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L^2-index formula for proper cocompact group actions

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Abstract. We study index theory of G-invariant elliptic pseudo-differential operators acting on a complete Riemannian manifold, where a unimodular, locally compact group G acts properly, cocompactly and isometrically. An L^2-index formula is obtained using the heat kernel method.

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Contents

1. Introduction

1.1. Main result. Let X be a complete Riemannian manifold acted on properly, cocompactly and isometrically by a locally compact unimodular group G and let E be a $\mathbb{Z}/2\mathbb{Z}$-graded G-vector bundle over X. Let

$$P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} : L^2(X, E) \to L^2(X, E)$$

be a 0-order properly supported elliptic pseudo-differential operator invariant under the group action. Such an operator has a real-valued L^2-index defined as the difference of the von Neumann traces of the projections onto the closed G-invariant subspaces $\operatorname{Ker} P_0, \operatorname{Ker} P_0^*$ of $L^2(X, E)$:

$$\operatorname{ind} P = \operatorname{tr}_G P_{\operatorname{Ker} P_0} - \operatorname{tr}_G P_{\operatorname{Ker} P_0^*}.$$

The paper is to prove that the L^2-index of P is calculated by the following topological formula:

$$\operatorname{ind} P = \int_{TX} (c \circ \pi) \cdot (\hat{A}(X))^2 \operatorname{ch}(\sigma_P). \tag{1.1}$$

Here $c \in C_c^\infty(X)$ is a non-negative function satisfying $\int_G c(g^{-1}x)\mathrm{d}g = 1$ for all $x \in X$, and $\pi : TX \to X$ is the projection.

1.2. Remarks on the result. The formula (1.1) generalizes the L^2-index formula for free cocompact group actions due to Atiyah [1] and the L^2-index formula for homogeneous spaces of unimodular Lie groups due to Connes and Moscovici [9]. The study of L^2-indices in general has implications in other areas of mathematics. For example, the non-vanishing of the L^2-index for the signature operator on X indicates the existence of L^2-harmonic forms on X. L^2-index is a key concept in the geometric realization of discrete series representations [9] and has been modified for use in a proof of the Novikov conjecture for hyperbolic groups [10].

Our index formula (1.1) is related to the type II theory of von Neumann algebras. The key feature of a type II index theory is that the elliptic operators being investigated are no longer Fredholm, but using some techniques analogous to those used in type II von Neumann theory, say, by formulating some trace, one may obtain generalized Fredholm indices associated to the elliptic operators. Refer to [24], [25] for another example of index for elliptic operators on open manifolds with bounded geometry by introducing a suitable trace.

When the orbit space X/G is an orbifold, the L^2-index discussed in this paper is not the same as the index for X/G as a compact orbifold [21]. For example, Dirac operators on a good orbifold are Fredholm and have integer indices, reflecting the information of the orbit space, while the L^2-indices of the Dirac operators lifted to the universal cover of the orbifold are rational numbers by definition. The integer indices and the rational indices are different in general [11]. They coincide on spacial cases, for example, when the orbit space is a smooth manifold [1]. Another example is that when both X and G are compact, (1.1) is the same as the Atiyah–Singer index formula for compact manifolds, regardless of the group G [3], while the index formula corresponding to the orbifold X/G involves group action [21]. Our formula is expected to have interesting applications when the group is not compact.

We also notice the existence of L^2-index formula (in some special cases) when X/G is a noncompact orbifold but has finite volume [30], where the analysis on the lower strata of X/G is heavily used. It is interesting to study the L^2-index (if exists) where the quotient is noncompact. However, our operator algebraic approach in finding the formula of L^2-index does not work for the case of noncompact quotient. The reason is that when G acts properly, cocompactly and isometrically on X, the group G and the manifold X are coarse equivalence; then we may use G, more precisely, $C^*(G)$ to study the elliptic operators on X invariant under the action of G

[22], [20]. However, when X/G is not compact, the group G is not the only factor affecting the L^2-indices for G-invariant elliptic operators on X.

Finally, (1.1) fits into the framework of the higher index formula taking values in cyclic theory. In [23], a general formula was proved and the indices of Dirac operators take values in the entire cyclic homology of some subalgebra of the group C^*-algebra $C^*(G)$. Formally, for Dirac operators, (1.1) can be obtained from [23], Corollary 1.2, by taking $g \in G$ to be the group identity and by taking $n = 0$. We would like to have a deeper investigation on the connection of the two results in future.

1.3. Idea of the proof. To prove (1.1), regard P as an element in the K-homology group $K^0_G(C_0(X))$, from which P has a higher index in $K_0(C^*(G))$, where $C^*(G)$ is the maximal group C^*-algebra. The L^2-index of P depends only on the equivalence class of its higher index in $K_0(C^*(G))$. This is proved in Section 4 by defining a trace on a dense holomorphic closed ideal $\mathcal{S}(\mathcal{E})$ in $\mathcal{K}(\mathcal{E})$, where $\mathcal{E}$ is a Hilbert $C^*(G)$-module having the same K-theory as $C^*(G)$. The trace is the von Neumann trace of a type II von Neumann algebra in the sense of Breuer [7]. A comprehensive discussion on the link between the L^2-index and the higher index may be found in [27].

Secondly, in Section 5, we reduce the problem of finding ind P into finding ind D for some Dirac type operator D, which has the same higher index as P. Kasparov's K-theoretic index formula [16] is essential in the argument. The formulation of Dirac type operators out of elliptic operators is also related to the vector bundle modification construction in the definition of geometric K-homology [4].

The final step is to calculate ind D using the heat kernel method. When D is a first order G-invariant operator of Dirac type on X, we have the McKean–Singer formula for the L^2-index:

$$\operatorname{ind} D = \operatorname{tr}_G e^{-tD^*D} - \operatorname{tr}_G e^{-tDD^*}, \quad t > 0. \tag{1.2}$$

In the case of a compact manifold without group action, a cohomological formula was obtained by studying the local invariants of metrics and connections [2], [13]. The proof of the local index formula was simplified by a rescaling argument of Getzler [12] on the asymptotic expansion of the heat kernel e^{-tD^2} around $t = 0$. Since the index ind D in (1.2) is local when $t \to 0+$, the new trace and the noncompactness of the manifold do not affect the calculation. The proof is based on a modification of the proofs in [26], [5] and is complete in Section 6.

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2. Preliminaries

Let G be a locally compact and *unimodular* group, that is, there is a bi-invariant Haar measure μ on G. For example, compact groups and discrete groups are unimodular. Set $\mathrm{d}g \doteq \mathrm{d}\mu(g)$ and we have

$$\mathrm{d}(tg) = \mathrm{d}g, \mathrm{d}(gt) = \mathrm{d}g \quad \text{and} \quad \mathrm{d}(g^{-1}) = \mathrm{d}g \quad \text{for any } g, t \in G.$$

Let X be a complete Riemannian manifold, on which G acts properly, cocompactly and isometrically, that is, the pre-image of any compact set under the continuous map

$$G \times X \to X \times X, \quad (g, x) \mapsto (g \cdot x, x),$$

is compact, the quotient space X/G is compact, and G respects the metric $\langle \cdot\,, \cdot \rangle$:

$$\langle x, y \rangle = \langle gx, gy \rangle \quad \text{for all } x, y \in X, \; g \in G.$$

The reason to consider proper cocompact actions is the existence of a cutoff function on X.

Definition 2.1. A non-negative function $c \in C_c^\infty(X)$ is a *cutoff function* if

$$\int_G c(g^{-1}x)\mathrm{d}g = 1.$$

for all $x \in X$.

Remark 2.2. A proper cocompact G-space has a cutoff function $c \in C_c^\infty(X)$ given by

$$c(x) = \frac{h(x)}{\int_G h(g^{-1}x)\mathrm{d}g},$$

where $h(x) \in C_c^\infty(X)$ is non-negative and has non-empty intersection with each orbit.

Example 2.3. Let G be a Lie group with a compact subgroup H, and let $X = G/H$ be the homogeneous space consisting of all the left cosets of H in G. The action of G on X is proper. Further, let E be a representation space of H. The induced representation $Y = G \times_H E$, which forms a G-vector bundle over X, is a proper G-space. According to the slice theorem, every proper space has such a local structure.

Theorem 2.4 (Slice theorem). *Let G be a locally compact group and X be a proper G-space. Then for any $x \in X$ and for any neighborhood O of x in X, there exists a compact subgroup K of G with $G_x \doteq \{g \in G \mid gx = x\} \subset K$ and a K-slice S such that $x \in S \subset O$. Here, a K-invariant subset $S \subset X$ is a K-slice in X if*

(1) *the union $G(S)$ (tubular set) of all orbits intersecting S is open;*

(2) *there is a G-equivariant map $f: G(S) \to G/K$ (the slicing map) such that $S = f^{-1}(eK)$.*

The slice theorem was established in the work of R. Palais in the 1960s. According to [6], Ch. II, Theorem 4.2, the tubular set $G(S) \subset X$ with a compact slicing subgroup K is G-homeomorphic to $G \times_K S$.

Remark 2.5. Since X is covered by G-invariant neighborhoods and since X/G is compact, X admits a finite sub-cover, that is,

$$X = \bigcup_{i=1}^{N} G \times_{K_i} S_i = \bigcup_{i=1}^{N} G(S_i). \tag{2.1}$$

The local structure (2.1) of X defines a G-invariant measure $\mathrm{d}x$ on X. In fact, The measure of a set in $G(S_i)$ is calculated from the measure on G and on S_i divided by the measure of K_i. Then the measure of a set $T \subset X$ is defined using a partition of unity argument. The 1-density on the Riemannian manifold X also defines the same measure.

In order to introduce ellipticity, we recall the following definitions concerning pseudo-differential operators. Let (E, p) be a finite dimensional complex G-vector bundle over X, that is, there is a smooth G action on E such that $p(gv) = gp(v)$ for $v \in E$ and the maps of the fibers $g: E_x \to E_{gx}$ are linear. Let $\pi: T^*X \to X$ be the projection map and π^*E over T^*X be the pull-back bundle of E. Here, $E = E_0 \oplus E_1$ is $\mathbb{Z}/2\mathbb{Z}$-graded and the G-action is grading preserving. The G-actions on E_0, E_1 give rise to a G-bundle $\mathrm{Hom}(\pi^*E_0, \pi^*E_1)$ over T^*X. A *symbol* function σ of order m is a continuous section of this G-bundle satisfying

$$\Big|\frac{\partial^a}{\partial x^{|a|}} \frac{\partial^b}{\partial \xi^{|b|}} \sigma(x, \xi)\Big| \leq C_{a,b,K}(1 + \|\xi\|)^{m-|b|}$$

for x in any compact set $K \subset X$ and ξ in the fiber T_xX, where $C_{a,b,K}$ is a constant depending on a, b, K. Here $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ and $|a| = \sum_{i=1}^n a_i, |b| = \sum_{i=1}^n b_i (\dim X = n)$. The set of all order m symbols is denoted by $S^m(X; E_0, E_1)$ and a *principal symbol* of order m is an element in the quotient $S^m(X; E_0, E_1)/S^{m-1}(X; E_0, E_1)$. We shall omit the word "principal" from now on.

Each symbol σ has an *amplitude* p defined by

$$p(x, y, \xi) = \alpha(x, y)\sigma(q(y, (x, \xi_x))),$$

where $\alpha \in C^\infty(X \times X)$ has support contained in a small neighborhood of the diagonal such that $\alpha(x, x) = 1$ and $\alpha(x, y) \geq 0$ for all $x, y \in X$ and $q: X \times T^*X \to T^*X$, $(y, (x, \xi_x)) \mapsto (y, \xi_y)$ (ξ_y is the parallel transport of ξ_x from x to y). Conversely,

$$\sigma(x, \xi) = p(x, x, \xi).$$

Denote by $C_c^\infty(X, E)$ the set of smooth sections of E with compact support in X and G acts on $C_c^\infty(X, E)$ by $(g \cdot f)(x) = g(f(g^{-1}x))$ for all $g \in G$, $f \in C_c^\infty(X, E)$. To each amplitude $p(x, y, \xi)$, we may construct a *pseudo-differential operator* $P_0 \colon C_c^\infty(X, E_0) \to C^\infty(X, E_1)$ by

$$P_0 u(x) = \int_{X \times T_x^* X} e^{i\Phi(x,y,\xi)} p(x, y, \xi) u(y) \mathrm{d}y \mathrm{d}\xi_x, \tag{2.2}$$

where $\Phi(x, y, \xi) = \langle \exp_x^{-1}(y), \xi_x \rangle$ is the phase function. The *Schwartz kernel* $K_{P_0}(x, y) \in \mathrm{Hom}(E_{0y}, E_{1x})$ of P_0, where

$$P_0 u(x) = \int_X K_{P_0}(x, y) u(y) \mathrm{d}y \quad \text{for all } u(x) \in C_c^\infty(X, E_0),$$

is expressed in the distributional sense:

$$K_P(x, y)(w) = \int_{X \times T^* X} e^{i\Phi(x,y,\xi)} p(x, y, \xi) w(x, y) \mathrm{d}x \mathrm{d}y \mathrm{d}\xi, \quad w \in C_c^\infty(X \times X).$$

We assume P_0 to be *G-invariant*, that is,

$$P_0(gf) = gP_0(f), \quad f \in C_c^\infty(X, E_0) \quad \text{for all } g \in G.$$

Clearly, the Schwartz kernel of a G-invariant operator P_0 satisfies

$$K_{P_0}(x, y) = K_{P_0}(gx, gy) \quad \text{for all } x, y \in X, \ g \in G. \tag{2.3}$$

In addition, assume P_0 to be *properly supported*, that is, for any compact subset $K \subset X$ the subsets $\mathrm{supp}\, K_P \cap (K \times X)$ and $\mathrm{supp}\, K_P \cap (X \times K)$ in $X \times X$ are compact. Proper supportness of P_0 in particular implies that P_0 maps $C_c^\infty(X, E_0)$ to $C_c^\infty(X, E_1)$.

Choose a G-invariant Hermitian structure on E and let $L^2(X, E)$ be the completion of $C_c(X, E)$ under inner product $\langle f, g \rangle_{L^2} = \int_X \langle f(x), g(x) \rangle_{E_x} \mathrm{d}x$. Let P be an essentially self-adjoint operator on $L^2(X, E)$ with odd grading in the form of

$$P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}.$$

Without loss of generality, P is assumed to be of order 0, and then P extends to a bounded self-adjoint operator on $L^2(X, E)$.

