \le \frac{\|V\|^2}{2h^2} \|u\|_{L^2}^2 + \frac{3h^2}{2} \|u'\|_{L^2}^2.$$

We thus conclude q is semibounded and closed.

By Friedrichs' result [42, Theorem 2.14], there is a unique (densely defined) self-adjoint operator $(A, \mathcal{D}_1)$ with

$$\begin{aligned} \mathcal{D}_1 = \{u \in H^1(\mathbb{R}) &: \text{there exists } \tilde{u} \in L^2 \\ &\text{with } q(u,v) = \langle \tilde{u}, v \rangle_{L^2} \text{ for all } v \in H^1(\mathbb{R})\}, \quad Au = \tilde{u}. \end{aligned}$$

Revisiting the calculation (22), we see that for any $u \in \mathcal{D}_1$, $\tilde{u} = -h^2 u'' + uV$ in the distributional sense. Thus, $(A, \mathcal{D}_1) \subseteq (P, \mathcal{D}_{\max})$, so $P^* \subseteq A^* = A \subseteq P$. Since we already showed $P \subseteq P^*$ (symmetricity), we conclude $P^* = P$ as desired. ■

4. Weighted estimate

The purpose of this section is to prove Theorem 1.1. As discussed in Section 1, we do so by means of a positive commutator argument that leverages the energy method.

Proof of Theorem 1.1. Our starting point is the pointwise energy F given by (10). As in the proof of Lemma 3.1, we fix with a continuous representative of $u \in \mathcal{D}$, and fix a representative of $u' \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that has locally bounded variation and $(u')^A = u'$ (thus $F^A = F$).

Since the measure $V = V(x;h)$ is finite, it can have only countably many point masses $\{x_j\}_j$, and moreover $\sum_j |V_j| < \infty$, where $V_j := V(\{x_j\})$. Let us decompose

$$V = V_c + V_d, \quad V_d = \sum_j V_j \delta_{x_j},$$

into its discrete and continuous parts. Here, δ_{x_j} denotes the Dirac measure concentrated at x_j. A key technical feature of the ensuing calculations is our use of the weight function

$$w = w_\eta(x) \\ := \exp\left(\int_{-\infty}^{x} \left[\frac{1}{E^{1/2}h}|V_c| + \left(\frac{1}{E^{1/2}h} V_{d,\eta}(x') + (|x'|+1)^{-1-\delta} \right) \right] dx' \right), \quad \eta > 0, \tag{23}$$

where $|V_c|$ denotes the total variation of V_c, and

$$V_{d,\eta}(x) := \pi^{-1/2}\eta^{-1} \sum_j |V_j| e^{-((x-x_j)/\eta)^2}. \tag{24}$$

The exponent of w is a continuous function, thus we may compute dw using (19). Nearing the end of the argument, we manage to control a term involving $|V_d|$ by sending $\eta \to 0^+$, essentially using that $\pi^{-1/2}\eta^{-1}\sum_j |V_j| e^{-((x-x_j)/\eta)^2} \to \sum_j |V_j| \delta_{x_j}$ in the distribution sense. We note also that

$$\sup_{\mathbb{R}} |w_\eta(x)| \le e^{C_1(V,E,h,\delta)}, \tag{25}$$

with $C_1(V, E, h, \delta)$ given by (6).

From (18) and $u^A = u$, $(u')^A = u'$, we find, in the sense of measures on $\mathbb{R}$,

$$\begin{aligned} dF &= 2h^2 \operatorname{Re}(\bar{u}' d(u')) + 2E \operatorname{Re}(u\bar{u}') \\ &= -2\operatorname{Re}\big(((P - E \pm i\varepsilon)u)\bar{u}'\big) \mp 2\varepsilon \operatorname{Im}(u\bar{u}') + 2\operatorname{Re}(u\bar{u}'V), \end{aligned}$$

where, to get the second line, we used that $-h^2 d(u') = Pu - uV$ as Borel measures. Using (18) again, this time to expand $d(wF)$,

$$\begin{aligned} d(wF) &= F dw + w dF \\ &= |hu'|^2 dw + E|u|^2 dw \\ &\quad - 2w \operatorname{Re}\big(((P - E \pm i\varepsilon)u)\bar{u}'\big) \mp 2\varepsilon w \operatorname{Im}(u\bar{u}') + 2\operatorname{Re} w(u\bar{u}'V) \\ &\ge -2w \operatorname{Re}\big(((P - E \pm i\varepsilon)u)\bar{u}'\big) \mp 2\varepsilon w \operatorname{Im}(u\bar{u}') \\ &\quad + (|hu'|^2 + E|u|^2) dw \\ &\quad - h^{-1} w (E^{1/2}|u|^2 + E^{-1/2}|hu'|^2)\Big(|V_c| + \sum_j |V_j| \delta_{x_j} \Big). \end{aligned} \tag{26}$$

By (19) and (23),

$$dw_\eta = h^{-1} w_\eta \Big(E^{-1/2}|V_c| \\ + E^{-1/2}\eta^{-1}\pi^{-1/2} \sum_j |V_j| e^{-((x-x_j)/\eta)^2} dx + h(|x|+1)^{-1-\delta} dx \Big). \tag{27}$$

Plugging (27) into the fifth line of (26) implies,

$$\begin{aligned} d(wF) \geq & -2w\,\mathrm{Re}(((P-E\pm i\varepsilon)u)\bar{u}') \mp 2\varepsilon w\,\mathrm{Im}(u\bar{u}') \\ & + w(|x|+1)^{-1-\delta}(E|u|^2+|hu'|^2)dx \\ & + \sum_j h^{-1}w(E^{1/2}|u|^2+E^{-1/2}|hu'|^2) \\ & \qquad\qquad \cdot |V_j|(\eta^{-1}\pi^{-1/2}e^{-((x-x_j)/\eta)^2}dx-\delta_{x_j}) \end{aligned} \tag{28}$$

Next, we note there exist sequences $\{a_n^{\pm}\}_{n=1}^{\infty}$ tending to $\pm\infty$, along which $F(a_n^{\pm}) = F^R(a_n^{\pm}) = F^L(a_n^{\pm}) \to 0$. This is because $F(x) \in L^1(\mathbb{R})$ and is continuous off of a countable set. So, we integrate both sides of (28) over $(a_n^-, a_n^+]$ and send $n \to \infty$. By (14), the left side of (28) becomes zero. Hence, from (25) and $w \geq 1$,

$$\begin{aligned} & \int (|x|+1)^{-1-\delta}(E|u|^2+|hu'|^2)dx \\ & \quad + \sum_j h^{-1}|V_j| \int w(E^{1/2}|u|^2+E^{-1/2}|hu'|^2)(\eta^{-1}\pi^{-1/2}e^{-((x-x_j)/\eta)^2}dx-\delta_{x_j}) \\ & \quad \leq e^{C_1(V,E,h,\delta)}\Big(\int \frac{1}{\gamma h^2}(|x|+1)^{1+\delta}|(P-E\pm i\varepsilon)u|^2+\gamma(|x|+1)^{-1-\delta}|hu'|^2 \\ & \qquad\qquad + 2\varepsilon\int |uu'|dx\Big), \quad \gamma,\, h>0. \end{aligned} \tag{29}$$

The goal of the following calculations is to show that the second line of (29) is nonnegative in the limit as $\eta \to 0^+$. First notice that as $\eta \to 0^+$,

$$\int_{-\infty}^{x_j} V_{d,\eta}(x')dx' = \pi^{-1/2}\sum_\ell |V_\ell| \int_{-\infty}^{(x_j-x_\ell)/\eta} e^{-(x')^2}dx' \to \frac{1}{2}|V_j| + \sum_{x_\ell<x_j}|V_\ell|,$$

$$\begin{aligned} \int_{-\infty}^{x_j+\eta x} V_{d,\eta}(x')dx' &= \pi^{-1/2}\sum_\ell |V_\ell| \int_{-\infty}^{\frac{x_j-x_\ell}{\eta}+x} e^{-(x')^2}dx' \\ & \to \pi^{-1/2}|V_j|\int_{-\infty}^{x} e^{-(x')^2}dx' + \sum_{x_\ell<x_j}|V_\ell|. \end{aligned}$$