We shall use the following notations and we omit E, F or X when it is clear in the context.

- $\Psi^n(X; E, F)$: the set of order n pseudo-differential operators from $C_c^\infty(X, E)$ to $C^\infty(X, F)$;
- $\Psi_G^n(X; E, F)$: the subset of G-invariant elements in $\Psi^n(X; E, F)$;

- $\Psi^n_{G,p}(X; E, F)$: the subset of properly supported elements in $\Psi^n_G(X; E, F)$;
- $\Psi^n_c(X; E, F)$: the subset of $\Psi^n(X; E, F)$ having compactly supported Schwartz kernels.

The symbol of an operator $P \in \Psi^*_{G,p}$ is G-invariant. Conversely, if $\sigma(x, \xi)$ is a G-invariant symbol, then there is an operator in $\Psi^*_{G,p}$ with symbol $\sigma(x, \xi)$. To do this we construct P using (2.2) and use the averaging operation from [9]:

$$\mathrm{Av}_G : \Psi^*_c \to \Psi^*_{G,p}, \quad P \mapsto \int_G gPg^{-1}\mathrm{d}g.$$

Then $\mathrm{Av}_G(cP) \in \Psi^*_{G,p}$, where c is a cutoff function for X, has the symbol $\sigma(x, \xi)$.

Definition 2.6 ([18]). A pseudo-differential operator $P \in \Psi^m(X; E, F)$ is *elliptic* if there exists $Q \in \Psi^{-m}(X; F, E)$ such that

$$\|\sigma_P(x,\xi)\sigma_Q(x,\xi) - I\| \to 0 \quad \text{and} \quad \|\sigma_Q(x,\xi)\sigma_P(x,\xi) - I\| \to 0 \tag{2.4}$$

uniformly in $x \in K$ as $\xi \to \infty$ in $T^*_x X$ for any compact subset K in X. Without loss of generality, we will consider order-0 elliptic pseudo-differential operators $P_0 \in \Psi^0_{G,p}(X; E_0, E_1)$ with the condition (2.4) replaced by

$$\|\sigma_{P_0}(x,\xi)\sigma_{P_0^*}(x,\xi) - I\| \to 0 \quad \text{and} \quad \|\sigma_{P_0^*}(x,\xi)\sigma_{P_0}(x,\xi) - I\| \to 0. \tag{2.5}$$

Proposition 2.7. (1) *If $P \in \Psi^n_c$, then $\mathrm{Av}_G(P) \in \Psi^n_{G,p}$.*

(2) *If $P \in \Psi^n_{G,p}(X)$ is elliptic, then there exists a parametrix $Q \in \Psi^{-n}_{G,p}(X)$ such that*

$$1 - PQ = S_1 \in \Psi^{-\infty}_{G,p}(X), 1 - QP = S_2 \in \Psi^{-\infty}_{G,p}(X),$$

where $\Psi^{-\infty}_{G,p}(X) = \bigcap_{n\in\mathbb{R}} \Psi^n_{G,p}(X)$ is the set of smoothing operators.

(3) *If $S \in \Psi^{-\infty}_{G,p}(X)$, then the kernel $K_S(\cdot, \cdot)$ is smooth and properly supported.*

Proof. (1) Clearly, $\mathrm{Av}_G(P) \in \Psi^*_{G,p}(X)$. If $p(x, y, \xi) \in S^m(X \times T^*X)$ is the amplitude, then $P \in \Psi^n_c$ implies that $K = \{(x, y) \in X \times X \mid p(x, y, \xi) \neq 0\}$ is compact. Using the fact that the Riemannian metric on T^*X is G-invariant and the measure on X is G-invariant, we calculate the amplitude for $\mathrm{Av}_G(P)$ as

$$\int_G p(g^{-1}x, g^{-1}y, \xi_{g^{-1}x})\mathrm{d}g,$$

which is of order n because the integral is taken over a set $\{g \in G \mid (g^{-1}x, g^{-1}y) \in K\}$ which is compact.

(2) Let $P \in \Psi^n_{G,p}(X)$ be elliptic and $c \in C^\infty_c(X)$ be a cutoff function for X. Cover X by finitely many bounded open balls $\{U_i\}_{i=1}^N$ such that $\mathrm{supp}(c) \subset \bigcup_{i=1}^N U_i$. Let $\{a_i\}_{i=1}^N$ be a partition of unity subordinate to the finite cover. Since P is elliptic, which implies that for any compact $K \subset X$, there exists a constant C_K such that

$|\sigma_P| \geq C(1+|\xi|)^n$ uniformly for all $|\xi| \geq C_K$. Then there exists $Q_i \in \Psi_c^{-n}(U_i)$, $1 \leq i \leq N$ such that

$$PQ_i - a_i = R_{1,i}, \quad Q_i P - a_i = R_{2,i}$$

are elements in $\Psi_c^{-\infty}(U_i)$. Extend the elements in $\Psi_c^*(U_i)$ to $\Psi_c^*(X)$, then

$$c \sum_{i=1}^N Q_i P - c = c \sum_{i=1}^N R_{2,i}.$$

Since $\sum_{i=1}^N Q_i \in \Psi_c^{-n}(X)$, $\sum_{i=1}^N R_{2,i} \in \Psi_c^{-\infty}(X)$, we set

$$Q = \int_G g\big(c \sum_{i=1}^N Q_i\big) \mathrm{d}g \in \Psi_{G,p}^{-n}(X)$$

and

$$S = \int_G g\big(c \sum_{i=1}^N R_{2,i}\big) \mathrm{d}g \in \Psi_{G,p}^{-\infty}(X).$$

Then

$$\begin{aligned} QP &= \int_G g\big(c \sum_{i=1}^N Q_i\big) P \mathrm{d}g \\ &= \int_G g(c) g\big(\sum_{i=1}^N Q_i P\big) \mathrm{d}g \\ &= \int_G g(c) \mathrm{d}g + \int_G g(c) g\big(\sum_{i=1}^n R_{2,i}\big) \mathrm{d}g = I + S. \end{aligned}$$

Similarly, there is are $Q' = \int_G g(\sum_{i=1}^N Q_i c) \mathrm{d}g \in \Psi_{G,p}^{-n}(X)$ and $S' \in \Psi_{G,p}^{-\infty}(X)$ such that $PQ' - I = S'$. Since $Q' + SQ' - Q = (1+S)Q' - Q = Q(PQ' - 1) = QS'$, we have $Q' - Q \in \Psi_{G,p}^{-\infty}(X)$. Hence there are $S_1, S_2 = S \in \Psi_{G,p}^{-\infty}(X)$ such that $PQ = 1 + S_1$, $QP = 1 + S_2$.

(3) If $S \in \Psi_{G,p}^{-\infty}(X)$, then $cS \in \Psi_c^{-\infty}(X)$.

We know that $cS \in \Psi_c^{-\infty}(X)$ is equivalent to the fact that $K_{cS}(x,y)$ is smooth and compactly supported in $X \times X$. Therefore the statement follows from the fact that

$$K_S(x,y) = K_{\mathrm{Av}_G(cS)}(x,y) = \int_G K_{cS}(g^{-1}x, g^{-1}y) \mathrm{d}g$$

and the fact that the integral vanishes outside a compact set in G. □

3. The G-trace and the L^2-index

When X is compact and when G is trivial, the dimensions of Ker P_0 and Ker P_0^* are finite and their difference defines the index of P. In our case we measure the size of Ker P_0 or Ker P_0^* by a real number in terms of von Neumann dimension. The L^2-index of P, analogous to the Fredholm index is defined, motivated by the L^2-index defined by Atiyah [1] and modified upon [9].

3.1. The G-trace of operators on X. Recall that a bounded operator T on a Hilbert space H is of *trace class* if $\sum_{i=1}^{\infty} |\langle |T|e_i, e_i\rangle| < \infty$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of the Hilbert space and its *trace* calculated by

$$\mathrm{tr}(T) = \sum_{i=1}^{\infty} \langle Te_i, e_i\rangle$$

is independent of the orthonormal basis.

Definition 3.1. A bounded operator $S\colon L^2(X, E) \to L^2(X, E)$ which commutes with the action of G is of *G-trace class* if $\phi|S|\psi$ is of trace class for all $\phi, \psi \in C_c^{\infty}(X)$.

If S is a G-trace class operator, we calculate the G-trace by the formula

$$\mathrm{tr}_G(S) = \mathrm{tr}(c_1 S c_2),$$

where $c_1, c_2 \in C_c^{\infty}(X)$ are non-negative, satisfying $c_1 c_2 = c$ for some cutoff function c on X.

Remark 3.2. When G is discrete, Definition 3.1 is essentially the definition of the G-trace class operator appearing in [1]. Similarly to Lemma 4.9 of [1], we prove in the following proposition that tr_G is well defined, that is, tr_G is independent of the choice of c_1, c_2 and c.

Proposition 3.3. *Suppose that S (bounded, G-invariant) is a G-trace class operator. Let further $c_1, c_2, d_1, d_2 \in C_c^{\infty}(X)$ be non-negative functions satisfying $\int_G c_1(g^{-1}x)c_2(g^{-1}x)\mathrm{d}g = 1$ and $\int_G d_1(g^{-1}x)d_2(g^{-1}x)\mathrm{d}g = 1$, which means that $c = c_1c_2$ and $d = d_1d_2$ are cutoff functions on X. Then* $\mathrm{tr}(c_1Sc_2) = \mathrm{tr}(d_1Sd_2)$.

Proof. Let $K = \{g \in G \mid \mathrm{supp}(g \cdot (d_1d_2)) \cap \mathrm{supp}\, c \neq \emptyset\}$. Then K is compact by the properness of the group action. Hence,

$$\begin{aligned}
\mathrm{tr}(c_1Sc_2) &= \mathrm{tr}\left(\int_G [g \cdot (d_1d_2)]c_1Sc_2\mathrm{d}g\right) = \mathrm{tr}\left(\int_K [g \cdot (d_1d_2)]c_1Sc_2\mathrm{d}g\right) \\
&= \int_K \mathrm{tr}([g \cdot d_1][g \cdot d_2]c_1Sc_2)\mathrm{d}g = \int_K \mathrm{tr}(c_1[g \cdot d_1]S[g \cdot d_2]c_2)\mathrm{d}g \\
&= \int_K \mathrm{tr}([g^{-1} \cdot c_1]d_1Sd_2[g^{-1} \cdot c_2])\mathrm{d}g \\
&= \mathrm{tr}\left(\left[\int_G g(c_1c_2)\mathrm{d}g\right]d_1Sd_2\right) = \mathrm{tr}(d_1Sd_2).
\end{aligned}$$

□

Using the fact that tr is a well-defined trace on compactly supported operators on X, it is easy to see that tr_G is linear, faithful, normal and semi-finite. The tracial property of tr_G is proved in the following proposition together with some other properties of tr_G.

Proposition 3.4. (1) *A properly supported smoothing operator $A \in \Psi_{G,p}^{-\infty}$ is of G-trace class. If $K_A\colon X \times X \to \operatorname{Hom} E$ is the kernel of A, then its G-trace is calculated by*

$$\mathrm{tr}_G(A) = \int_X c(x) \operatorname{Tr} K_A(x,x) \mathrm{d}x,$$

where c is a cutoff function and Tr *is the matrix trace of* $\operatorname{Hom} E$. *In fact, this formula holds for all G-invariant operators having smooth integral kernel.*

(2) *If $A \in \Psi_G^*$ is of G-trace class, so is A^*.*

(3) *If $A \in \Psi_G^*$ is of G-trace class and $B \in \Psi_G^*$ is bounded, then AB and BA are of G-trace class.*

(4) *If AB and BA are of G-trace class, then* $\mathrm{tr}_G(AB) = \mathrm{tr}_G(BA)$.

Proof. Let $\phi, \psi \in C_c^\infty(X)$ and let $\{\alpha_i^2\}_{i=1}^N$ be the G-invariant partition of unity in Proposition 2.7.

(1) It follows from Proposition 2.7 (3) that $A \in \Psi_{G,p}^{-\infty}$ has smooth kernel. Then $K_{\phi A \psi}(x,y) = \phi(x) K_A(x,y) \psi(y)$, is smooth and compactly supported, which means that $\phi A \psi$ is of trace class. The integral formula for smoothing operators is classical. A proof may be found at [28], Section 2.21.

(2) Because $\bar{\psi} A \bar{\phi}$ has finite trace by definition, $\phi A^* \psi = (\bar{\psi} A \bar{\phi})^*$ is of trace class.

(3) Assume that we have a G-trace class operator $A \in \Psi_{G,p}^*$. Since $\operatorname{supp} \psi$ is compact and A is properly supported, there is a compact set K such that $\operatorname{supp} A\psi \subset K$. Choose $\eta, \zeta \in C_c^\infty(X)$ with $K \subset \operatorname{supp} \eta$ and $\eta\zeta = \eta$. Then $\eta A \psi = A\psi$, and for a bounded $B \in \Psi_G^*$ we have $\phi B A \psi = \phi B \zeta \eta A \psi = (\phi B \zeta)(\eta A \psi)$. Since $\phi B \zeta$ is bounded operator with compact support and $\eta A \psi$ is trace class operator, their product is also a trace class operator. So BA is of G-trace class. AB is of G-trace class because $B^* A^*$ is of G-trace class.

If $A \in \Psi_G^*$, then we have $A = A_1 + A_2$ where $A_1 \in \Psi_{G,p}^*$ and A_2 has smooth kernel (which follows from a classical statement saying that the Schwartz kernel is smooth off the diagonal). Then the statement follows from the fact that $\phi A_2 \psi$ has smooth, compactly supported Schwartz kernel.