This implies

$$\begin{aligned} w_\eta(x_j) &\to e^{\Gamma(E,h,j)}\exp\Big(\frac{|V_j|}{2E^{1/2}h}\Big), \\ w_\eta(x_j+\eta x) &\to e^{\Gamma(E,h,j)}\exp\Big(\frac{|V_j|}{(\pi E)^{1/2}h}\int_{-\infty}^{x} e^{-(x')^2}dx'\Big), \end{aligned}$$

where

$$e^{\Gamma(E,h,j)} := \exp\Big(\frac{1}{E^{1/2}h}\sum_{x_\ell<x_j}|V_\ell| + \int_{-\infty}^{x_j}\Big[\frac{1}{E^{1/2}h}|V_c| + (|x'|+1)^{-1-\delta}dx'\Big]\Big).$$

Therefore,

$$\begin{aligned}
&h^{-1}|V_j|\int w_\eta(E^{1/2}|u|^2 + E^{-1/2}|hu'|^2)\delta_{x_j}\\
&\quad= h^{-1}|V_j|w_\eta(x_j)(E^{1/2}|u(x_j)|^2 + E^{-1/2}|hu'(x_j)|^2)\\
&\qquad\to h^{-1}|V_j|(E^{1/2}|u(x_j)|^2 + E^{-1/2}|hu'(x_j)|^2)e^{\Gamma(E,h,j)}\exp\Big(\frac{|V_j|}{2E^{1/2}h}\Big),
\end{aligned}$$

while

$$\begin{aligned}
&h^{-1}\eta^{-1}\pi^{-1/2}|V_j|\int w_\eta(E^{1/2}|u|^2 + E^{-1/2}|hu'|^2)e^{-((x-x_j)/\eta)^2}dx\\
&\quad= h^{-1}\pi^{-1/2}|V_j|\int w_\eta(x_j+\eta x)(E^{1/2}|u(x_j+\eta x)|^2\\
&\qquad+ E^{-1/2}|hu'(x_j+\eta x)|^2)e^{-x^2}dx,\\
&\qquad\to (E^{1/2}|u(x_j)|^2 + E^{-1/2}|hu'(x_j)|^2)e^{\Gamma(E,h,j)}\\
&\qquad\quad\cdot h^{-1}\pi^{-1/2}|V_j|\int \exp\Big(\frac{|V_j|}{(\pi E)^{1/2}h}\int_{-\infty}^{x} e^{-(x')^2}dx'\Big)e^{-x^2}dx\\
&\quad= E^{1/2}(E^{1/2}|u(x_j)|^2 + E^{-1/2}|hu'(x_j)|^2)e^{\Gamma(E,h,j)}\Big(\exp\Big(\frac{|V_j|}{E^{1/2}h}\Big) - 1\Big).
\end{aligned}$$

In summary, we have shown that the second line of (29), upon sending $\eta \to 0^+$, converges to

$$\begin{aligned}
&\sum_j (E|u(x_j)|^2 + |hu'(x_j)|^2)e^{\Gamma(E,h,j)}\\
&\qquad\cdot\Big(\exp\Big(\frac{|V_j|}{E^{1/2}h}\Big) - 1 - \frac{|V_j|}{E^{1/2}h}\exp\Big(\frac{|V_j|}{2E^{1/2}h}\Big)\Big) \ge 0.
\end{aligned}\tag{30}$$

The nonnegativity follows from the fact that $e^x - 1 - xe^{x/2} \ge 0$ for all $x \ge 0$.

Returning to (29), we fix $\gamma = 2^{-1}e^{-C_1(V,E,h,\delta)}$, so that we may absorb the first term in line three into the left side, and invoke (30), implying

$$\begin{aligned}
&\int (|x|+1)^{-1-\delta}\Big(E|u|^2 + \frac{1}{2}|hu'|^2\Big)dx\\
&\quad\le e^{C_1(V,E,h,\delta)}\Big(\frac{2e^{C_1(V,E,h,\delta)}}{h^2}\int (|x|+1)^{1+\delta}|(P-E\pm i\varepsilon)u|^2dx\\
&\qquad\qquad + 2\varepsilon\int |uu'|dx\Big), \quad h>0.
\end{aligned}\tag{31}$$

For the term in (31) having the factor of ε,

$$2\int |uu'|dx \le \frac{1}{h}\int |u|^2 dx + \frac{1}{h}\int |hu'|^2 dx, \quad h>0, \tag{32}$$

and

$$\begin{aligned}\int |hu'|^2 dx &= \mathrm{Re}\int ((P-E\pm i\varepsilon)u)\bar{u}dx + E\int |u|^2 dx + \int |u|^2 V\\ &\le \frac{1}{2}\int |(P-E\pm i\varepsilon)u|^2 dx + \Big(\frac{1}{2}+E\Big)\int |u|^2 dx + \|V\|\|u\|_{L^2}\|u'\|_{L^2}\\ &\le \frac{1}{2}\int |(P-E\pm i\varepsilon)u|^2 dx + \Big(\frac{1}{2}+E+\frac{\|V\|}{2\gamma h^2}\Big)\int |u|^2 dx\\ &\quad + \frac{\gamma}{2}\|V\|\int |hu'|^2 dx, \quad \gamma, h>0. \end{aligned}\tag{33}$$

Fixing $\gamma = \|V\|^{-1}$ in (33) implies

$$\int |hu'|^2 dx \le \int |(P-E\pm i\varepsilon)u|^2 dx + \Big(1+2E+\frac{\|V\|^2}{h^2}\Big)\int |u|^2 dx, \quad h>0. \tag{34}$$

Now, replace $\int |hu'|^2 dx$ in (32) by the right side of (34). From this, we get a bound for $2\int |uu'|dx$, which we insert into the in the last line of (31). We conclude

$$\begin{aligned}&\int (|x|+1)^{-1-\delta}(E|u|^2 + \frac{1}{2}|hu'|^2)dx\\ &\le e^{C_1(V,E,h,\delta)}\Big(\Big(\frac{2e^{C_1(V,E,h,\delta)}}{h^2}+\frac{1}{h}\Big)\int (|x|+1)^{1+\delta}|(P-E\pm i\varepsilon)u|^2 dx\\ &\qquad + \frac{\varepsilon}{h}\Big(2+2E+\frac{\|V\|^2}{h^2}\Big)\int |u|^2 dx\Big), \quad \varepsilon\in[0,1],\ h>0. \end{aligned}\tag{35}$$

We absorb the last term of (35) into the left side by estimating

$$\begin{aligned}\varepsilon\|u\|_{L^2}^2 &= \mp \mathrm{Im}\langle (P-E\pm i\varepsilon)u, u\rangle_{L^2}\\ &\le \frac{\gamma^{-1}}{2}\|(|x|+1)^{\frac{1+\delta}{2}}(P-E\pm i\varepsilon)u\|_{L^2}^2 + \frac{\gamma}{2}\|(|x|+1)^{-\frac{1+\delta}{2}}u\|_{L^2}^2, \end{aligned}\tag{36}$$

and then choosing $\gamma = Eh^{-1}(2+2E+h^{-2}\|V\|^2)^{-1}e^{-C_1(V,E,h,\delta)}$. We then have

$$\begin{aligned}&\int (|x|+1)^{-1-\delta}\Big(\frac{E}{2}|u|^2 + \frac{1}{2}|hu'|^2\Big)dx\\ &\le e^{C_1(V,E,h,\delta)}\Big(\frac{2e^{C_1(V,E,h,\delta)}}{h^2}+\frac{1}{h}+\frac{1}{2Eh^2}\Big(2+2E+\frac{\|V\|^2}{h^2}\Big)^2 e^{C_1(V,E,h,\delta)}\Big)\\ &\quad\cdot\int (|x|+1)^{1+\delta}|(P-E\pm i\varepsilon)u|^2 dx, \quad \varepsilon\in[0,1],\ h>0, \end{aligned}$$

which implies (4). ∎

5. Exterior estimate

Proof of Theorem 1.7. We again start from (10), considering the case $\varepsilon \in (0,1]$ and putting

$$f := (P(h) - E \pm i\varepsilon)^{-1}(|x|+1)^{-(1+\delta)/2}u.$$

This time, we pair F with a much simpler weight w. In particular, we take w to be the continuous, odd function vanishing on $[-R_0, R_0]$, and obeying

$$\begin{aligned} w(x) &= 1 - \frac{(1+R_0)^\delta}{(1+x)^\delta}, \quad x > R_0 \\ \implies dw &= w'(x) = \delta(1+R_0)^\delta(1+|x|)^{-1-\delta}\mathbf{1}_{\geq R_0}. \end{aligned}$$

Note that the same w was used in the proof of [14, Theorem 2]. Since $wV = 0$, we find, proceeding as in (26),

$$\begin{aligned} d(wF) = &-2w^A \operatorname{Re}(((P - E \pm i\varepsilon)u)\bar{u}') \mp 2\varepsilon w^A \operatorname{Im}(u\bar{u}') \\ &+ |hu'|^2 dw + E|u|^2 dw, \end{aligned}$$

and thus

$$\begin{aligned} &\delta(1+R_0)^\delta \int_{\mathbb{R}\setminus[-R_0,R_0]} (|x|+1)^{-1-\delta}(E|u|^2 + |hu'|^2) \\ &\leq \frac{1}{\gamma h^2} \int_{\mathbb{R}\setminus[-R_0,R_0]} |f|^2 + \gamma \int_{\mathbb{R}\setminus[-R_0,R_0]} (|x|+1)^{-1-\delta}|hu'|^2 + 2\varepsilon \int_{\mathbb{R}\setminus[-R_0,R_0]} w|uu'|, \quad \gamma, h > 0. \end{aligned} \tag{37}$$

Taking $\gamma = 2^{-1}\delta(1+R_0)^\delta$, we absorb the second term on the right side of (37) into the left side. To handle the term involving ε, we proceed to find

$$2\int_{\mathbb{R}\setminus[-R_0,R_0]} w|uu'| \leq \frac{1}{h} \int_{\mathbb{R}\setminus[-R_0,R_0]} |u|^2 + \frac{1}{h} \int_{\mathbb{R}\setminus[-R_0,R_0]} w^2|hu'|^2, \quad h > 0, \tag{38}$$

and

$$\begin{aligned} &\int_{\mathbb{R}\setminus[-R_0,R_0]} w^2|hu'|^2 \\ &= 2h^2 \operatorname{Re} \int_{\mathbb{R}\setminus[-R_0,R_0]} ww'u'\bar{u} + \operatorname{Re} \int_{\mathbb{R}\setminus[-R_0,R_0]} w^2(-h^2u'')\bar{u} \leq \delta(1+R_0)^\delta h \int_{\mathbb{R}\setminus[-R_0,R_0]} (|u|^2 + w^2|hu'|^2) \\ &\quad + \operatorname{Re} \int_{\mathbb{R}\setminus[-R_0,R_0]} w^2\big((P - E \pm i\varepsilon)u\big)\bar{u} + E \int_{\mathbb{R}\setminus[-R_0,R_0]} |u|^2 \end{aligned}$$

$$\leq \frac{1}{2} \int_{\mathbb{R}\setminus[-R_0,R_0]} |f|^2 + \Big(\frac{1}{2} + E + \delta(1+R_0)^\delta h\Big) \int_{\mathbb{R}\setminus[-R_0,R_0]} |u|^2$$
$$+ \delta(1+R_0)^\delta h \int_{\mathbb{R}\setminus[-R_0,R_0]} w^2|hu'|^2, \quad h > 0. \tag{39}$$

Putting $h_0 := 2^{-1}\delta^{-1}(1+R_0)^{-\delta}$ and restricting $h \in (0, h_0]$ in (39) allows us to bound $\int_{\mathbb{R}\setminus[-R_0,R_0]} w^2|hu'|^2$ in (38) by twice the fourth line of (39). Inserting the resulting estimate for $2\int_{\mathbb{R}\setminus[-R_0,R_0]} w|u\bar{u}'|$ into the right side of (37) yields, for $\varepsilon \in (0,1]$ and $h \in (0, h_0]$,

$$\int_{\mathbb{R}\setminus[-R_0,R_0]} (|x|+1)^{-1-\delta}|u|^2 \leq \Big(\frac{2}{\delta^2 E(1+R_0)^{2\delta}h^2} + \frac{1}{E\delta(1+R_0)^\delta h}\Big) \int_{\mathbb{R}\setminus[-R_0,R_0]} |f|^2$$
$$+ \varepsilon \frac{3+2E}{E\delta(1+R_0)^\delta h} \int_{\mathbb{R}\setminus[-R_0,R_0]} |u|^2. \tag{40}$$

The last term in line two of (40) may be estimated in manner similar (36), leading to (9). ■

6. Uniform resolvent estimate and resonance free strips

In this section, we prove Theorem 1.8 as an application of Corollary 1.4. We are concerned with the self-adjoint operator

$$H := -\partial_x^2 + V : \mathcal{D} \to L^2(\mathbb{R}),$$

where V remains a finite signed Borel measure, and has support in $[-R_0, R_0]$ for some $R_0 > 0$.

In this situation, H is a *black box Hamiltonian* in the sense of Sjöstrand and Zworski [40], as defined in [16, Definition 4.1]. More precisely, in our setting this means the following. First, if $u \in \mathcal{D}$, then $u|_{\mathbb{R}\setminus[-R_0,R_0]} \in H^2(\mathbb{R} \setminus [-R_0, R_0])$. Second, for any $u \in \mathcal{D}$, we have $(Hu)|_{\mathbb{R}\setminus[-R_0,R_0]} = -(u|_{\mathbb{R}\setminus[-R_0,R_0]})''$. Third, any $u \in H^2(\mathbb{R})$ which vanishes on a neighborhood of $[-R_0, R_0]$ is also in $\mathcal{D}$. Fourth, $\mathbf{1}_{[-R_0,R_0]}(H+i)^{-1}$ is compact on $\mathcal{H}$; this last condition follows from the fact that $\mathcal{D} \subseteq H^1(\mathbb{R})$.

Then, by the analytic Fredholm theorem (see [16, Theorem 4.4]), we have the following. In $\operatorname{Im}\lambda > 0$, the resolvent $(H-\lambda^2)^{-1}$ is meromorphic $L^2(\mathbb{R}) \to \mathcal{D}$; λ is a pole of $(H-\lambda^2)^{-1}$, if and only if $\lambda^2 < 0$ is an eigenvalue of H. Furthermore, for $\chi \in C_0^\infty(\mathbb{R}; [0,1])$ with $\chi = 1$ near $[-R_0, R_0]$, the cutoff resolvent $\chi(H-\lambda^2)^{-1}\chi$

continues meromorphically $L^2(\mathbb{R}) \to \mathcal{D}$ from $\operatorname{Im}\lambda > 0$ to $\mathbb{C}$. The poles of the continuation are known as its *resonances*.

Proof of Theorem 1.8. Throughout the proof, we use $C(\|V\|, \lambda_0)$ to denote a positive constant which may depend on $\|V\|$ and λ_0, and whose value may change from line to line, but is always independent of λ.