(4) We first prove a special case when AB and BA have smooth integral kernels. Use the slice Theorem 2.1 to get $\{G \times_{K_i} S_i = G(S_i)\}_{i=1}^N$, G-invariant tubular open sets covering X. Then there exist G-invariant maps $\alpha_i\colon X \to [0,1]$ with $\operatorname{supp} \alpha_i \subset G(S_i)$ such that $\sum_{i=1}^N \alpha_i^2 = 1$. In fact, let $\widetilde{\alpha_i}^2$ be a partition of unity of X/G subordinate to the open sets $G(S_i)/G$. Lift $\widetilde{\alpha_i}$ to α_i on X, then $\{\alpha_i^2\}$ is a

G-invariant partition of unity of X. Then

$$
\begin{aligned}
\operatorname{tr}_G(AB) &= \int_X \int_X c(x) \operatorname{Tr}(K_A(x,y)K_B(y,x)) \mathrm{d}y \mathrm{d}x \\
&= \sum_{i,j} \int_{G\times_{K_i} S_i} \int_{G\times_{K_j} S_j} \alpha_i^2(x)\alpha_j^2(y)c(x) \operatorname{Tr}(K_A(x,y)K_B(y,x)) \mathrm{d}y \mathrm{d}x \\
&= \sum_{i,j} \frac{1}{\mu(K_i)\mu(K_j)} \int_{S_i} \int_{S_j} \alpha_i^2(\bar{s})\alpha_j^2(\bar{t}) \\
&\qquad \int_G \int_G c(\overline{ht}) \operatorname{Tr}(K_A(\overline{gs}, \overline{ht})K_B(\overline{ht}, \overline{gs})) \mathrm{d}g \mathrm{d}s \mathrm{d}h \mathrm{d}t \\
&= \sum_{i,j} \frac{1}{\mu(K_i)\mu(K_j)} \int_{S_i} \int_{S_j} \alpha_i^2(\bar{s})\alpha_j^2(\bar{t}) \\
&\qquad \int_G \operatorname{Tr}(K_B(\overline{ht}, \bar{s})K_A(\bar{s}, \overline{ht})) \mathrm{d}g \mathrm{d}s \mathrm{d}h \mathrm{d}t = \operatorname{tr}_G(BA).
\end{aligned}
$$

Note that in the third equality, $\overline{gs} \doteq (g,s)K_i = x \in G \times_{K_i} S_i$ and $\overline{ht} \doteq (h,t)K_j = y \in G \times_{K_j} S_j$ and by definition $\alpha_i(\bar{s}) = \alpha_i(\overline{gs}), \alpha_j(\bar{t}) = \alpha_j(\overline{ht})$. Also, we have used (2.3), $\mathrm{d}h^{-1} = \mathrm{d}h, \mathrm{d}(h^{-1}g) = \mathrm{d}g$, and change of variable in the fourth equality.

If either A or B are properly supported, (say A), then $\operatorname{tr}_G(AB) = \operatorname{tr}(c_1ABc_2) = \operatorname{tr}(\int_G c_1Ag\cdot(c_1c_2)Bc_2)$. So the set $\{g \in G \mid c_1Ag\cdot c_1 \neq 0\}$ is compact in K, which allows us to interchange tr and $\int_K$, and to use tracial property of tr and G-invariance of A and B to prove $\operatorname{tr}_G(AB) = \operatorname{tr}_G(BA)$

In general let $A = A_1+A_2$ and $B = B_1+B_2$ where A_1, B_1 are properly supported and A_2, B_2 are bounded and have smooth kernel. Then $\operatorname{tr}_G(AB) = \operatorname{tr}_G(BA)$ using the special cases discussed above. □

Remark 3.5. Let S be a bounded G-invariant operator with smooth integral kernel and define $T_i \doteq \alpha_i S\alpha_i \in \Psi_c^{-\infty}(X; E, E)$. Then $\alpha_i^2 S$ is of G-trace class by Proposition 3.4 (3). We may calculate $\operatorname{tr}_G(S)$ as follows:

$$
\begin{aligned}
\operatorname{tr}_G(S) &= \operatorname{tr}_G(\sum_{i=1}^N \alpha_i^2 S) \\
&= \sum_{i=1}^N \int_{G\times_{K_i} S_i} \alpha_i(x)c(x) \operatorname{Tr} K_S(x,x)\alpha_i(x)\mathrm{d}x \\
&= \sum_{i=1}^N \int_{G\times_{K_i} S_i} c(x) \operatorname{Tr} K_{T_i}(x,x)\mathrm{d}x \\
&= \sum_{i=1}^N \mu(K_i)^{-1} \int_{G\times S_i} c((g,s)) \operatorname{Tr} K_{T_i}((g,s),(g,s))\mathrm{d}g\mathrm{d}s
\end{aligned}
$$

$$= \sum_{i=1}^{N} \mu(K_i)^{-1} \int_{G\times S_i} c((g,s)) \operatorname{Tr} K_{T_i}((e,s),(e,s)) \mathrm{d}g\mathrm{d}s$$
$$= \sum_{i=1}^{N} \mu(K_i)^{-1} \int_{S_i} \operatorname{Tr} K_{T_i}(s,s)\mathrm{d}s.$$

The above trace formula coincides with the trace formulas in the special cases.

(1) If $X = G \times U$, for a bounded positive self-adjoint operator S with smooth kernel we obtain
$$\operatorname{tr}_G(S) = \int_U \operatorname{Tr} K_S(x,x)\mathrm{d}x.$$

(2) For a homogeneous space of a Lie group $X = G/H$ and for $S \in \Psi^{-\infty}_{G,p}(X)$, we have $\operatorname{tr}_G(S) = K_S(e,e)$, where e is the group identity. Here, we assumed the measure of the compact set H to be 1.

Proposition 3.6. *If $P_0 \in \Psi^m_{G,p}$ is an elliptic operator, then $P_{\operatorname{Ker} P_0} \in \Psi^{-\infty}_G$ is of G-trace class.*

Proof. By Proposition 2.7, there is a $Q \in \Psi^{-m}_{G,p}$ such that $1 - QP_0 = S \in \Psi^{-\infty}_{G,p}$. Then apply it to $P_{\operatorname{Ker} P_0}$ and get $P_{\operatorname{Ker} P_0} = SP_{\operatorname{Ker} P_0} \in \Psi^{-\infty}_G$. The statement is proved using (1) and (3) of Proposition 3.4. □

Remark 3.7. Let $\{\alpha_i^2\}_{i=1}^N$ be the G-invariant partition of unity as in the proof of Proposition 3.4 (4). Then by the same property and for any bounded operator $T \in \Psi^{-\infty}_G$ we have
$$\operatorname{tr}_G T = \sum_{i=1}^{N} \operatorname{tr}_G(\alpha_i T \alpha_i),$$
where every summand $\alpha_i T \alpha_i$ is G-invariant and restricts to a slice $G \times_{K_i} S_i$ in X.

The action of G on the vector bundle E is induced by the action of its subgroup K_i on $V \doteq E|_{S_i}$, the restriction of the bundle E over a subset $\{(e,s)K_i \mid s \in S_i\}$ of X.

$$\begin{tikzcd} V = E|_{S_i} \arrow[r, "k\in K_i"] \arrow[d] & V = E|_{S_i} \arrow[d] \\ \{e\} \times S_i \arrow[r, "k\in K_i"] & \{e\} \times S_i. \end{tikzcd}$$

Then we have the identification of the Hilbert spaces $L^2(G \times_{K_i} S_i, E) = (L^2(G) \otimes L^2(S_i, V))^{K_i}$, which consists of the elements of $L^2(G) \otimes L^2(S, V)$ invariant under the action of K_i, where the action of $k \in K_i$ is given by
$$k(f(g), h(s)) = (f(gk^{-1}), k \cdot h(s))$$

for $g \in G$, $s \in S_i$, $f \in L^2(G)$, $h \in L^2(S_i, V)$. The G-invariance of $\ker P_0$ implies that $\alpha_i P_{\ker P_0} \alpha_i$ is an element of

$$\mathcal{R}(L^2(G)) \otimes \mathcal{B}(L^2(S_i, V)).$$

This element commutes with the action of the group K_i on $\mathcal{R}(L^2(G)) \otimes \mathcal{B}(L^2(S_i, V))$. Here $\mathcal{R}(L^2(G))$ is the weak closure of the right regular representation of G (or, more precisely, $L^1(G)$) represented on $L^2(G)$. On this set there is a natural von Neumann trace determined by

$$\tau(R(f)^* R(f)) = \int_G |f(g)|^2 \mathrm{d}g,$$

where $f \in L^2(G) \cap L^1(G)$ and $R(f) = \int_G f(g) R(g) \mathrm{d}g$. Here $R(g)$ is the right regular representation of $g \in G$ on $L^2(G)$. Also $\mathcal{B}(L^2(S_i, V))$ also has a subset where an operator trace tr can be defined. There is a natural normal, semi-finite and faithful trace defined on $\mathcal{R}(L^2(G)) \otimes \mathcal{B}(L^2(S_i, V))$ given by $\tau \otimes \mathrm{tr}$ on algebraic tensors.

This trace coincides with the G-trace in Definition 3.1 on the set of bounded G-invariant operators with smooth kernel. In fact, by a partition of unity argument, such an operator is approximated by finite sums of operators of form $S = A \otimes B \in \mathcal{R}(L^2(G)) \otimes \mathcal{B}(L^2(S_i, V))$, which commutes with the action of K_i, where A and B have smooth kernel. In [9], it has been shown that $\tau(A) = K_A(e, e)$. Let $d \in C_c^\infty(G)$ be any cutoff function for G. Then $\tau(A) = \int_G d(g) K_A(g, g) \mathrm{d}g$. Hence,

$$\begin{aligned}
\tau(A)\,\mathrm{tr}(B) &= \int_G d(g) K_A(g, g) \mathrm{d}g \int_{S_i} \mathrm{Tr} K_B(s, s) \mathrm{d}s \\
&= \int_{G \times S_i} \frac{1}{\mu(K_i)} c((g, s)) \mathrm{Tr} K_S((g, s), (g, s)) \mathrm{d}g \mathrm{d}s \\
&= \int_{G \times_{K_i} S_i} c(x) \mathrm{Tr} K_S(x, x) \mathrm{d}x.
\end{aligned}$$

Therefore we have proved the following proposition.

Proposition 3.8. *On $\Psi_{G,p}^{-\infty}(X; E, E)$, the G-trace equals the natural von Neumann trace on the von Neumann algebra $\mathcal{R}(L^2(X, E))$, the weak closure of all the natural bounded operators on $L^2(X, E)$ which commute with the action of G. The L^2-index is the difference of the von Neumann traces of $P_{\mathrm{Ker}\, P_0}$ and $P_{\mathrm{Ker}\, P_0^*}$.*

Example 3.9. When G is a discrete group acting on itself by left translations, define

$$c(g) = \begin{cases} 1, & g = e, \\ 0, & g \neq e. \end{cases}$$

Then

$$\mathrm{tr}\, cT = \sum_{g \in G} \langle cT\delta_g, \delta_g \rangle = \langle \sum_{g \in G} g^{-1}(cT) g \delta_e, \delta_e \rangle = \langle \mathrm{Av}(cT) \delta_e, \delta_e \rangle = \mathrm{tr}_G\, \mathrm{Av}(cT).$$

In general, $\mathrm{Av}(c\,\cdot): \mathcal{B}(L^2(G)) \to \mathcal{R}(L^2(G))$ extends the map $\Psi_c^* \to \Psi_{G,p}^*: cT \to \mathrm{Av}(cT)$, which preserves the corresponding trace. When T is G-invariant, $T = \mathrm{Av}(cT)$ and then $\mathrm{tr}_G\, T = \mathrm{tr}_G\, \mathrm{Av}(cT) = \mathrm{tr}\, cT$. This is a motivation for the formula for tr_G.

3.2. The L^2-index of elliptic operators on X. According to Proposition 3.6, we define a real valued G-*dimension* of K, a closed G-invariant subspace of $L^2(X, E)$, by

$$\dim_G K = \mathrm{tr}_G\, P_K$$

where P_K is the projection from $L^2(X, E)$ onto K, and is G-invariant.

Definition 3.10. The L^2-*index* of the elliptic operator $P \in \Psi_{G,p}^*$ is

$$\mathrm{ind}\, P = \dim_G \mathrm{Ker}\, P_0 - \dim_G \mathrm{Ker}\, P_0^*.$$

An immediate computation of the L^2-index is given by the following proposition,

Proposition 3.11. *Let $P \in \Psi_{G,p}^m$ be elliptic and Q be an operator such that $1 - QP_0 = S_1$, $1 - P_0Q = S_2$ are of G-trace class. Then*

$$\mathrm{ind}\, P = \mathrm{tr}_G\, S_1 - \mathrm{tr}_G\, S_2.$$

Proof. The proof is similar to the one in [1]. We have

$$S_1 P_{\mathrm{Ker}\, P_0} = P_{\mathrm{Ker}\, P_0} \text{ and } P_{\mathrm{Ker}\, P_0^*} S_2 = P_{\mathrm{Ker}\, P_0^*}$$

by composing $QP_0 = 1 - S_1$ with $P_{\mathrm{Ker}\, P_0}$ and by composing $P_{\mathrm{Ker}\, P_0^*}$ with $1 - S_2 = P_0Q$ respectively. Also, $P_0(QP_0) = (P_0Q)P_0$ implies that $P_0S_1 = S_2P_0$. Set $R = \delta_0(P_0^*P_0)P_0^*$, where $\delta_0(0) = 1, \delta_0(x) = 0$ for $x \neq 0$, so

$$RP_0 = 1 - P_{\mathrm{Ker}\, P_0},\ P_0R = 1 - P_{\mathrm{Ker}\, P_0^*}.$$

On one hand, $\mathrm{tr}_G\, S_1 - \mathrm{tr}_G\, P_{\mathrm{Ker}\, P_0} = \mathrm{tr}_G\, S_1(1 - P_{\mathrm{Ker}\, P_0}) = \mathrm{tr}_G(S_1RP_0)$. On the other hand, $\mathrm{tr}_G\, S_2 - \mathrm{tr}_G\, P_{\mathrm{Ker}\, P_0^*} = \mathrm{tr}_G\, S_2(1 - P_{\mathrm{Ker}\, P_0^*}) = \mathrm{tr}_G(S_2P_0R) = \mathrm{tr}_G(P_0S_1R)$. Therefore, $\mathrm{tr}_G\, S_1 - \mathrm{tr}_G\, S_2 = \mathrm{tr}_G\, P_{\mathrm{Ker}\, P_0} - \mathrm{tr}_G\, P_{\mathrm{Ker}\, P_0^*}$ by Proposition 3.4. ☐

From the last proposition we derive the following *McKean-Singer formula*.