We first show (11) for $k = 0$, $\operatorname{Im}\lambda > 0$, and $|\operatorname{Re}\lambda| \geq \lambda_0$. In this case, let us expand

$$\begin{aligned}\chi(H-\lambda^2)^{-1}\chi &= \chi(\partial_x^2 + V - (\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2 - i2\operatorname{Re}\lambda\cdot\operatorname{Im}\lambda)^{-1}\chi \\ &= (\operatorname{Re}\lambda)^{-2}\chi\big((\operatorname{Re}\lambda)^{-2}\partial_x^2 + (\operatorname{Re}\lambda)^{-2}V - 1 + (\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda)^{-2} \\ &\qquad - i2(\operatorname{Re}\lambda)^{-1}\operatorname{Im}\lambda\big)^{-1}\chi. \end{aligned} \tag{41}$$

If $\operatorname{Im}\lambda \geq |\operatorname{Re}\lambda|/2$, then by the spectral theorem for self-adjoint operators,

$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq |\operatorname{Re}\lambda|^{-2} \leq \lambda_0^{-1}|\operatorname{Re}\lambda|^{-1},$$

fer $\operatorname{Im}\lambda \geq |\operatorname{Re}\lambda|/2$, $|\operatorname{Re}\lambda| \geq \lambda_0$. If $\operatorname{Im}\lambda < |\operatorname{Re}\lambda|/2$, we apply (8) to (41) (the notational correspondence is $\delta = 1$, $E = 1 - (\operatorname{Im}\lambda)^2(\operatorname{Re}\lambda)^{-2} \geq 3/4$, $\varepsilon = 2|\operatorname{Re}\lambda|^{-1}\operatorname{Im}\lambda \in (0,1]$, $h = |\operatorname{Re}\lambda|^{-1}$, and $V(x,h) = h^2 V$). Therefore,

$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq C(\|V\|,\lambda_0)|\operatorname{Re}\lambda|^{-1}, \quad \operatorname{Im}\lambda > 0,\ |\operatorname{Re}\lambda| \geq \lambda_0. \tag{42}$$

Next, we adapt the proof of [6, Proposition 2.5] to show

$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to H^1} \leq C(\|V\|,\lambda_0), \quad 0 < \operatorname{Im}\lambda \leq 1,\ |\operatorname{Re}\lambda| \geq \lambda_0. \tag{43}$$

We employ the notation

$$(H-\lambda^2)u = \chi f, \quad 0 < \operatorname{Im}\lambda \leq 1,\ |\operatorname{Re}\lambda| \geq \lambda_0,\ f \in L^2(\mathbb{R}),\ u \in \mathcal{D}, \tag{44}$$

and make use of additional cutoffs

$$\chi_1,\ \chi_2 \in C_0^\infty(\mathbb{R};[0,1]), \quad \chi_1 = 1 \text{ on } \operatorname{supp}\chi, \quad \chi_2 = 1 \text{ on } \operatorname{supp}\chi_1. \tag{45}$$

Observe

$$\|\chi(H-\lambda^2)^{-1}\chi f\|_{H^1} \leq \|\chi u\|_{L^2} + \|(\chi u)'\|_{L^2} \leq C(\|\chi_2 u\|_{L^2} + \|\chi_1 u'\|_{L^2}),$$

where here and below, C is a positive constant, which may depend on $\|V\|$ and the derivatives of the cutoffs, and which may change between lines, but stays independent of λ. So, by (42) it suffices to show

$$\|\chi_1 u'\|_{L^2}^2 \leq C\big((|\operatorname{Re}\lambda|+1)^2\|\chi_2 u\|_{L^2}^2 + \|\chi_2 f\|_{L^2}^2\big). \tag{46}$$

Multiplying (44) by $\chi_1^2\bar{u}$ and integrating gives

$$\int \chi_1^2\chi f\bar{u}dx = \int \chi_1^2|u'|^2dx + 2\int \chi_1'\bar{u}\chi_1 u'dx + \int \chi_1^2|u|^2V - \lambda^2\int \chi_1^2|u|^2dx,$$

where (22) was used, and consequently

$$\begin{aligned}\int \chi_1^2|u'|^2dx &\le \int |\chi_1^2\chi f\bar{u}|dx + 2\int |\chi_1'u'\chi_1\bar{u}|dx + \|V\|\|\chi_1 u\|_{L^2}\|(\chi_1 u)'\|_{L^2}\\ &\quad + (|\operatorname{Re}\lambda|+1)^2\int \chi_1^2|u|^2dx\\ &\le C\big(\|\chi_2 f\|_{L^2}^2 + (|\operatorname{Re}\lambda|+1)^2\|\chi_2 u\|_{L^2}^2\big) + \frac{1}{2}\int \chi_1^2|u'|^2dx. \qquad (47)\end{aligned}$$

Absorbing the last term on the right side into the left side confirms (46).

With (42) and (43), for $0 < \operatorname{Im}\lambda \le 1$, $|\operatorname{Re}\lambda| \ge \lambda_0$, and $f \in L^2(\mathbb{R})$,

$$\begin{aligned}\|\chi(H-\lambda^2)^{-1}\chi f\|_{\mathcal{D}} &\le \|\chi(H-\lambda^2)^{-1}\chi f\|_{L^2} + \|H\chi(H-\lambda^2)^{-1}\chi f\|_{L^2}\\ &\le \|\chi(H-\lambda^2)^{-1}\chi f\|_{L^2} + \|[-\partial_x^2,\chi]\chi_1(H-\lambda^2)^{-1}\chi f\|_{L^2}\\ &\quad + \|\chi((H-\lambda^2)+\lambda^2)(H-\lambda^2)^{-1}\chi f\|_{L^2}\\ &\le \|\chi(H-\lambda^2)^{-1}\chi f\|_{L^2} + \|f\|_{L^2}\\ &\quad + \|[-\partial_x^2,\chi]\chi_1(H-\lambda^2)^{-1}\chi f\|_{L^2}\\ &\quad + (|\operatorname{Re}\lambda|+1)^2\|\chi(H-\lambda^2)^{-1}\chi f\|_{L^2}\\ &\le C(\|V\|,\lambda_0)(|\operatorname{Re}\lambda|+1)\|f\|_{L^2}.\end{aligned}$$

This implies (12) for $0<\operatorname{Im}\lambda\le 1$, $|\operatorname{Re}\lambda|\ge\lambda_0$, and that continued resolvent $L^2(\mathbb{R})\to\mathcal{D}$ has no poles in $\mathbb{R}\setminus\{0\}$ (since $\lambda_0>0$ is arbitrary).

Now, we turn to showing (12) in strips in the lower half plane. For this, we use a resolvent identity argument due to Vodev [44, Theorem 1.5], adapted to the non-semiclassical case. It yields holomorphicity of $\chi(H-\lambda^2)^{-1}\chi\colon L^2(\mathbb{R})\to\mathcal{D}$ in $|\operatorname{Re}\lambda|\ge\lambda_0$, $-\varepsilon_0<\operatorname{Im}\lambda<0$ ($\varepsilon_0>0$ sufficiently small), with bounds in these strips of the form (11) and (12).

Fix $\tilde{\chi}\in C_0^\infty(\mathbb{R};[0,1])$ such that $\tilde{\chi}=1$ near $[-R_0,R_0]$ and $\chi=1$ near $\operatorname{supp}\tilde{\chi}$. We are going to develop several resolvent identities, and let us work initially with λ,μ such that $\operatorname{Im}\lambda$, $\operatorname{Im}\mu>0$, $|\operatorname{Re}\lambda|$, $|\operatorname{Re}\mu|\ge\lambda_0$ (before sending these parameters into the lower half plane). By the first resolvent identity,

$$\begin{aligned}(H-\lambda^2)^{-1}-(H-\mu^2)^{-1} &= (\lambda^2-\mu^2)(H-\lambda^2)^{-1}(H-\mu^2)^{-1}\\ &= (\lambda^2-\mu^2)(H-\lambda^2)^{-1}\tilde{\chi}(2-\tilde{\chi})(H-\mu^2)^{-1}\\ &\quad + (\lambda^2-\mu^2)(H-\lambda^2)^{-1}(1-\tilde{\chi})^2(H-\mu^2)^{-1}.\end{aligned}$$