Corollary 3.12. *If $D = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix} \in \Psi_G^1(X; E, E)$ is a first order essentially self-adjoint elliptic differential operator, then*

$$\mathrm{ind}\, D = \mathrm{tr}_G(e^{-tD_0^*D_0}) - \mathrm{tr}_G(e^{-tD_0D_0^*}) \quad \textit{for all } t > 0, \tag{3.1}$$

which in particular means that $\mathrm{ind}\, D$ *is independent of* $t > 0$.

To prove (3.1) we need the following lemma.

Lemma 3.13. *Let D_0 be as above. Then $e^{-tD_0D_0^*}$ and $e^{-tD_0^*D_0}$ are of G-trace class.*

Proof. It is sufficient to prove the case when $t = 1$. The proof is based on the ideas in [14], [9].

If $\lambda \in \mathbb{C} - [0, \infty)$, then $\lambda I - D_0^* D_0$ is invertible. Let $L = \{\lambda \in \mathbb{C} \mid d(\lambda, \mathbb{R}_+) = 1\}$ be clock-wise oriented. Then

$$e^{-D_0^* D_0} = \frac{1}{2\pi i} \int_L \frac{e^{-\lambda}}{\lambda I - D_0^* D_0} \mathrm{d}\lambda.$$

Let $\phi, \psi \in C_c^\infty(X)$ be supported in a compact set $K \subset X$ and let $\{\alpha_i\}_{i=1}^N$ be a partition of unity subordinated to an open cover of K of local coordinate charts. We approximate $\phi e^{-D_0^* D_0} \psi$ by an operator in $\Psi_c^{-\infty}$ (with smooth and compactly supported Schwartz kernel) by inverting $\lambda I - D_0^* D_0$ "locally".

Let p_i be the full symbol of $\alpha_i \phi (\lambda I - D_0^* D_0)^{-1} \psi$, having the asymptotic sum

$$p_i \sim \sum_{j=2}^{\infty} a_{-j} \tag{3.2}$$

on a local coordinate, that is,

$$\mathrm{Op}(p_i - \sum_{j=2}^{m} a_{-j}) \in \Psi_c^{-m-1} \quad \text{for all } m > 1,$$

where Op is the operator corresponding to the local symbol.

For any $l > 0$ and $n > 0$, choose a large enough M and set the operator approximating $\alpha_i \phi (\lambda I - D_0^* D_0)^{-1} \psi$ to be

$$P_i(\lambda) = \mathrm{Op}(\sum_{j=2}^{M} a_{-j}) \tag{3.3}$$

in the sense that $P_i(\lambda)$ is analytic in λ and

$$\|(P_k(\lambda) - \alpha_i \phi (\lambda I - D_0^* D_0)^{-1} \psi) u\|_l \le C(1 + |\lambda|)^{-n} \tag{3.4}$$

for any fixed $u \in L^2(X, E)$, where the norm is the Sobolev l-norm $\|\cdot\|_l$. The estimate (3.4) is made possible by the asymptotic sum (3.2). In fact, let $r(x, \xi)$ be the symbol of $R \doteq P_i(\lambda) - \alpha_i \phi (\lambda I - D_0^* D_0)^{-1} \psi$ which is in S^{-M-1}. Then the left-hand side of (3.4) is $\|Ru\|_l = \int (1 + |\xi|^2)^l |\widehat{Ru}(\xi)| \mathrm{d}\xi$, where $Ru(x) = \int e^{\langle x-y, \xi \rangle} r(x, \xi) u(y) \mathrm{d}y \mathrm{d}\xi$ can be controlled by the right-hand side of (3.4) when $M \gg 2l + 2n$. This is because by the definition of $r(x, \xi)$ there is a constant C such that $|r(x, \xi)| < C(1 + |\xi|)^{-M-1}$.

Set

$$E(\lambda)=\sum_{i=1}^{N}E_i(\lambda)=\sum_{i=1}^{N}\frac{1}{2\pi i}\int_L e^{-\lambda}P_i(\lambda)\mathrm{d}\lambda. \tag{3.5}$$

Then the following two observations prove that $\phi e^{-D_0^*D_0}\psi$ is of trace class.

(1) The operator $E(\lambda)$ is a compactly supported operator with smooth Schwartz kernel.

Proof of claim. We need to show that the Schwartz kernel of $E_k(\lambda)$ is smooth. In view of (3.3) and (3.5), it is sufficient to show that $\mathrm{Op}(a_j)$, $j\le -2$ has smooth kernel and $\int_L e^{-\lambda}\partial^\beta(\mathrm{Op}(a_j)u)\mathrm{d}\lambda$ is integrable for all β. This claim can be proved by the symbolic calculus ([14]). The crucial part in the argument is that by the symbolic calculus, all a_j, $j\le -2$ contain the factor $e^{-\sigma_2(D_0^*D_0)}$ and the fact that $e^{-t\sigma_2(D_0^*D_0)}$ is rapidly decreasing in ξ.

(2) The function $(E(\lambda)-\phi e^{-D_0^*D_0}\psi)u$ is in H^l for any fixed $u\in L^2$.

Proof of claim. Using (3.4) and fixing a $u\in L^2(X,E)$, we have

$$\begin{aligned}&\|(E(\lambda)-\phi e^{-D_0^*D_0}\psi)u\|_l\\&\quad\le\frac{1}{2\pi}\sum_{i=1}^{N}\int_L e^{-\lambda}\|(P_i(\lambda)-\alpha_i\phi(D_0^*D_0-\lambda I)^{-1}\psi)u\|_l\mathrm{d}\lambda\\&\quad\le C\int_L e^{-\lambda}(1+|\lambda|)^{-n}\mathrm{d}\lambda\to 0\end{aligned}$$

as $n\to\infty$.

Note that $E(\lambda)$ depends on the number M, which is chosen based on l,n, and it has a compactly supported smooth kernel by the first claim and hence $E(\lambda)u\in C_c^\infty\subset H^l$. The second claim shows that $\phi e^{-D_0^*D_0}\psi$ is in H^l. (When $n\to\infty$, there is a sequence of $E(\lambda)\in H^l$ approaching $\phi e^{-D_0^*D_0}\psi$ in $\|\cdot\|_l$ norm.)

Let $l\to\infty$. Then by the Sobolev Embedding Theorem $(\phi e^{-D_0^*D_0}\psi)u$ is smooth for all $u\in L^2$. Therefore $\phi e^{-D_0^*D_0}\psi$ has a compactly supported smooth kernel and is a trace class operator. □

Proof of Corollary 3.12. Let $Q=\int_0^t e^{-sD_0^*D_0}D_0^*\mathrm{d}s$, which is the parametrix of D_0. In fact

$$1-QD_0=e^{-tD_0^*D_0},\quad I-D_0Q=e^{-tD_0D_0^*},$$

and they are of G-trace class by the lemma. The statement follows from Proposition 3.11. The independence of t can be carried out by a modification of the second proof of [5], Theorem 3.50. □

4. The connection of the L^2-index to the K-theoretic index

Let $f \in C_0(X)$ be identified as an operator on $L^2(X, E)$ by point-wise multiplication. Let $A \in \Psi_p^0(X; E, E)$ be elliptic in the sense of Definition 2.6. Using the Rellich lemma one may check that $A_0 : L^2(X, E_0) \to L^2(X, E_1)$ satisfies the following conditions:

- $(A_0 A_0^* - I) f \in \mathcal{K}(L^2(X, E_1))$, $(A_0^* A_0 - I) f \in \mathcal{K}(L^2(X, E_0))$, ;
- $Af - fA \in \mathcal{K}(L^2(X, E))$;
- $A_0 - g \cdot A_0 \in \mathcal{K}(L^2(X, E_1), L^2(X, E_2))$ for all $g \in G$.

Hence A represents an element in the K-homology group $K_G^0(C_0(X))$.

Topologically, the *K-theoretic index* of $[A] \in K_G^0(C_0(X))$, according to [16], is defined by

$$\operatorname{Ind}_t A \doteq [p] \otimes_{C^*(G, C_0(X))} j^G([A]) \in K_0(C^*(G)),$$

which is the image of $[A]$ under the descent map

$$j^G\colon\ \mathrm{KK}^G(\mathbb{C}, C_0(X)) \to \mathrm{KK}(C^*(G), C^*(G, C_0(X)))$$

composed with the intersection product with $[p] \in \mathrm{KK}(\mathbb{C}, C^*(G, C_0(X)))$,

$$[p] \otimes_{C^*(G, C_0(X))}\colon\ \mathrm{KK}(C^*(G, C_0(X)), C^*(G)) \to \mathrm{KK}(\mathbb{C}, C^*(G)).$$

Here $p \doteq (c \cdot g(c))^{\frac{1}{2}}$ is an idempotent in $C_c(G, C_0(X))$, being the image of 1 under the $*$-homomorphism $\mathbb{C} \to C^*(G, C_0(X))$ and defining an element in $K_0(C^*(G, C_0(X)))$.

Analytically, the K-theoretic index of A is constructed explicitly as follows [17]. First of all, embed $C_c(X, E)$ in a larger Hilbert $C^*(G)$-module $C^*(G, L^2(X, E))$ and after completion under the norm of the Hilbert module, we obtain a $C^*(G)$-module $\mathcal{E}$ containing $C_c(X, E)$ as a dense subalgebra. Note that $\mathcal{E}$ is a direct summand of $C^*(G, L^2(X, E))$ and is obtained by compressing the $C^*(G)$-module $C^*(G, L^2(X, E))$ with the idempotent p.

Then the operator

$$\bar{A} \doteq \mathrm{Av}(cA)\colon C_c(X, E) \to C_c(X, E)$$

in $\Psi_{G,p}^0(X; E, E)$ extends to two bounded maps $\bar{A}\colon L^2(X, E) \to L^2(X, E)$ and $\bar{A}\colon \mathcal{E} \to \mathcal{E}$ with $\|\bar{A}\|_{\mathcal{E}} \le \|\bar{A}\|_{L^2(X,E)}$. Denote by $\mathcal{B}(\mathcal{E})$ the C^*-algebra of all bounded operators on $\mathcal{E}$ having an adjoint and being $C^*(G)$-module maps. Then $\bar{A}\colon \mathcal{E} \to \mathcal{E}$ defines an element in $\mathcal{B}(\mathcal{E})$ according to [18]. On the Hilbert $C^*(G)$ module $\mathcal{E}$, for $e, e_1, e_2 \in C_c(X, E)$, a *rank one* operator is defined by

$$\theta_{e_1,e_2}(e)(x) = e_1(e_2, e)(x) = \int_X \left(\int_G \theta_{g(e_1)(x), g(e_2)(y)} \mathrm{d}g \right) e(y) \mathrm{d}y$$

for all $x \in X$. The closure of the linear combinations of the rank one operators under the norm of $\mathcal{B}(\mathcal{E})$ is the set of *compact operators*, denoted by $\mathcal{K}(\mathcal{E})$. The elements of $\mathcal{K}(\mathcal{E})$ can be identified with the integral operators with G-invariant continuous kernel and with proper support. The following proposition indicates some features of elements from $\mathcal{B}(\mathcal{E}), \mathcal{K}(\mathcal{E})$.

Proposition 4.1. [18] *If the symbol of the G-invariant properly supported operator P of order 0 is bounded in the cotangent direction by a constant, then the norm of P in $\mathcal{B}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ does not exceed that constant. The operator P is compact, i.e., $P \in \mathcal{K}(\mathcal{E})$, if the symbol of P is 0 at infinity (in the cotangent direction).*

Since $\bar{A}$ is elliptic, which means that $\|\sigma_{\bar{A}}(x,\xi)^2 - 1\| \to 0$ as $\xi \to 0$, $x \in K$, uniformly for any compact set $K \subset X$, according to Proposition 4.1 we have $\bar{A}^2 - \mathrm{Id} \in \mathcal{K}(\mathcal{E})$. Let us set $\bar{A} = \begin{pmatrix} 0 & \bar{A}_0{}^* \\ \bar{A}_0 & 0 \end{pmatrix}$. Then $[\bar{A}_0] \in K_1(\mathcal{B}(\mathcal{E})/\mathcal{K}(\mathcal{E}))$. The analytical *K-theoretic index*, $\mathrm{Ind}_a A$, is the image of this class in the K-theory of the quotient algebra under the boundary map $\partial\colon K_*(\mathcal{B}(\mathcal{E})/\mathcal{K}(\mathcal{E})) \to K_{*+1}(\mathcal{K}(\mathcal{E}))$ of the six term exact sequence associated to the short exact sequence $0 \to \mathcal{K}(\mathcal{E}) \to \mathcal{B}(\mathcal{E}) \to \mathcal{B}(\mathcal{E})/\mathcal{K}(\mathcal{E}) \to 0$.

Remark 4.2. The set of finite rank $\mathcal{K}(\mathcal{E})$-valued projections forms a finite generated projective $C^*(G)$-module. Then Theorem 3 of Section 6 in [15] implies that $K_*(\mathcal{K}(\mathcal{E})) \simeq K_*(C^*(G))$. Hence, $\mathrm{Ind}_a A \in K_0(C^*(G))$.

As a generalization of the Atiyah–Singer index theorem, Kasparov proved that Ind_a and Ind_t coincide [16], [18]. We will simply use Ind to denote the K-theoretic index. In summary, under the homomorphism $\mathrm{Ind}\colon K^0_G(C_0(X)) \to \mathrm{KK}(\mathbb{C}, C^*(G)) \simeq K_0(\mathcal{K}(\mathcal{E}))$ the K-theoretic index is calculated by

$$\begin{aligned}[(L^2(X,E),A)] &\mapsto [(\mathcal{E},\bar{A})] \\ &\mapsto \left[\begin{pmatrix} \bar{A}_0\bar{A}_0^* & \bar{A}_0\sqrt{1-\bar{A}_0^*\bar{A}_0} \\ \sqrt{1-\bar{A}_0^*\bar{A}_0}\,\bar{A}_0^* & 1-\bar{A}_0^*\bar{A}_0 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].\end{aligned} \tag{4.1}$$

Note that the second arrow is the Fredholm picture of $\mathrm{KK}(\mathbb{C}, C^*(G))$ via boundary map.

Given the K-theoretic index $\mathrm{Ind}\, A \in K_0(\mathcal{K}(\mathcal{E}))$, we will define the a homomorphism $K_0(\mathcal{K}(\mathcal{E})) \to \mathbb{R}$. To do this we find a dense subalgebra $\mathcal{S}(\mathcal{E})$ of $\mathcal{K}(\mathcal{E})$ on which a "trace" can be defined and which is closed under holomorphic functional calculus. Since $\mathcal{K}(\mathcal{E})$ is generated by G-invariant operators with continuous and properly supported kernel, we define $\mathcal{S}(\mathcal{E})$ to be the subset of the bounded G-invariant operators with smooth kernels. Let $S\colon C_c^\infty(X,E) \to C_c^\infty(X,E)$ be a G-invariant smoothing operator. Extend S to an operator $\bar{S} \in \mathcal{B}(\mathcal{E})$, then $\bar{S} \in \mathcal{S}(\mathcal{E})$. Define the trace on $\bar{S} \in \mathcal{S}(\mathcal{E})$ by $\mathrm{tr}_G(S)$ and still denote by tr_G. The trace is well defined for all the elements of $\mathcal{S}(\mathcal{E})$. An element of $\mathcal{S}(\mathcal{E})$ is viewed as matrices with $C^*(G)$-entries. The trace on such a matrix is the Breuer von Neumann trace [7] on the image of the following map $\mathcal{S}(\mathcal{E}) \mapsto \mathcal{S}(\mathcal{E} \otimes_{C^*(G)} \mathcal{R}(L^2(G))) \subset \mathcal{R}(L^2(X, E))$. Here $\mathcal{S}(\mathcal{E} \otimes_{C^*(G)} \mathcal{R}(L^2(G)))$ is a subset of all G-trace class operators and its elements are represented as matrices with $\mathcal{R}(L^2(G))$-entries. Recall (Remark 3.7) that tr_G is defined on a dense subset of the G-invariant operators on $L^2(X, E)$,which can be represented as elements of

$$\mathcal{R}(L^2(G)) \otimes (\oplus_{i,j} \mathcal{B}(L^2(U_i, E), L^2(U_j, E)),$$

and an element of this set can be expressed in terms of a $\mathcal{R}(L^2(G))$-valued matrix.

Proposition 4.3. *We have a canonical isomorphism* $K_0(\mathcal{K}(\mathcal{E})) \simeq K_0(\mathcal{S}(\mathcal{E}))$.
The G-trace tr_G *on* $\mathcal{S}$ *defines a group homomorphism*

$$\mathrm{tr}_{G*}\colon K_0(\mathcal{K}(\mathcal{E})) \to \mathbb{R}.$$

Proof. Proposition 3.4 (4) shows that $\mathcal{S}(\mathcal{E})$ is an ideal of $\mathcal{B}(\mathcal{E})$. Since $\mathcal{S}(\mathcal{E})$ contains the rank one operators, $\mathcal{K}(\mathcal{E})$ is the C^*-closure of $\mathcal{S}(\mathcal{E})$. Let $J = \mathcal{K}(\mathcal{E})$, $J_0 = \mathcal{S}(\mathcal{E})$ and let $\tilde{J}$, $\tilde{J}_0$ be obtained by adjoining a unit. Note that $\tilde{J} = \mathcal{B}(\mathcal{E})$. We claim that J_0 is stable under holomorphic functional calculus. To show the claim we essentially need to prove that if $a \in \tilde{J}_0$ is invertible in $\tilde{J}$, then $a^{-1} \in \tilde{J}_0$ [8]. Let $a^{-1} = \lambda I + r$, where $\lambda \in \mathbb{C}$, I is the unit and $r \in J$. Choose an $s \in J_0$ such that $\|a^{-1} - \lambda I - s\| < \min\{\frac{1}{\|a\|}, 1\}$. Then $\|1 - \lambda a - as\| < 1$ implies that $a(\lambda I + s)$ is invertible. Thus $\lambda I + s$ is also invertible and so $a^{-1} = (\lambda I + s)[a(\lambda I + s)]^{-1}$. Since $\tilde{J}_0$ is an ideal of $\tilde{J}$, we only need to show that $[a(\lambda I + s)]^{-1} \in \tilde{J}_0$. Let $x = a(\lambda I + s) \in J_0$. Then $\|1 - x\| < 1$ and so $x^{-1} = [1 - (1 - x)]^{-1} = \sum_{i=0}^{\infty} (1 - x)^i \in \tilde{J}_0$. The claim is proved. Hence $\mathcal{S}(\mathcal{E})$ is a dense subalgebra of $\mathcal{K}(\mathcal{E})$ closed under holomorphic functional calculus, which implies that $K_*(\mathcal{K}(\mathcal{E})) = K_*(\mathcal{S}(\mathcal{E}))$.

An element of $K_0(\mathcal{S}(\mathcal{E}))$ is represented by projection matrix with entries in $\mathcal{S}(\mathcal{E})$, on which there is a natural trace consisting of the composition of the matrix trace with the G-trace on $\mathcal{S}(\mathcal{E})$. Note that if the element was represented by the difference of two classes of matrices with entries in $\mathcal{S}(\mathcal{E})^+$, the algebra defined by adding a unit. Then we define the trace of this extra unit to be 0. Hence we obtain a homomorphism $\mathrm{tr}_{G*}\colon K_*(\mathcal{S}(\mathcal{E})) \to \mathbb{R}$ by the properties of the trace tr_G. □

Composing with the K-theoretic index, P has a numerical index given by the image of the map

$$K^0_G(C_0(X)) \xrightarrow{\text{K-theoretic index}} K_0(\mathcal{S}) \xrightarrow{\mathrm{tr}_{G*}} \mathbb{R},$$

and this number depends only on the symbol class and the manifold according to Kasparov's K-theoretic index formula (Theorem 5.1). We show that this number is in fact the L^2-index.

Proposition 4.4. *Let $P \in \Psi^0_{G,p}(X; E, E)$ be elliptic. Then its L^2-index coincides with the trace of its K-theoretic index, i.e.,* $\operatorname{ind} P = \operatorname{tr}_{G*}(\operatorname{Ind}[P])$.

Proof. Let $P = A$. Then $P = \bar{A} = \operatorname{Av}(cA)$ in (4.1). Then

$$\operatorname{Ind} P = \left[\begin{pmatrix} P_0 P_0^* & P_0\sqrt{1 - P_0^* P_0} \\ \sqrt{1 - P_0^* P_0} P_0^* & 1 - P_0^* P_0 \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

We shall alter the matrix representatives without changing the equivalence class, so that we may apply tr_G to the 2×2-matrices.

Given $P_0 \in \Psi^0_{G,p}(X; E_0, E_1)$ and using Proposition 2.7, there is a $Q \in \Psi^0_{G,p}$ such that $1 - QP_0 = S_0$, $1 - P_0 Q = S_1$. According to the boundary map construction in [10], Section 2, we lift $\begin{pmatrix} 0 & -Q \\ P_0 & 0 \end{pmatrix}$, which is invertible in $M_2(\mathcal{B}(\mathcal{E})/\mathcal{S}(\mathcal{E}))$, to an invertible element $u = \begin{pmatrix} S_0 & -(1+S_0)Q \\ P_0 & S_1 \end{pmatrix}$ in $M_2(\mathcal{B}(\mathcal{E}))$ such that

$$\operatorname{Ind} P \doteq \left[u\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}u^{-1}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \left[\begin{pmatrix} S_0^2 & S_0(1+S_0)Q \\ P_0 S_1 & 1 - S_1^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Therefore, $\operatorname{tr}_{G*}(\operatorname{Ind} P) = \operatorname{tr}_G(S_0^2) + \operatorname{tr}_G(1 - S_1^2) - \operatorname{tr}_G(1) = \operatorname{tr}_G(S_0^2) - \operatorname{tr}_G(S_1^2)$. Choose another $Q' \doteq 2Q - QP_0Q$, then $1 - Q'P_0 = S_0^2$, $1 - P_0Q' = S_1^2$ with S_0^2, S_1^2 being smoothing operators. Then using Proposition 3.11, we conclude that $\operatorname{tr}_G(S_0^2) - \operatorname{tr}_G(S_1^2) = \operatorname{ind} P$. Hence $\operatorname{tr}_{G*}(\operatorname{Ind} P) = \operatorname{ind} P$. □

Remark 4.5. Let $X = G/H$ be a homogeneous space of a unimodular Lie group G (where H is a compact subgroup). In [9], Section 3, it was shown directly that the L^2-index depends only on the symbol class $[\sigma_P]$ of P in $K_0^G(C_0(T^*X))$. In addition, there exists a homomorphism $i\colon K_0^G(C_0(T^*X)) \to \mathbb{R}$ such that $i[\sigma_P] = \operatorname{ind} P$. Note that the Poincaré duality between K-homology and K-theory gives rise to $K_0^G(C_0(T^*X)) \simeq K_G^0(C_0(X))$. So L^2-index essentially gives a homomorphism

$$\operatorname{ind}\colon K_G^0(C_0(X)) \to \mathbb{R}.$$

Remark 4.6. In this section we work on the cycles in $K_G^0(C_0(X))$ determined by odd self-adjoint elliptic pseudo-differential operators on X. If Y is another proper cocompact G-manifold and if E is a G-bundle where $L^2(Y, E)$ admits a $C_0(X)$-representation such that $[(L^2(Y, E), Q)] \in K_G^0(C_0(X))$ with $Q \in \Psi^0_{G,p}(Y; E, E)$, we may carry out similar constructions to define the L^2-index of Q. However, it is not clear how to define L^2-index for an arbitrary representative (A, F) in a general representing cycle $[(A, F)] \in K_G^0(C_0(X))$, where A is a $C_0(X)$-algebra and F is a general elliptic operator. The reason is that we do not know the way to define pseudo-differential calculus for the C^*-algebra A and we do not have Proposition 2.7 for F, which is essential for calculating the L^2-index. But it should be possible to find a proper cocompact G-manifold Y and pseudo-differential operator Q on Y such that $[(A, F)] = [(L^2(Y, E), Q)] \in K_G^0(C_0(X))$.

5. Reduction to the L^2-index of a Dirac type operator

We shall show in this section that for any elliptic operator $P \in \Psi^0_{G,p}(X; E, E)$, there is a Dirac type operator $\widetilde{D}$ satisfying ind $P = \operatorname{ind} \widetilde{D}$. To do this, we show that P and $\widetilde{D}$ have the same K-theoretic index and then apply Proposition 4.4.

Theorem 5.1 ([16], [18]). *Let X be a complete Riemannian manifold and let G be a locally compact group acting on X properly and isometrically. Let P be a G-invariant elliptic operator on X of order* 0. *Then*

$$[P] = [\sigma_P] \otimes_{C_0(T^*X)} [D] \in K^*_G(C_0(X)), \tag{5.1}$$

where $[D]$ is the equivalence class defined by the Dolbeault operator on T^*X.

Remark 5.2. In (5.1), the ellipticity of $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix} \in \Psi^0_{G,p}(X; E, E)$ (Definition 2.6) implies that the symbol $\sigma_P = \begin{pmatrix} 0 & \sigma_{P_0} \\ \sigma_{P_0} & 0 \end{pmatrix}$ defines an element of $\mathrm{KK}^G(C_0(X), C_0(T^*X))$. In fact, using the Hermitian structure on $E = E_0 \oplus E_1$, we obtain $C_0(T^*X, \pi^*E)$, a Hilbert module over $C_0(T^*X)$, and the set of "compact operators" is $C_0(T^*X, \mathrm{Hom}(\pi^*E, \pi^*E))$. Also $C_0(X)$ acts on $C_0(T^*X, \pi^*E_0 \oplus \pi^*E_1)$ by pointwise multiplication. Hence for all $f \in C_0(X)$, $(\sigma_P^2 - I)f$ is compact by (2.5) and $[\sigma_P, f] = 0$. Therefore, the symbol $\sigma_P : \pi^*E \to \pi^*E$ defines the following element in KK-theory:

$$\Big[\Big(C_0(T^*X, \pi^*E_0 \oplus \pi^*E_1), \begin{pmatrix} 0 & \sigma_{P_0}^* \\ \sigma_{P_0} & 0 \end{pmatrix}\Big)\Big] \in \mathrm{KK}^G(C_0(X), C_0(T^*X)).$$

In (5.1), the *Dolbeault operator* D is a first order differential operator $D = \sqrt{2}(\bar\partial + \bar\partial^*)$ acting on smooth sections of $\bigwedge^{0,*}(T^*(T^*X))$, where $\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial x})$. Denote by H the Hilbert space of L^2-forms of bi-degree $(0, *)$ on T^*X graded by the odd and even forms. Then D is an order 1 essentially self-adjoint operator on H. The C^*-algebra $C_0(T^*X)$ acts on H by point-wise multiplication. *The Dolbeault element* is the K-homological cycle given by

$$[(H, \tfrac{D}{\sqrt{1+D^2}})] \in K^0_G(C_0(T^*X)) = \mathrm{KK}^G(C_0(T^*X), \mathbb{C}).$$

Remark 5.3. Theorem 5.1 says that $[P]$ is given by the index pairing of the symbol with some fundamental (Dolbeault) operator on T^*X. This is the essence of the Atiyah–Singer index theorem. When X is compact with trivial group action, apply the map

$$C^* \colon K^0(C(X)) \to K^0(\mathbb{C})$$

induced by the constant map $C : \mathbb{C} \to C(X)$ to both sides of (5.1). The left-hand side of (5.1) is then the Fredholm index of P and the right-hand side is the intersection product of $[\sigma_P] \in K_0(C_0(T^*X))$ with $[D] \in K^0(C_0(T^*X))$. It is classical fact that

$[\sigma_P]$ is viewed as some equivalence class of a vector bundle V. Then the intersection product is the well-known Fredholm index of the Dirac operator D with coefficients in V.

The following K-theoretic index formula serves as an important corollary to Theorem 5.1.

Theorem 5.4 ([16]). *Let X be a complete Riemannian manifold, on which a locally compact group G acts properly and isometrically with compact quotient. Let P be a properly supported G-invariant elliptic operator on X of order 0. Then*

$$\begin{aligned}\operatorname{Ind} P &= [p] \otimes_{C^*(G,C_0(X))} j^G([P]) \\ &= [p] \otimes_{C^*(G,C_0(X))} j^G([\sigma_P]) \otimes_{C^*(G,C_0(T^*X))} j^G([D]) \in K_*(C^*(G)).\end{aligned}$$

Here p is the idempotent in $C^*(G, L^2(X, E))$ defined by $p = (c \cdot g(c))^{\frac{1}{2}}$ and $[D]$ is the Dolbeault element.