As operators on $H^2(\mathbb{R})$,

$$\begin{aligned}(1-\tilde{\chi})(-\partial_x^2-\lambda^2)-(H-\lambda^2)(1-\tilde{\chi}) &= [-\partial_x^2,\tilde{\chi}]\\ \implies (H-\lambda^2)^{-1}(1-\tilde{\chi})-(1-\tilde{\chi})(-\partial_x^2-\lambda^2)^{-1}&\\ =(H-\lambda^2)^{-1}[-\partial_x^2,\tilde{\chi}](-\partial_x^2-\lambda^2)^{-1},&\end{aligned}$$

while as operators on $\mathcal{D}$,

$$\begin{aligned}(-\partial_x^2-\mu^2)(1-\tilde{\chi})-(1-\tilde{\chi})(H-\mu^2) &= [\partial_x^2,\tilde{\chi}]\\ \implies (1-\tilde{\chi})(H-\mu^2)^{-1}-(-\partial_x^2-\mu^2)^{-1}(1-\tilde{\chi})&\\ =(-\partial_x^2-\mu^2)^{-1}[\partial_x^2,\tilde{\chi}](H-\mu^2)^{-1}.&\end{aligned}$$

Using $\chi=1$ on $\operatorname{supp}\tilde{\chi}$ and the three previous calculations,

$$\begin{aligned}&\chi(H-\lambda^2)^{-1}\chi-\chi(H-\mu^2)^{-1}\chi\\ &\quad=(\lambda^2-\mu^2)\chi(H-\lambda^2)^{-1}\chi\tilde{\chi}(2-\tilde{\chi})(H-\mu^2)^{-1}\chi\\ &\qquad+(\lambda^2-\mu^2)\chi\big((1-\tilde{\chi})(-\partial_x^2-\lambda^2)^{-1}+(H-\lambda^2)^{-1}[-\partial_x^2,\tilde{\chi}](-\partial_x^2-\lambda^2)^{-1}\big)\\ &\qquad\quad\cdot\big((-\partial_x^2-\mu^2)^{-1}(1-\tilde{\chi})+(-\partial_x^2-\mu^2)^{-1}[\partial_x^2,\tilde{\chi}](H-\mu^2)^{-1}\big)\chi\\ &\quad=(\lambda^2-\mu^2)\chi(H-\lambda^2)^{-1}\chi\tilde{\chi}(2-\tilde{\chi})\chi(H-\mu^2)^{-1}\chi\\ &\qquad+\big(1-\tilde{\chi}-\chi(H-\lambda^2)^{-1}\chi[\partial_x^2,\tilde{\chi}]\big)\big(\chi(-\partial_x^2-\lambda^2)^{-1}\chi-\chi(-\partial_x^2-\mu^2)^{-1}\chi\big)\\ &\qquad\quad\cdot\big(1-\tilde{\chi}+[\partial_x^2,\tilde{\chi}]\chi(H-\mu^2)^{-1}\chi\big).\end{aligned}\tag{48}$$

To get the equality sign in line four of (48), we expanded the terms appearing in lines two and three, and repeatedly applied the first resolvent identity to $(\lambda^2-\mu^2)(-\partial_x^2-\lambda^2)^{-1}(-\partial_x^2-\mu^2)^{-1}$.

Before proceeding to use (48) to estimate $\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}}$ in the lower half plane, we quote a well-known estimate for the difference of continued free resolvents (see [44, Section 5] or [36, Section 3.2]):

$$\begin{aligned}&\|\chi(-\partial_x^2-\lambda^2)^{-1}\chi-\chi(-\partial_x^2-\mu^2)^{-1}\chi\|_{H^{k_1}\to H^{k_2}}\\ &\quad\le C(\lambda_0)|\lambda-\mu|\sup_{\lambda'\in\Gamma_{\lambda,\mu}}|\lambda'|^{k_2-k_1-1},\end{aligned}\tag{49}$$

for $k_1\in\{0,1\}$, $k_2\in\{0,1,2\}$, $|\operatorname{Re}\lambda|,\ |\operatorname{Re}\mu|\ge\lambda_0$, $\operatorname{Im}\lambda,\operatorname{Im}\mu\ge-1$, where $\Gamma_{\lambda,\mu}$ denotes the line segment connecting λ and μ.

Identity (48) continues to hold after meromorphically continuing both sides ($L^2\to\mathcal{D}$) to $\lambda,\ \mu\in\mathbb{C}$. Now, fix $\mu\in\mathbb{R}$ with $|\mu|\ge\lambda_0$; Assume λ in the lower half plane is not a pole of the continued cutoff resolvent, and obeys $|\lambda-\mu|\le\min(1,\lambda_0/2,\gamma)$, for suitable $0<\gamma\ll1$ to be chosen. Then $\|\chi(H-\mu)^{-1}\chi\|_{L^2\to H^k}\le C(\|V\|,\lambda_0)|\mu|^{k-1}$, $k=0,1$, $\|\chi(H-\mu)^{-1}\chi\|_{L^2\to\mathcal{D}}\le C(\|V\|,\lambda_0)|\mu|$, (48), and (49) imply

$$\begin{aligned}
&\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}} \\
&\quad\le C(\|V\|,\lambda_0)\big(|\mu| + |\lambda^2-\mu^2||\mu|^{-1}\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}} \\
&\qquad\qquad + \big\|(1-\tilde{\chi})\chi\big((-h^2\partial_x^2-\lambda^2)^{-1}-(-\partial_x^2-\mu^2)^{-1}\big)\chi\big\|_{L^2\to\mathcal{D}} \\
&\qquad\qquad + \big\|\chi\big((-h^2\partial_x^2-\lambda^2)^{-1}-(-\partial_x^2-\mu^2)^{-1}\big)\chi\big\|_{L^2\to H^1} \\
&\qquad\qquad\quad \cdot \|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}} \\
&\qquad\qquad + |\mu|\big\|\chi\big((-h^2\partial_x^2-\lambda^2)^{-1}-(-\partial_x^2-\mu^2)^{-1}\big)\chi\big\|_{L^2\to L^2} \\
&\qquad\qquad\quad \cdot \|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}}\big) \\
&\quad\le C(\|V\|,\lambda_0)(|\mu| + \gamma\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}}).
\end{aligned} \tag{50}$$

Fixing γ small enough (depending on λ_0) allows us to absorb the term involving $\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}}$ on the right side of (50) into the left side. This precludes resonances in the region $\operatorname{Im}\lambda \le 0$, $|\lambda-\mu| \le \min(1,\lambda_0/2,\gamma)$, and in this region we have $\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to\mathcal{D}} \le C(\|V\|,\lambda_0)|\mu|$.

Starting from (48), we use the same strategy to show (11) in strips in the lower half plane. Thanks to (42), (43), and (49), more negative powers of $|\mu|$ appear while making an estimate similar to (50), since now we need only use operator norms $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ or $L^2(\mathbb{R}) \to H^1(\mathbb{R})$. ∎

To conclude this section, we consider the two by two matrix operator

$$G := -i\begin{pmatrix} 0 & 1 \\ -H & 0 \end{pmatrix} : \mathcal{D} \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R}),$$

which arises naturally from rewriting (13) as a first order system. A short computation yields

$$(G+\lambda)^{-1} = \begin{pmatrix} -\lambda(H-\lambda^2)^{-1} & -i(H-\lambda^2)^{-1} \\ i\lambda^2(H-\lambda^2)^{-1} + i & -\lambda(H-\lambda^2)^{-1} \end{pmatrix} \tag{51}$$

when $\operatorname{Im}\lambda > 0$ and $(H-\lambda^2)^{-1}$ exists.

The following corollary of Theorem 1.8 is essentially well known, and is an important input to the proof of Theorem 7.1 in Section 7.

Corollary 6.1. *Let $\chi \in C_0^\infty(\mathbb{R};[0,1])$ be identically one near $[-R_0,R_0]$. The operator*

$$\chi(G+\lambda)^{-1}\chi : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to \mathcal{D} \oplus L^2(\mathbb{R})$$

given by

$$\chi(G+\lambda)^{-1}\chi : \chi(G+\lambda)^{-1}\chi := \begin{pmatrix} -\lambda\chi(H-\lambda^2)^{-1}\chi & -i\chi(H-\lambda^2)^{-1}\chi \\ i\lambda^2\chi(H-\lambda^2)^{-1}\chi + i\chi^2 & -\lambda\chi(H-\lambda^2)^{-1}\chi \end{pmatrix} \tag{52}$$

continues meromorphically from $\operatorname{Im}\lambda > 0$ *to* $\mathbb{C}$, *without poles on* $\mathbb{R} \setminus \{0\}$. *For any* $\lambda_0 > 0$, *there exist* C, λ_0, $\varepsilon_0 > 0$ *so that*

$$\|\chi(G+\lambda)^{-1}\chi\|_{H^1(\mathbb{R})\oplus L^2(\mathbb{R})\to H^1(\mathbb{R})\oplus L^2(\mathbb{R})} \le C, \quad |\operatorname{Re}\lambda| \ge \lambda_0, \ |\operatorname{Im}\lambda| \le \varepsilon_0. \tag{53}$$

If $\chi(G+\lambda)^{-1}\chi$ *has a pole at* $\lambda = 0$, *it is a simple pole. More precisely, if* $w_0 \in H^1(\mathbb{R})$ *and* $w_1 \in L^2(\mathbb{R})$, *then*

$$\begin{aligned}\lim_{\lambda\to 0} \lambda\chi(G+\lambda)^{-1}\chi\begin{pmatrix} w_0 \\ w_1\end{pmatrix} &= \begin{pmatrix} -i\lim_{\lambda\to 0}\lambda\chi(H-\lambda^2)^{-1}\chi w_1 \\ 0\end{pmatrix} \\ &= \begin{pmatrix} \langle \chi u_0, w_1\rangle_{L^2}\chi u_0 \\ 0 \end{pmatrix}\end{aligned} \tag{54}$$

for some real valued $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}) \cap H^2_{\mathrm{loc}}(\mathbb{R} \setminus [-R_0, R_0])$ *with* $Hu_0 = 0$ *in the sense of distributions.*

Proof. By the blackbox formalism (see [16, Definition 4.1 and Theorem 4.4]) and Theorem 1.8, $\chi(H-\lambda^2)^{-1}\chi\colon L^2(\mathbb{R}) \to \mathcal{D}$ continues meromorphically from $\operatorname{Im}\lambda > 0$ to $\mathbb{C}$, and has no poles in $\mathbb{R}\setminus\{0\}$. This implies that each entry of (52) continues meromorphically as an operator between the appropriate spaces, again without poles in $\mathbb{R}\setminus\{0\}$.

With (11) already in hand, to establish (53), we need to show for any $\lambda_0 > 0$, there exist C, $\varepsilon_0 > 0$ so that

$$\|\lambda^2\chi(H-\lambda^2)^{-1}\chi + \chi^2\|_{H^1\to L^2} = \|\chi H(H-\lambda^2)^{-1}\chi\|_{H^1\to L^2} \le C, \tag{55}$$

$$\|\lambda\chi(H-\lambda^2)^{-1}\chi\|_{H^1\to H^1} \le C, \tag{56}$$

for $|\operatorname{Re}\lambda| \ge \lambda_0$ and $|\operatorname{Im}\lambda| \le \varepsilon_0$. First, we first prove (55) for $|\operatorname{Re}\lambda| \ge \lambda_0$ and $0 < \operatorname{Im}\lambda \le \varepsilon_0$, and then handle the remaining cases.

Let us use the notation

$$u = (H-\lambda^2)^{-1}\chi f \in \mathcal{D}, \quad f \in H^1(\mathbb{R}),\ |\operatorname{Re}\lambda| \ge \lambda_0 \text{ and } \operatorname{Im}\lambda > 0. \tag{57}$$

Let $\chi_1 \in C_0^\infty(\mathbb{R})$ with $\chi_1 = 1$ near $\operatorname{supp}\chi$. As we showed in the proof of Lemma 3.1, the form domain of $(H, \mathcal{D})$ is $H^1(\mathbb{R})$, so there exists a sequence $f_k \in \mathcal{D}$ converging to f in $H^1(\mathbb{R})$, and corresponding functions $u_k := (H-\lambda^2)^{-1}\chi f_k$ converging to u in $(\mathcal{D}, \|\cdot\|_{\mathcal{D}})$. Since $Hu_k = (H-\lambda^2)^{-1}\chi_1 H\chi f_k$,

$$\|\chi H u\|_{L^2} = \lim_{k\to\infty}\|\chi H u_k\|_{L^2} \le \lim_{k\to\infty}\|\chi_1(H-\lambda^2)^{-1}\chi_1 H\chi f_k\|_{L^2}. \tag{58}$$

Furthermore, by (22), there exists $C(\|V\|) > 0$ depending on $\|V\|$ so that for any $v \in L^2(\mathbb{R})$,

$$\begin{aligned}|\langle \chi_1(H-\lambda^2)^{-1}\chi_1 H\chi f_k, v\rangle_{L^2}| &= |\langle H\chi f_k, \chi_1(H-(-\bar\lambda)^2)^{-1}\chi_1 v\rangle_{L^2} \\ &\le C(\|V\|)\|\chi f_k\|_{H^1}\|\chi_1(H-(-\bar\lambda)^2)^{-1}\chi_1 v\|_{H^1}.\end{aligned}$$

Since (11) gives $\|\chi_1(H-(-\bar{\lambda})^2)^{-1}\chi_1 v\|_{H^1} \le C(\|V\|,\lambda_0)\|v\|_{L^2}$ (provided $\operatorname{Im}\lambda$ is small), we conclude $\|\chi_1(H-\lambda^2)^{-1}\chi_1 H\chi f_k\|_{L^2} \le C(\|V\|,\lambda_0)\|\chi f_k\|_{H^1}$. Returning to (58), we now find

$$\|\chi H u\|_{L^2} \le C(\|V\|,\lambda_0) \lim_{k\to\infty} \|\chi f_k\|_{H^1} \le C(\|V\|,\lambda_0)\|f\|_{H^1},$$

with $|\operatorname{Re}\lambda| \ge \lambda_0$, $0 < \operatorname{Im}\lambda \le \varepsilon_0$. In turn,

$$\|\lambda^2\chi(H-\lambda^2)^{-1}\chi\|_{H^1\to L^2} = \|\chi H(H-\lambda^2)^{-1}\chi - \chi^2\|_{H^1\to L^2} \le C(\|V\|,\lambda_0), \tag{59}$$

with $|\operatorname{Re}\lambda| \ge \lambda_0$ and $0 < \operatorname{Im}\lambda \le \varepsilon_0$, which is (55) for $|\operatorname{Re}\lambda| \ge \lambda_0$ and $0 < \operatorname{Im}\lambda \le \varepsilon_0$.

Next, with u as in (57), we slightly modify the method of estimation in (47), this time finding

$$\int \chi_1^2|u'|^2dx \le C(\|V\|,\lambda_0)\big((|\operatorname{Re}\lambda|+1)^{-2}\|\chi_2 f\|_{L^2}^2 + (|\operatorname{Re}\lambda|+1)^2\|\chi_2 u\|_{L^2}^2\big),$$

and where we recall from (45) that $\chi_2 = 1$ on $\operatorname{supp}\chi_1$. Combining this with (59) establishes (56) when $|\operatorname{Re}\lambda| \ge \lambda_0$ and $0 < \operatorname{Im}\lambda \le \varepsilon_0$

To show that (56) and (55) hold for $|\operatorname{Re}\lambda| \ge \lambda_0$ and $-\varepsilon_0 < \operatorname{Im}\lambda < 0$, we revisit (48) and multiply by the appropriate power of λ. We then perform an estimate similar to (50). As needed, we invoke (49) and $\|\mu^2\chi(H-\mu^2)^{-1}\chi\|_{H^1\to L^2}$, $\|\mu\chi(H-\mu^2)^{-1}\chi\|_{H^1\to H^1} \le C$ $(\mu\in\mathbb{R}, |\mu|\ge\lambda_0)$.