Analogous to the vector bundle construction mentioned in Remark 5.3 (see also [2], Section 7), we define a G-bundle $V(\sigma_P)$ using the symbol σ_P as follows. Let $B(X) \subset T^*X$ be the unit ball bundle with its boundary, that is, the sphere bundle $S(X) \subset T^*X$. A new manifold ΣX is obtained by gluing two copies of $B(X)$ along their boundaries:

$$\Sigma X = B(X) \cup_{S(X)} B(X).$$

The action of G on T^*X extends naturally to ΣX because G acts on X isometrically. The ellipticity of P implies the invertibility of $\sigma_P|_{S(X)}$, the symbol restricted to $S(X)$. Define a G-vector bundle over ΣX by the gluing map σ_P on the boundary, that is,

$$V(\sigma_P) = \pi^*E|_{B(X)} \cup_{\sigma_P|_{S(X)}} \pi^*E|_{B(X)}. \tag{5.2}$$

Here $V(\sigma_P)$ defines an element in $\mathrm{RKK}^0_G(X; C_0(X), C_0(\Sigma X))$, the representable KK-theory. $V(\sigma_P)$ is $\mathbb{Z}/2\mathbb{Z}$-graded and is the direct sum of two bundles: $V(\sigma_{P_0}) = \pi^*E_0|_{B(X)} \bigcup_{\sigma_{P_0}|_{S(X)}} \pi^*E_1|_{B(X)}$ and $V(\sigma_{P_0^*}) = \pi^*E_1|_{B(X)} \bigcup_{\sigma_{P_0^*}|_{S(X)}} \pi^*E_0|_{B(X)}$. There is a natural homomorphism

$$\mathrm{RKK}^0_G(X; C_0(X), C_0(\Sigma X)) \to \mathrm{KK}^G(C_0(X), C_0(\Sigma X)).$$

Let $[V(\sigma_P)]$ be the equivalence class of $V(\sigma_P)$ either in $\mathrm{RKK}^0_G(X; C_0(X), C_0(\Sigma X))$ and or in $\mathrm{KK}^G(C_0(X), C_0(\Sigma X))$. We shall not distinguish the notations when it is clear from the context. In the proof Proposition 5.6 we shall see that as a KK-cycle,

$$[V(\sigma_P)] = [(C_0(\Sigma X, V(\sigma_P)), 0)] \text{ in } \mathrm{KK}^G(C_0(X), C_0(\Sigma X)).$$

Remark 5.5. When X is compact and when $G = \{e\}$, the inclusion $\mathbb{C} \to C(X)$ further reduces σ_P to an element of $\mathrm{KK}(\mathbb{C}, C_0(T^*X))$ by "forgetting" the action of $C(X)$

on the Hilbert-$C(X)$ module $C_0(T^*X)$. Therefore, $[\sigma_P] \in \mathrm{KK}(\mathbb{C}, C_0(T^*X))) \simeq K_0(C_0(T^*X))$ maps to a vector bundle, trivial at infinity in T^*X. The bundle is constructed by gluing $\pi^*E|_{B(X)}$ and $\pi^*E|_{T^*X-B(X)^\circ}$ along the boundaries using the invertible map $\sigma_P|_{S(X)}$ and is the restriction of $V(\sigma_P)$ to T^*X.

Proposition 5.6. *The homomorphism*

$$\mathrm{KK}^G(C_0(X), C_0(T^*X)) \to \mathrm{KK}^G(C_0(X), C_0(\Sigma X)),$$
$$[(C_0(T^*X, \pi^*E), \sigma_P)] \mapsto [(C_0(\Sigma X, V(\sigma_P)), 0)],$$

is induced by the inclusion map $i\colon C_0(T^*X) \to C_0(\Sigma X)$.

Proof. The cycle $(C_0(\Sigma X, V(\sigma_P)), 0)$ defines an element of $\mathrm{KK}^G(C_0(X), C_0(\Sigma X))$ because $f \cdot (0^2 - \mathrm{Id}_{C_0(\Sigma X, V(\sigma_P))})$ is compact in the Hilbert-$C_0(\Sigma X)$-module $C_0(\Sigma X, V(\sigma_P))$. Here, the compactness of the fiber of ΣX over X is important. The argument fails when replacing ΣX by T^*X. For example, $(C_0(T^*X, V(\sigma_P)|_{T^*X}), 0)$ does not define an element in $\mathrm{KK}^G(C_0(X), C_0(T^*X))$.

Without loss of generality, we may assume σ_P satisfies that

$$\sigma_P^2 = 1 \text{ on } S(X) \quad \text{and} \quad \|\sigma_P\| \leq 1.$$

Using the standard boundary map construction in the exact sequence of K-theory, we obtain the following projection Q using the unitary $u = \left(\begin{smallmatrix} \sigma_P & -\sqrt{1-\sigma_P^2} \\ \sqrt{1-\sigma_P^2} & \sigma_P \end{smallmatrix}\right) \in M_2(C_0(T^*X, \pi^*E))$:

$$Q \doteq u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u^{-1} = \begin{pmatrix} \sigma_P^2 & \sigma_P\sqrt{1-\sigma_P^2} \\ \sqrt{1-\sigma_P^2}\,\sigma_P & 1-\sigma_P^2 \end{pmatrix}.$$

Recall that the bundle $V(\sigma_P)|_{T^*X}$ glued by σ_P is given by the image of the projection Q on $\pi^*E \oplus \pi^*E$, that is,

$$C_0(T^*X, V(\sigma_P)|_{T^*X}) = QC_0(T^*X, \pi^*E \oplus \pi^*E).$$

Let $w = uxu^* = u\left(\begin{smallmatrix} \sigma_P & 0 \\ 0 & 1 \end{smallmatrix}\right)u^*$. Then $QwQ = w$, $(1-Q)w(1-Q) = 1-Q$, and $Qw(1-Q) = (1-Q)wQ = 0$. Then

$$\begin{aligned}
&[(C_0(T^*X, \pi^*E), \sigma_P)] = [(C_0(T^*X, \pi^*E), \sigma_P)] + [(C_0(T^*X, \pi^*E), 1)] \\
&\quad = [(C_0(T^*X, \pi^*E \oplus \pi^*E), \left(\begin{smallmatrix} \sigma_P & 0 \\ 0 & 1 \end{smallmatrix}\right))] = [(C_0(T^*X, \pi^*E \oplus \pi^*E), w)] \\
&\quad = [(QC_0(T^*X, \pi^*E \oplus \pi^*E), x)] + [((1-Q)C_0(T^*X, \pi^*E \oplus \pi^*E), 1-Q)] \\
&\quad = [(QC_0(T^*X, \pi^*E \oplus \pi^*E), x)] \\
&\quad \to [(C_0(\Sigma X, V(\sigma_P)), \tilde{x})] = [(C_0(\Sigma X, V(\sigma_P)), 0)].
\end{aligned}$$

Here, the arrow in the last line comes from the following fact. The Hilbert $C_0(T^*X)$-module $C_0(T^*X, V(\sigma_P)|_{T^*X})$ maps to the Hilbert $C_0(\Sigma X)$-module $C_0(\Sigma X, V(\sigma_P))$ under the map $i_*\colon \mathrm{KK}^G(C_0(X), C_0(T^*X)) \to \mathrm{KK}^G(C_0(X), C_0(\Sigma X))$ induced from the inclusion i. The last equality follows from the operator homotopy $t \to t\tilde{x}$ and the observation that $(C_0(\Sigma X, V(\sigma_P)), t\tilde{x})$ is a Kasparov $(C_0(X), C_0(\Sigma X))$ module for all $t \in [0, 1]$. The proof is complete. □

The Dolbeault operator on T^*X extends to the proper cocompact G-manifold ΣX, which also has an almost complex structure. We just glue two Dolbeault operators on $B(X) \subset T^*X$ along the boundary (the normal directions of $S(X)$ in $B(X)$ need to switch signs on different pieces). The new Dolbeault operator $\bar{D}$ is clearly G-invariant and defines an element

$$[\bar{D}] = \Big[(L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))), \frac{\bar{D}}{\sqrt{1+\bar{D}^2}})\Big] \tag{5.3}$$

in $\mathrm{KK}^G(C_0(\Sigma X), \mathbb{C})$. (In Section 6 we shall not distinguish $[\bar{D}]$ and $[D]$.) The following proposition is obvious.

Proposition 5.7. *The inclusion $i\colon C_0(T^*X) \to C_0(\Sigma X)$ induces the natural map*

$$i^*\colon \mathrm{KK}^G(C_0(\Sigma X), \mathbb{C}) \to \mathrm{KK}^G(C_0(T^*X), \mathbb{C}), \quad [\bar{D}] \mapsto [D].$$

Corollary 5.8. *Assuming the same notations and conditions in the K-homological formula in Theorem 5.1, we have:*

(1) *The elliptic pseudo-differential operator P is in the same K-homology class as the intersection product $[V(\sigma_P)] \otimes [\bar{D}]$ in the image of the map*

$$\mathrm{KK}^G(C_0(X), C_0(\Sigma X)) \times \mathrm{KK}^G(C_0(\Sigma X), \mathbb{C}) \to \mathrm{KK}^G(C_0(X), \mathbb{C}).$$

(2) *The operator P relates to a Dirac type operator $\bar{D}_{V(\sigma_P)}$, that is, the Dolbeault operator $\bar{D}$ on ΣX twisted by the bundle $V(\sigma_P)$ over ΣX in the sense*

$$[P] = j^*[\bar{D}_{V(\sigma_P)}],$$

where $j^*\colon \mathrm{KK}^G(C_0(\Sigma X), \mathbb{C}) \to \mathrm{KK}^G(C_0(X), \mathbb{C})$ is induced by the inclusion $j\colon C_0(X) \to C_0(\Sigma X)$.

Proof. The first statement is a result of Theorem 5.1 as well as the functorality of intersection products

$$\begin{aligned}[P] &= [\sigma_P] \otimes_{C_0(TX)} [D] \\ &= [\sigma_P] \otimes_{C_0(TX)} i^*[\bar{D}] \\ &= i_*[\sigma_P] \otimes_{C_0(\Sigma X)} [\bar{D}] \\ &= [V(\sigma_P)] \otimes_{C_0(\Sigma X)} [\bar{D}].\end{aligned}$$

To prove the second statement, we calculate

$$\begin{aligned}&[V(\sigma_P)] \otimes_{C_0(\Sigma X)} [\bar{D}] \\ &\quad= [(C_0(\Sigma X, V(\sigma_P)), \phi_1, 0)] \otimes_{C_0(\Sigma X)} [(L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))), \phi_2, F)],\end{aligned} \tag{5.4}$$

where $F \doteq \frac{\bar{D}}{\sqrt{1+\bar{D}^2}}$. We denote by $[(H, \eta, I)]$ the KK-product appeared in (5.4).

According to the definition of KK-product,

$$H = C_0(\Sigma X, V(\sigma_P)) \otimes_{C_0(\Sigma X)} L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))),$$

and the operator I needs to satisfy the following two conditions [29]:

(1) I is an F-connection;

(2) I has the property $\eta(a)[0 \otimes 1, I]\eta(a) \geq 0$ modulo $\mathcal{K}(H)$.

By Kasparov's stabilization theorem, there is a $C_0(\Sigma X)$-valued projection Q such that $C_0(\Sigma X, V(\sigma_P)) = Q(\oplus_1^\infty C_0(\Sigma X))$. Therefore,

$$\begin{aligned}H &= Q(\oplus_1^\infty C_0(\Sigma X)) \otimes_{C_0(\Sigma X)} L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))) \\ &= \phi_2(Q)(\oplus_1^\infty L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))),\end{aligned}$$

where, $\phi_2(Q)$, by definition, acts by matrix multiplication and point-wise multiplication.

We claim that

$$I = \phi_2(Q)(\oplus_1^\infty F)\phi_2(Q) \tag{5.5}$$

The statement is proved if (5.5) is true. In fact, one needs only to observe that

$$H = \phi_2(Q)(\oplus_1^\infty L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X))) = L^2(\Sigma X, \textstyle\bigwedge^{0,*}(T^*(\Sigma X)) \otimes V(\sigma_P))$$

and

$$\phi_2(Q)(\oplus_1^\infty \bar{D})\phi_2(Q) = \bar{D}_{V(\sigma_P)} \text{ on } H.$$

To prove the claim (5.5), it is sufficient to show the following observations.