Finally, to show (54), we proceed as in the proof of [16, Theorem 2.7]. We omit the details, but take care to note that this argument does require that, for each $x_0 \in \mathbb{R}\setminus[-R_0,R_0]$ and $a,b\in\mathbb{C}$, there is a unique solution f to $Hf = 0$ satisfying $f(x_0) = a$ and $f'(x_0) = b$; Even though V is only a measure in our setting, such well-posedness still holds for the initial value problem, see [19, Theorem 3.1]. (In general, it is not necessary to prescribe the initial conditions outside the support of the measure, but this is sufficient for our purpose). The result is that near $\lambda = 0$,

$$\chi(H-\lambda^2)^{-1}\chi w_1 = \frac{i}{\lambda}\langle\chi u_0, w_1\rangle_{L^2}\chi u_0 + A(\lambda)w_1, \quad w_1\in L^2(\mathbb{R}),$$

where $A(\lambda)\colon L^2(\mathbb{R}) \to \mathcal{D}$ is holomorphic near zero, and for some $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}) \cap H^2_{\mathrm{loc}}(\mathbb{R}\setminus[-R_0,R_0])$ with $Hu_0 = 0$ in the sense of distributions. Hence, we have (54). ∎

7. Wave decay

In this section, we combine Corollary 6.1 with an argument similar to those appearing in [43, Section 3] and [15, Section 4]. We establish exponential local energy decay, modulo negative eigenvalues and a possible zero resonance, for solutions of the wave equation (13).

First, we represent the solution to (13) via the spectral theorem of for self-adjoint operators. Additionally, we use that the proof of Lemma 3.1 shows the form domain of $(H, \mathcal{D})$ (i.e., the domain of $|H|^{1/2}$) is $H^1(\mathbb{R})$. Thus, given initial conditions $w_0 \in \mathcal{D}$, $w_1 \in H^1(\mathbb{R})$, the unique function $w \in C^2((0, \infty), \mathcal{H})$ with $w(0) = w_0$, $\partial_t w(0) = w_1$, $w(t) \in \mathcal{D}(H)$ for all $t > 0$, and $\partial_t^2 w(t) + H w(t) = 0$, is

$$w(t) = w(\cdot, t) = \mathbf{1}_{\geq 0}(H) w(\cdot, t) + \mathbf{1}_{<0}(H) w(\cdot, t), \tag{60a}$$

$$\mathbf{1}_{\geq 0}(H) w(\cdot, t) = \mathbf{1}_{\geq 0}(H)\Big(\cos(t|H|^{1/2}) w_0 + \frac{\sin(t|H|^{1/2})}{|H|^{1/2}} w_1\Big), \tag{60b}$$

$$\mathbf{1}_{<0}(H) w(\cdot, t) = \mathbf{1}_{<0}(H)\Big(\cos(it|H|^{1/2}) w_0 + \frac{\sin(it|H|^{1/2})}{i|H|^{1/2}} w_1\Big). \tag{60c}$$

Theorem 7.1. *Suppose $w_0 \in \mathcal{D}$, $w_1 \in H^1(\mathbb{R})$, and* supp w_0, supp $w_1 \subseteq (-R, R)$ *for some $R > 0$. Let $w(t)$ be given by (60). For any $R_1 > 0$, there exist $C, c > 0$ so that*

$$\begin{aligned} &\|\mathbf{1}_{\geq 0}(H) w(\cdot, t) - w_\infty(x)\|_{H^1(-R_1, R_1)} + \|\partial_t \mathbf{1}_{\geq 0}(H) w(\cdot, t)\|_{L^2(-R_1, R_1)} \\ &\quad \leq C e^{-ct}(\|w_0\|_{H^1(\mathbb{R})} + \|w_1\|_{L^2(\mathbb{R})}), \quad t > 0. \end{aligned}$$

Furthermore, if $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ is identically one near $[-R_1, R_1] \cup [-R_0, R_0] \cup [-R, R]$, then the function $w_\infty(x)$ may be written as

$$w_\infty(x) := \chi(x) u_0(x) \int_{\mathbb{R}} \chi u_0 w_1,$$

for some real valued $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}) \cap H^2_{\mathrm{loc}}(\mathbb{R} \setminus [-R_0, R_0])$ with $H u_0 = 0$ in the sense of distributions (and if the continued operator (52) does not have a pole at $\lambda = 0$, we may take $u_0 \equiv 0$.)

Proof. Choose $\chi \in C_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ near the interval $[-R_1, R_1] \cup [-R, R] \cup [R_0, R_0]$ (where as before supp $V \subseteq [-R_0, R_0]$). From Corollary 6.1, for any $\lambda_0 > 0$, there exist $C, \varepsilon_0 > 0$ such that

$$\|\chi(G + \lambda)^{-1} \chi f\| \leq C\|f\|, \tag{61}$$

whenever $|\operatorname{Re} \lambda| \geq \lambda_0$ and $|\operatorname{Im} \lambda| \leq \varepsilon_0$, where here and for the rest of this section, all norms are $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ unless otherwise specified.

We have

$$\begin{aligned} \mathbf{1}_{\geq 0}(H) w(t) &= \mathbf{1}_{\geq 0}(H)\big(\cos(t|H|^{1/2}) w_0 + \sin(t|H|^{1/2})|H|^{-1/2} w_1\big), \\ \partial_t \mathbf{1}_{\geq 0}(H) w(t) &= \mathbf{1}_{\geq 0}(H)\big(-\sin(t|H|^{1/2})|H|^{1/2} w_0 + \cos(t|H|^{1/2}) w_1\big), \\ \partial_t^2 \mathbf{1}_{\geq 0}(H) w(t) &= -H \mathbf{1}_{\geq 0}(H) w(t). \end{aligned}$$

Consequently, after defining

$$f := \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}, \quad U(t)f := \begin{pmatrix} \mathbf{1}_{\geq 0}(H)w(t) \\ \partial_t \mathbf{1}_{\geq 0}(H)w(t) \end{pmatrix},$$

we have

$$\|U(t)f\| \leq C(1+|t|)\|f\|, \quad \partial_t U(t)f = iGU(t)f, \quad U(t)U(s)f = U(t+s)f, \tag{62}$$

for all real t and s, and for some $C > 0$ independent of t and f.

Take $\varphi \in C^\infty(\mathbb{R};[0,1])$ which is 0 near $(-\infty,1]$ and 1 near $[2,\infty)$ and put

$$W(t)f := \varphi(t)U(t)f = \int\limits_{\operatorname{Im}\lambda=\varepsilon} e^{-it\lambda}\check{W}(\lambda)\,d\lambda, \quad \check{W}(\lambda) := \frac{1}{2\pi}\int\limits_{\mathbb{R}} e^{is\lambda}W(s)f\,ds, \quad \varepsilon > 0.$$

We compute $\partial_t W(t)f = \varphi'(t)U(t)f + iGW(t)f$, and therefore find

$$W(t)f = \int\limits_{\operatorname{Im}\lambda=\varepsilon} e^{-it\lambda}(G+\lambda)^{-1}(i\varphi' Uf)\check{\ }(\lambda)\,d\lambda, \quad \varepsilon > 0. \tag{63}$$

Since w_0, w_1, and V have compact support, finite propagation speeds holds for the solution (60). Therefore, increasing $R > 0$ if necessary, we have that $x \mapsto U(t)f$ is supported in $(-R,R)$ for all $t \in [0,2]$. By continuity of integration, the same is true of $x \mapsto (i\varphi' Uf)\check{\ }(\lambda)$ for every λ. Hence, $\lambda \mapsto (i\varphi' Uf)\check{\ }(\lambda)$ is entire and rapidly decaying as $|\operatorname{Re}\lambda| \to \infty$ with $|\operatorname{Im}\lambda|$ remaining bounded and further $(i\varphi' Uf)\check{\ }(\lambda) = \chi(i\varphi' Uf)\check{\ }(\lambda)$.

By (61), there exists $\varepsilon > 0$ small enough so that, within the strip $|\operatorname{Im}\lambda| < 2\varepsilon$, either $\chi(G+\lambda)^{-1}\chi$ has no poles, or just a pole at $\lambda = 0$. Deforming the contour in (63), by the residue theorem, we find

$$\begin{aligned}\chi W(t)f &= -2\pi i \operatorname{Res}_{\lambda=0}(e^{-it\lambda}\chi(G+\lambda)^{-1}\chi(i\varphi' Uf)\check{\ }(\lambda)) \\ &\quad + \int\limits_{\operatorname{Im}\lambda=-\varepsilon} e^{-it\lambda}\chi(G+\lambda)^{-1}\chi(i\varphi' Uf)\check{\ }(\lambda)d\lambda. \\ &= \lim_{\lambda\to 0}\lambda\chi(G+\lambda)^{-1}\chi\int\limits_{\mathbb{R}}\varphi'(s)U(s)f\,ds \\ &\quad + \int\limits_{\operatorname{Im}\lambda=-\varepsilon} e^{-it\lambda}\chi(G+\lambda)^{-1}\chi(i\varphi' Uf)\check{\ }(\lambda)d\lambda.\end{aligned}$$

To simplify this, use (54) and put

$$W_1(t)f := \int_{-\infty}^{\infty} e^{-it\lambda}\chi(G+\lambda-i\varepsilon)^{-1}\chi(i\varphi' Uf)\check{\ }(\lambda-i\varepsilon)\,d\lambda,$$

to obtain

$$\chi W(t)f = \begin{pmatrix} \chi u_0 \int_{\mathbb{R}} \int_0^2 \chi(x)u_0(x)\varphi'(s)\partial_s w(s,x)dsdx \\ 0 \end{pmatrix} + e^{-\varepsilon t} W_1(t)f.$$

To simplify the first term, we integrate by parts in s, using $\varphi' = -(1-\varphi)'$, to obtain

$$\begin{aligned} &\int_{\mathbb{R}} \int_0^2 \chi(x)u_0(x)\varphi'(s)\partial_s w(s,x)\,ds\,dx \\ &\qquad = \int_{\mathbb{R}} \chi u_0 w_1 + \int_{\mathbb{R}} \int_0^2 \chi(x)(1-\varphi(s))\partial_s^2 w(s,x)\,ds\,dx. \end{aligned}$$

Now, observe that $\partial_s^2 w = -Hw$, $\chi u_0 \in \mathcal{D}$, so

$$\begin{aligned} \langle \chi u_0, Hw(s)\rangle_{L^2} &= \langle H\chi u_0, w(s)\rangle_{L^2} \\ &= \langle ([H,\chi]+H)u_0, w(s)\rangle_{L^2} = 0, \qquad \text{for } s \in [0,2], \end{aligned}$$

the last equality following from $\chi = 1$ near $[-R,R]$ and $\operatorname{supp} w(s) \subseteq (-R,R)$ for $s \in [0,2]$. Thus,

$$\chi W(t)f = \begin{pmatrix} \langle \chi u_0, w_1\rangle_{L^2}\chi u_0 \\ 0 \end{pmatrix} + e^{-\varepsilon t} W_1(t)f.$$

It now suffices to show that

$$\|W_1(t)f\| \le Ce^{\varepsilon t/2}\|f\|.$$

To prove this, we first use Plancherel's theorem, along with the fact that by (62), the operator norm $\|U(t)\|_{H^1(\mathbb{R})\oplus L^2(\mathbb{R})\to H^1(\mathbb{R})\oplus L^2(\mathbb{R})}$ is uniformly bounded for all $t \in \mathbb{R}$, as well as the fact that by Corollary 6.1, the operator norm $\|\chi(G+\lambda-i\varepsilon)^{-1}\chi\|_{H^1(\mathbb{R})\oplus L^2(\mathbb{R})\to H^1(\mathbb{R})\oplus L^2(\mathbb{R})}$ is uniformly bounded for all $\lambda \in \mathbb{R}$, to obtain

$$\begin{aligned} \int \|W_1(t)f\|^2\,dt &= C\int \|\chi(G+\lambda-i\varepsilon)^{-1}\chi(\varphi' Uf)\check{\ }(\lambda-i\varepsilon)\|^2\,d\lambda \\ &\le C_\varepsilon \int \|(\varphi' Uf)\check{\ }(\lambda-i\varepsilon)\|^2\,d\lambda \\ &= C_\varepsilon \int e^{2\varepsilon t}\|\varphi'(t)U(t)f\|^2\,dt \le C_\varepsilon\|f\|^2. \end{aligned} \tag{64}$$

Next, let $\tilde{\chi} \in C_0^\infty(\mathbb{R};[0,1])$ with $\tilde{\chi} = 1$ on $\operatorname{supp}\chi$. Observe that

$$G\chi(G+\lambda)^{-1}\chi = [G,\chi]\tilde{\chi}(G+\lambda+i\varepsilon)^{-1}\tilde{\chi}\chi - \lambda\chi(G+\lambda)^{-1}\chi + \chi^2 \tag{65}$$

holds initially for $\operatorname{Im}\lambda \gg 1$ by (51), and continues meromorphically to $\mathbb{C}$ by (52). In particular, decreasing ε if necessary, (53) implies that (65) holds for everywhere in $|\operatorname{Im}\lambda| < 2\varepsilon$, except possibly at $\lambda = 0$. Therefore, setting,

$$\widetilde{W}_1(t)f := \int_{-\infty}^{\infty} e^{-it\lambda}\tilde{\chi}(G+\lambda-i\varepsilon)^{-1}\tilde{\chi}(i\varphi' Uf)\check{}(\lambda-i\varepsilon)\,d\lambda,$$

we have

$$\begin{aligned}&(\partial_t - iG)W_1(t)f\\ &\quad= -i[G,\chi]\widetilde{W}_1(t)f + \varepsilon W_1(t)f - i\int e^{-it\lambda}(i\varphi' Uf)\check{}(\lambda-i\varepsilon)\,d\lambda =: W_2(t)f.\end{aligned}$$

Integrating both sides of

$$\begin{aligned}\partial_s(U(t-s)W_1(s)f) &= -iGU(t-s)W_1(s)f + U(t-s)\big(iGW_1(s)f + W_2(s)f\big)\\ &= U(t-s)W_2(s)f\end{aligned}$$

from $s = 0$ to $s = t$ gives

$$W_1(t)f = U(t)W_1(0)f + U(t)\int_0^t U(-s)W_2(s)f\,ds.$$

Thus,

$$\begin{aligned}\|W_1(t)f\| &\le C(1+t)\bigg(\|f\| + \int_0^t (1+s)\|W_2(s)f\|ds\bigg)\\ &\le C(1+t)\bigg(\|f\| + \Big(\frac{t^3}{3}+t^2+t\Big)^{1/2}\Big(\int_0^t \|W_2(s)f\|^2 ds\Big)^{1/2}\bigg).\end{aligned}$$

Now, check that, since $\|[G,\chi]\widetilde{W}_1(t)f\| \le C\|\widetilde{W}_1(t)f\|$, calculating as in (64), we obtain $\int \|W_2(s)f\|^2\,ds \le C\|f\|^2$, and hence

$$\|W_1(t)f\| \le C(1+t^{5/2})\|f\|$$

as desired. ∎

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Andrés Larraín-Hubach
Department of Mathematics, University of Dayton, 300 College Park, Dayton,
OH 45469-2316, USA; alarrainhubach1@udayton.edu

Jacob Shapiro
Department of Mathematics, University of Dayton, 300 College Park, Dayton,
OH 45469-2316, USA; jshapiro1@udayton.edu