$(I^2 - 1)\eta(f) \in \mathcal{K}(H)$ for all $f \in C_0(X)$;

$[I, \eta(f)] \in \mathcal{K}(H)$ for all $f \in C_0(X)$;

$[\widetilde{T}_\xi, F \oplus I] \in \mathcal{K}(L^2(\Sigma X, \bigwedge^*(\Sigma X)) \oplus H)$ for all $\xi \in C_0(\Sigma X, V(\sigma_P))$, where $\widetilde{T}_\xi = \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix} \in \mathcal{B}(L^2(\Sigma X, \bigwedge^{0,*}(T^*(\Sigma X)) \oplus H)$ and

$T_\xi \in \mathcal{B}(L^2(\Sigma X, \bigwedge^{0,*}(T^*(\Sigma X)), H)$ is defined by $T_\xi(\eta) = \xi \mathbin{\widehat{\otimes}} \eta \in H$. □

Proposition 5.9. *Let P be a properly supported G-invariant elliptic pseudo-differential operator of order 0, $\bar{D}$ be the Dolbeault operator on ΣX defined in (5.3) and $V(\sigma_P)$ be the G-vector bundle over ΣX defined in (5.2) Then P and $D_{V(\sigma_P)}$ have the same L^2-index, that is,*

$$\operatorname{ind} P = \operatorname{ind} D_{V(\sigma_P)}.$$

Proof. In view of Corollary 5.8, the cycle

$$[(L^2(\Sigma X, \wedge^{0,*}(T^*(\Sigma X)) \otimes V(\sigma_P)), \bar{D}_{V(\sigma_P)})]$$

represents as the intersection product $[V(\sigma_P)] \otimes [\bar{D}]$, which is the same as $[(L^2(X,E),P)]$ in $K^0_G(C_0(X))$. This implies that $\operatorname{Ind} P = \operatorname{Ind} D_{V(\sigma_P)}$ and the statement is proved by taking the trace of the K-theoretic indices. $\square$

6. Local index formula

6.1. L^2-index of Dirac type operators. Using Proposition 5.9, to find a cohomological formula for the L^2-index of P, it is sufficient to figure out a formula for Dirac type operators. Let M be an even-dimensional ($\dim M = n$) proper cocompact G-manifold with a G-Clifford bundle V, which is a $\mathbb{C}\mathrm{l}(T^*M)$-module via Clifford multiplication. Here $\mathbb{C}\mathrm{l}(T^*M) = \mathrm{Cl}(T^*M) \otimes \mathbb{C}$ is the complex Clifford algebra generated by T^*M. We construct $\mathcal{D}$, a Dirac type operator acting on sections in V. Let ∇ be the G-invariant Levi-Civita connection on TM, which can be extended to $\mathbb{C}\mathrm{l}(T^*M)$. Let ∇^V be the G-invariant *Clifford connection* on V, i.e., $[\nabla^V, \mathrm{c}(a)] = \mathrm{c}(\nabla a), a \in C_c^\infty(M, \mathbb{C}\mathrm{l}(T^*M))$. A *Dirac operator* $\mathcal{D}\colon C_c^\infty(M,V) \to C_c^\infty(M,V)$ is defined as the composition of the connection ∇^V and the Clifford multiplication $\mathrm{c}\colon C_c^\infty(M, T^*M \times V) \to C_c^\infty(M,V)$ by

$$\mathcal{D} = \sum_i \mathrm{c}(e^i)\nabla^V_{e_i},$$

where $\{e_i\}$ forms an orthonormal basis of the bundle TM and $\{e^i\}$ is the dual basis of T^*M. Here, $V = V_0 \oplus V_1$ is $\mathbb{Z}/2\mathbb{Z}$ graded and $\mathcal{D}$ is essentially self-adjoint with an odd grading, in particular, $\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_0^* \\ \mathcal{D}_0 & 0 \end{pmatrix}\colon L^2(M,V) \to L^2(M,V)$. The L^2-index of $\mathcal{D}$ is expressed by the McKean–Singer formula (3.1), which is independent of t:

$$\operatorname{ind} \mathcal{D} = \operatorname{str}_G(e^{-t\mathcal{D}^2}).$$

Here $\operatorname{str}_G\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \operatorname{tr}_G(a) - \operatorname{tr}_G(d)$ and $\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_0^*\mathcal{D}_0 & 0 \\ 0 & \mathcal{D}_0\mathcal{D}_0^* \end{pmatrix}$.

Let $R^V = (\nabla^V)^2 \in \wedge^2(M, \operatorname{Hom} V)$ be the curvature tensor of the Clifford connection ∇^V. Then

$$\begin{aligned}\mathcal{D}^2 &= -\sum_i (\nabla^V_{e_i})^2 + \sum_i \nabla^V_{\nabla_{e_i} e_i} + \sum_{i<j} \mathrm{c}(e^i)\mathrm{c}(e^j)R^V(e_i,e_j)\\ &\doteq \Delta^V + \sum_{i<j} \mathrm{c}(e^i)\mathrm{c}(e^j)R^V(e_i,e_j)\end{aligned}$$

is a generalized Laplacian. Let S be the spinor (irreducible) representation of $\mathbb{C}\mathrm{l}(T_x^*M)$. It is a standard fact that $\operatorname{Hom} S = S \otimes S^* = \mathbb{C}\mathrm{l}(T_x^*M)$. The fiber of the Clifford module V at x has the decomposition $V_x = S \otimes W$. Here W is the set of vectors in V_x that commute with the action of $\mathbb{C}\mathrm{l}(T_x^* M)$. Therefore on the endomorphism level we have

$$\mathrm{Hom}\, V_x = \mathbb{C}\mathrm{l}(T_x^* M) \otimes \mathrm{Hom}\, W. \tag{6.1}$$

Here $\mathrm{Hom}_{\mathbb{C}\mathrm{l}(T_x^* M)}(V_x) \doteq \mathrm{Hom}\, W$ is made of the transformations of V_x that commute with $\mathbb{C}\mathrm{l}(T_x^* M)$. According to [5], Proposition 3.43, the curvature R^V decomposes under the isomorphism (6.1) into

$$R^V = R^S + F^{V/S},$$

where $R^S(e_i, e_j) = \frac{1}{4}\sum_{kl}(R(e_i, e_j)e_k, e_l)\mathrm{c}^k \mathrm{c}^l$ is the action of the Riemannian curvature $R \doteq \nabla^2$ of M on the bundle V and $F^{V/S} \in \bigwedge^2(M, \mathrm{Hom}_{\mathbb{C}\mathrm{l}}\, V)$ is the twisting curvature of the Clifford connection ∇^V. According to the Lichnerowicz Formula, [5], Proposition 3.52, the generalized Laplacian is calculated by

$$\mathcal{D}^2 = -\sum_{i=1}^{n} (\nabla^V_{e_i})^2 + \textstyle\sum_i \nabla^V_{\nabla_{e_i} e_i} + \frac{1}{4} r_M + \displaystyle\sum_{i<j} F^{V/S}(e_i, e_j)\mathrm{c}(e_i)\mathrm{c}(e_j), \tag{6.2}$$

where $F^{V/S}(e_i, e_j) \in \mathrm{Hom}_{\mathbb{C}\mathrm{l}}\, V$ are the coefficients of the twisting curvature $F^{V/S}$.

Let the *heat kernel* k_t be the Schwartz kernel of the solution operator $e^{-t\mathcal{D}^2}$ of the heat equation $\frac{\partial}{\partial t} u(t,x) + \mathcal{D}^2 u(t,x) = 0$. It is a smooth map $M \times M \to \mathrm{Hom}(V, V)$ satisfying $e^{-t\mathcal{D}^2} f(x) = \int_M k_t(x,y) f(y) \mathrm{d}y$. Hence

$$\mathrm{ind}\, \mathcal{D} = \int_M c(x)\, \mathrm{str}\, k_t(x,x) \mathrm{d}x.$$

We have the following properties of the heat kernel.

Lemma 6.1. (1) *For $f(x) \in L^2(M)$, $e^{-t\mathcal{D}^2} f$ is a smooth section;*

(2) *The kernel $k_t(x,y)$ of $e^{-t\mathcal{D}^2}$ tends to the δ function weakly, i.e., $e^{-t\mathcal{D}^2} s(x) = \int_M k_t(x, x_0) s(x_0) \mathrm{d}x_0 \to s(x)$ uniformly on a compact set in M as $t \to 0$.*

Proof. We have proved that the Schwartz kernel of $ce^{-t\mathcal{D}^2}$ is smooth in Lemma 3.13. So

$$(e^{-t\mathcal{D}^2} f)(x) = \int_{G \times M} c(g^{-1}x) k_t(x,y) f(y) \mathrm{d}y \mathrm{d}g \doteq \int_G h_g(x) \mathrm{d}g,$$

where $h_g(x) = \int_M c(g^{-1}x) k_t(x,y) f(y) \mathrm{d}y$ is smooth in $x \in M$ for fixed $g \in G$. Using the fact that $e^{-t\mathcal{D}^2}$ is a bounded operator and that $c(x)$ is smooth and compactly supported, we conclude that $h_g(x)$ depends smoothly on $g \in G$. Let K be any compact neighborhood of x. Then by the properness of the group action, the set

$$Z \doteq \{g \in G \mid c(g^{-1}x) \neq 0,\ x \in K,\ g \in G\}$$

is compact and so $(e^{-t\mathcal{D}^2}f)(x) = \int_Z h_g(x)\mathrm{d}g$ is smooth for $x \in K$. Therefore the first statement is proved.

To prove the second one, suppose that u is a smooth function with norm 1. Then $\langle e^{-t\mathcal{D}^2}u, u\rangle = \int_{\lambda\in\mathrm{sp}(\mathcal{D})} e^{-t\lambda^2}\mathrm{d}P_{u,u}$, where $\mathrm{sp}(\mathcal{D})$ means the spectrum of $\mathcal{D}$. Since the set of integrals for $0 < t \leq 1$ is bounded by 1, by the dominated convergence theorem,

$$\langle e^{-t\mathcal{D}^2}u, u\rangle \to \int_{\lambda\in\mathrm{sp}(\mathcal{D})} 1\mathrm{d}P_{u,u} = \langle u, u\rangle \quad \text{as } t \to 0.$$

□

The heat kernel on $\mathbb{R}^n$ of $u_t - \sum_{i=1}^n \frac{\partial^2}{\partial^2 x_i} = 0$, which is

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t}, \tag{6.3}$$

suggests a first approximation for the heat kernel on M. The small time behavior of the heat kernel $k_t(x, y)$ for x near y depends on the local geometry of x near y. This is made precise by the *asymptotic expansion* for $k_t(x, y)$.

Definition 6.2 ([26]). Let B be a Banach space with norm $\|\cdot\|$ and $f : \mathbb{R}^+ \to B$, $t \mapsto f(t)$, be a function. A formal series $\sum_{k=0}^\infty a_k(t)$ with $a_k(t) \in E$ is called an *asymptotic expansion* for f, denoted by $f(t) \sim \sum_{i=0}^\infty a_k(t)$, if for any $m > 0$, there are M_m and $\epsilon_m > 0$. So that for all $l \geq M_m, t \in (0, \epsilon_m]$, we have

$$\|f(t) - \sum_{k=0}^{l} a_k(t)\| \leq Ct^m.$$

When M is compact and when $B = C^0(M, \mathrm{Hom}(V, V))$ has C^0-norm $\|f\| = \sup_{x\in M}|f(x)|$, it is the standard fact that the heat kernel $k_t(x, x)$ of $e^{-t\mathcal{D}^2}$ has an asymptotic expansion

$$k_t(x, x) \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^\infty t^j a_j(x),$$

where $a_j(x) \in \mathrm{Hom}(V_x, X_x), x \in M$ are smooth sections ([26], Theorem 7.15). In our case, this theorem is formulated as follows.

Theorem 6.3. *Let M be a proper cocompact Riemannian G-manifold and $\mathcal{D}$ be an equivariant Dirac type operator acting on the sections of a Clifford bundle V, and let k_t be the heat kernel of $\mathcal{D}$. There is an asymptotic expansion for $c(x)k_t(x, x)$ under the C^0-norm $\|f\| = \sup_{x\in M}|f(x)|$:*

$$c(x)k_t(x, x) \sim c(x)\frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^\infty t^j a_j(x) \tag{6.4}$$

where $a_j \in C^\infty(M, \mathrm{Hom}\, V)$ and $a_j(x)$ depends only on the geometry at x (involving metrics, connection coefficients and their derivatives). In particular $a_0(x) = 1$. The asymptotic expansion works for any C^l-norm for $l \geq 0$. (We only need and prove the case when $l = 0$.)

To prove Theorem 6.3 we construct an "approximating heat kernel". The proof is a modification of the case of operators on compact manifold ([26], Theorem 7.15, or [5], Chapter 2). Now $k_t(x, y)$ satisfies the heat equation:

$$\frac{\partial}{\partial t} k_t(x, y) + \mathcal{D}^2 k_t(x, y) = 0, k_0(x, y) = \delta_y(x) \tag{6.5}$$

where $\mathcal{D}$ operates on the x-coordinate only. We fix y and denote it by x_0 and solve this equation locally on a coordinate neighborhood O_{x_0} of x_0 with $x \in O_{x_0}$. We approximate the heat kernel $k_t(x, x_0), x \in O_{x_0}$ locally by looking for a formal solution

$$p_t(x, x_0) \sum_{i=0}^{\infty} t^i b_i(x) \tag{6.6}$$

to the equation (6.5), where $p_t(x, x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}}$ with $r = |\boldsymbol{x}| = d(x, x_0)$ is the heat kernel on Euclidean space (6.3). Denote by $s_t(x, x_0) = \sum_{i=0}^{\infty} t^i b_i(x)$ in (6.6) and so the heat kernel writes

$$k_t(x, x_0) = p_t(x, x_0) s_t(x, x_0). \tag{6.7}$$

According to [26], eq. 7.16, $\mathcal{D}^2$ in (6.2) on O_{x_0} is calculated by

$$\left[\frac{\partial}{\partial t} + \mathcal{D}^2\right](p_t s_t) = p_t \left[\frac{\partial}{\partial t} + \mathcal{D}^2 + \frac{r}{4gt}\frac{\partial g}{\partial r} + \frac{1}{t}\nabla_{r\frac{\partial}{\partial r}}\right] s_t \tag{6.8}$$

when operating on (6.7), where $r = |\boldsymbol{x}|$, $g = \det(g_{ij})$ and (g_{ij}) is the Riemannian metric on M. To find the formal solution (6.6), set the right-hand side of (6.8) to be 0. Then the comparison of the coefficients of terms containing t^i for each $i \geq 0$ enables us to find b_i inductively via [26], eq. (7.17):

$$\nabla_{\frac{\partial}{\partial r}}(r^i g^{\frac{1}{4}} b_i(x)) = \begin{cases} 0, & i = 0, \\ -r^{i-1} g^{\frac{1}{4}} \mathcal{D}^2 b_{i-1}(x), & i > 0. \end{cases} \tag{6.9}$$

(1) (Solve $b_0(x)$) It is trivial to see that $p_t(x, x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}} \to \delta_{x_0}(x)$ uniformly as $t \to 0+$. From Lemma 6.1, $k_t(x, x_0) \to \delta_{x_0}(x)$ uniformly as $t \to 0+$ for all $x \in K$, where $K \subset X$ is any compact subset. Therefore $b_0(x_0) = 1$ necessarily. The first line in (6.9) indicates that $g^{\frac{1}{4}} b_0(x) = g(x_0)^{\frac{1}{4}} b_0(x_0) = 1$, and then $b_0(x) = g^{-\frac{1}{4}}(x)$ is determined by $b_0(x_0)$.

(2) (Solve $b_i(x), i > 0$) Inductively the smoothness of b_i implies the uniqueness of the smooth solution b_{i+1}. In fact, when solving the equation in (6.9), the constant term has to be 0 otherwise b_{i+1} is not smooth at $r = 0$. Then b_{i+1} is smooth except that it may blow up at 0. But by setting $r = 0$ in the second line in (6.9) we have $b_{i+1}(x_0) = -\frac{1}{j}(\mathcal{D}^2 b_i)(x_0)$ which makes sense if b_i is smooth. Therefore, there exists a sequence of smooth sections $\{b_i(x)\}$ in $\mathrm{Hom}(V_{x_0}, V_x)$ uniquely determined by $b_0(x_0) = 1$.

Note that b_is are defined on a coordinate neighborhood O_{x_0} and depend smoothly on the local geometry around x_0. For example, $b_1(x) = \frac{1}{6}k(x) - K(x)$, where k is scalar curvature and K satisfies $\mathcal{D}^2 = \Delta + K$.

Denote $b_i(x)$ by $b_i(x, x_0), x \in O_{x_0}$. Now for any $x_0 \doteq y \in M$, we obtain a formal solution $b_i(x, y)$ which smoothly depends on both x and y for $x \in O_y$. Choose $O' \subset M \times M$ such that $\{(x, x) \mid x \in M\} \subset O' \subset \bigcup_{y \in M} O_y$ and choose

$$\phi(x, y) \in C^\infty(M \times M) \text{ such that } \phi(x, y) = \begin{cases} 1, & (x, y) \in O', \\ 0, & (x, y) \notin \bigcup_{y \in M} O_y. \end{cases}$$

This definition is based on a cutoff function used to define the approximate heat kernel in [5], Definition 2.28.

Definition 6.4. Let (6.7) be the true heat kernel. The *approximating heat kernel* is

$$h_t^n(x, y) = p_t(x, y) \sum_{i=0}^{n} t^i a_i(x, y), \tag{6.10}$$

where $a_i(x, y) = \phi(x, y) b_i(x, y) \in C^\infty(M \times M)$, supported in a neighborhood of the diagonal.

With the previous set up we may state the following lemma, which implies Theorem 6.3 when setting $x = y$.

Lemma 6.5. *Let $k_t(x, y)$ be the heat kernel and $h_t^n(x, y)$ be the one in (6.10). Let $c \in C_c^\infty(M)$ be a cutoff function of the proper cocompact G-manifold M. Choose $\bar{c} \in C_c^\infty(M)$ satisfying $c(x)\bar{c}(x) = c(x)$, $x \in M$. For all $m > 0$, there is N_m such that*

$$\|c(x)h_t^l(x, y)\bar{c}(y) - c(x)k_t(x, y)\bar{c}(y)\| < Ct^m$$

for all $l > N_m$ and $t \in (0, 1]$, where $\|f\| = \sup_{x,y \in M} |f(x, y)|$.

Proof. For all m, let $N_m > \max\{n + 1, m + \frac{n}{2}\}$, where $n = \dim M$. By definition $h_t^{N_m}(x, y)$ approximately satisfies the heat equation in the sense that

$$(\tfrac{\partial}{\partial t} + \mathcal{D}^2)h^{N_m} = t^{N_m} p_t(x, y)\mathcal{D}^2 a_{N_m}(x, y) + O(t^\infty) \doteq r_t(x, y), \tag{6.11}$$

where the first term in (6.11) comes from the calculation of the formal solution. In fact, using (6.8) and (6.9), it follows that $(\frac{\partial}{\partial t} + \mathcal{D}^2)[p_t(x,y)\sum_{j=0}^{N_m} t^j b_j(x,y)] = t^{N_m} p_t(x,y)\mathcal{D}^2 b_{N_m}(x,y)$. $O(t^\infty)$ is of order t^∞ because this term contains the derivatives of ϕ, which are of 0-value for x near y, and $p_t(x,y)$, $x \neq y$, which decreases faster than any positive power t^k as $t \to 0+$. $r_t(x,y)$ has the following properties:

(1) The remainder $r_t(x,y)$ is smooth for any fixed $t > 0$. This is because $p_t(x,y)$ in (6.3) and $a_i(x,y)$s in Definition 6.4 are smooth functions, for all $t > 0$.

(2) Denote the k-th Sobolev norm on $C^m(M \times M)$ by $\|\cdot\|_k$. Then

$$\|c(x)r_t(x,y)\bar{c}(y)\|_k$$

exists for all fixed $t > 0$ and for all k. This is because $c(x)r_t(x,y)\bar{c}(y)$ is smooth and compactly supported on $M \times M$.

(3) We have the estimate

$$\|c(x)r_t(x,y)\bar{c}(y)\|_{\frac{n}{2}+1} < Ct^m$$

uniformly for all $t \in (0,1]$. In fact, in the first term of $c(x)r_t(x,y)\bar{c}(y)$, $c(x)t^{N_m} p_t(x,y)(\mathcal{D}^2 a_{N_m}(x,y))\bar{c}(y)$, only $t^{N_m} p_t(x,y)$ depends on t; it is sufficient to know the order of t in the k-th derivative (in x or y) of $t^{N_m} p_t(x,y)$, where $k \leq \frac{n}{2}+1$. It is $t^{N_m} t^{-\frac{n}{2}} t^{-k} = t^{N_m - \frac{n}{2} - k}$. So

$$\|c(x)t^{N_m} p_t(x,y)(\mathcal{D}^2 a_{N_m}(x,y))\bar{c}(y)\|_{\frac{n}{2}+1} \leq \sum_{k=0}^{\frac{n}{2}+1} c_k t^{N_m - \frac{n}{2} - k}.$$

Since $N_m > n+1$, there are no terms of non-positive order in t on the right-hand side. In addition, since $N_m > \frac{n}{2} + m$, for all $t \in (0,1]$ there is a constant C_1 such that

$$\|c(x)t^{N_m} p_t(x,y)(\mathcal{D}^2 a_{N_m}(x,y))\bar{c}(y)\|_{\frac{n}{2}+1} \leq C_1 t^{N_m - \frac{n}{2}} \leq C_1 t^m.$$

The derivatives of $c(x)O(t^\infty)\bar{c}(y)$ do not have any terms containing negative power of t so $\|c(x)O(t^\infty)\bar{c}(y)\|_{\frac{n}{2}+1} < C_2 t^m$ for all $t \in (0,1]$. So property (3) is proved.

Next we use $r_t(x,y)$ to relate $k_t(x,y)$ and $h_t^{N_m}(x,y)$ in the following claim:
Claim: There is a unique smooth solution for the equation

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{D}^2)u_t(x,y) = r_t(x,y), \\ u_0(x,y) = 0. \end{cases} \tag{6.12}$$

Here $u_t(x,y)$ is regarded as a function of t and x.

In fact, It is trivial to check that $u_1 = \int_0^t e^{-(t-\tau)\mathcal{D}^2} r_\tau(x, x_0)\mathrm{d}\tau$ is smooth and satisfies the equation. If u_2 is another smooth solution, then $u = u_1 - u_2$ satisfies $(\frac{\partial}{\partial t} + \mathcal{D}^2)u = 0, u_0 = u(t = 0) = 0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle u, u\rangle = -\langle u, \mathcal{D}^2 u\rangle - \langle \mathcal{D}^2 u, u\rangle = -2\|\mathcal{D}u\|_{L^2}^2$$

implies that $\|u\|^2$ is non-decreasing in t, and so $\|u(t = 0)\| = 0$ forces $u = u_1 - u_2 = 0$. So the claim is proved.

Since $h_t^{N_m}(x, y) - k_t(x, y)$ is also a solution to the equation (6.12), by the uniqueness of solution we have that $h_t^{N_M}(x, y) - k_t(x, y) = \int_0^t e^{-(t-\tau)\mathcal{D}^2} r_\tau(x, y)\mathrm{d}\tau$. Then for all $t \in (0, 1]$,

$$\begin{aligned}&\|c(x)k_t(x, y)\bar{c}(y) - c(x)h_t^{N_m}(x, y)\bar{c}(y)\|_{\frac{n}{2}+1}\\ &\quad \leq t \sup\{\|c(x)r_\tau(x, y)\bar{c}(y)\|_{\frac{n}{2}+1} | 0 \leq \tau \leq t\} \leq Ct^m,\end{aligned}$$

where the second inequality is because of property (3).

By the Sobolev embedding theorem, for all $p > \frac{n}{2}$, $\|u\| \leq C_0\|u\|_p$ for $u \in H^p$, where $\|\cdot\|$ is the C^0 sup norm and $\|\cdot\|_p$ is the Sobolev p-norm. Therefore,

$$\begin{aligned}&\|c(x)k_t(x, y)\bar{c}(y) - c(x)h_t^{N_m}(x, y)\bar{c}(y)\|\\ &\quad \leq C'\|c(x)k_t(x, y)\bar{c}(y) - c(x)h_t^{N_m}(x, y)\bar{c}(y)\|_{\frac{n}{2}+1} \leq C'Ct^m.\end{aligned}$$

In fact, since $c(x)$ and $\bar{c}(x_0)$ are compactly supported, the function in the norm is supported in a compact set in $M \times M$, where the embedding theorem can be applied. □

Remark 6.6. From (6.4) it follows that

$$\lim_{t\to 0+} c(x) \operatorname{str} k_t(x, x) = \lim_{t\to 0+} c(x)\frac{1}{(4\pi t)^{n/2}}\sum_{j=0}^{l} t^j \operatorname{str} a_j(x)$$

for sufficiently large l. Here, $a_j(x) = a_j(x, x)$. To calculate the left-hand side it is sufficient to investigate a_js on the right-hand side.

If $a \in \operatorname{Hom} V_x$, then a has a decomposition $a = b \otimes c$, $b \in \mathbb{C}\mathrm{l}(T_x^*M)$, $c \in \operatorname{Hom} W$ as in (6.1). The super-trace $\operatorname{str} a$ is then calculated by $\operatorname{str}(b \otimes c) = \tau(b) \cdot \operatorname{str}^{V/S}(c)$ where $\operatorname{str}^{V/S}$ is the super-trace on $\mathbb{C}$-linear endomorphisms of W under the identification $\operatorname{Hom}_{\mathbb{C}\mathrm{l}(T_x^*M)}(V_x) = \operatorname{Hom}_{\mathbb{C}}(W)$ and τ_s is the super-trace on $\operatorname{Hom} S = S \otimes S^* = \mathbb{C}\mathrm{l}(T_x^*M)$. The super-trace τ_s on $\mathbb{C}\mathrm{l}(T_x^*M)$ is explicitly calculated by [5], Proposition 3.21. Let $c = \sum c_{i_1 i_2 \ldots i_k} e^{i_1} e^{i_2} \ldots e^{i_k}$ be an element in $\mathbb{C}\mathrm{l}(T_x^*M) = \operatorname{Hom}(S)$, where $c_{i_1 i_2 \ldots i_k}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, is the coefficient of the element $e^{i_1} e^{i_2} \ldots e^{i_k}$ in $\mathbb{C}\mathrm{l}(T_x^*M)$. Then

$$\tau_s(c) = (-2i)^{\frac{n}{2}} c_{12\ldots n}. \tag{6.13}$$

The Clifford algebra $\mathbb{C}\mathrm{l}(T_x^*M)$ is a filtered algebra, more specifically, $\mathbb{C}\mathrm{l}(T_x^*M) = \mathbb{C}\mathrm{l}(\mathbb{R}^n) = \bigcup_{i=0}^n \mathbb{C}\mathrm{l}_i$. Here $\mathbb{C}\mathrm{l}_i$ is the linear combination of $e^{j_1}\dots e^{j_k}, k \le i$. In proving Theorem 6.3, the following lemma is obtained as a corollary.

Lemma 6.7. *Let $a_i(x)$ be the i-th term in the asymptotic expansion. Then*

$$a_i(x) \in \mathbb{C}\mathrm{l}_{2i} \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{l}(T_x^*M)}(S). \tag{6.14}$$

Proof. We defined $a_i(x) = a_i(x,x)$ to be $b_i(x,x)$. We need to show that $b_i(y,y) \in \mathbb{C}\mathrm{l}_{2i} \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(V_y)$. Put $x = y$ in (6.9). Then

$$b_0(y,y) = 1 \quad \text{and} \quad b_j(y,y) = -\tfrac{1}{j}(\mathcal{D}^2 b_{j-1})(y,y),$$

with $b_0(y,y) = 1 \in \mathbb{C}\mathrm{l}_0 \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{l}}(V_y)$. Inductively, the fact that $\mathcal{D}^2$ contains the factor $\mathrm{c}(e_i)\mathrm{c}(e_j)$, makes sure that the degree of $b_i(x)$ does not increase by more than 2 compared to that of $b_{i-1}(x)$. □

Remark 6.8. As a consequence of (6.13) and (6.14) we have $\mathrm{str}\, a_i(x) = 0$ for $i \le \frac{n}{2}$. Therefore $\mathrm{ind}\,\mathcal{D} = \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum_{i \ge \frac{n}{2}} t^i \int_M c(x)\,\mathrm{str}(a_i(x))\mathrm{d}x$. Furthermore, since the index is independent of t and n is even, we have the following theorem.

Theorem 6.9. *The index of the graded Dirac operator $\mathcal{D}$ is equal to*

$$\mathrm{ind}\,\mathcal{D} = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_M c(x)\,\mathrm{str}(a_{n/2}(x))\mathrm{d}x. \tag{6.15}$$

The element $\mathrm{str}(a_{\frac{n}{2}}(x))$ in (6.15) can be calculated analytically in terms of differential forms on M. To calculate $\mathrm{str}(a_{n/2}(x)) \in \mathrm{Hom}(V_x)$, we localize the operator $\mathcal{D}$ and the heat kernel $k_t(x,y)$ at a point x. Because the local calculation is irrelevant to M being compact or not, we use the classical calculation of $\mathrm{str}\, a_{\frac{n}{2}}$ on a compact manifold without modification. Therefore, $\mathrm{str}(a_{\frac{n}{2}}(x))$ is the n form part of $\det^{\frac{1}{2}}(\frac{R/2}{\sinh R/2})\,\mathrm{tr}^{V/S}(e^{-F})$. For details, please refer to [5], Chapter 4. Finally we obtain the following main theorem of this subsection.

Theorem 6.10. *Let R be the curvature 2-form with respect to the Levi-Civita connection on the manifold (on TM). Then*

$$\mathrm{ind}\,\mathcal{D} = \int_M c(x)\hat{A}(M)\cdot \mathrm{ch}(V/S),$$

where $\hat{A}(M) = \det^{\frac{1}{2}}(\frac{R/4\pi i}{\sinh R/4\pi i})$ is the $\hat{A}$-class of TM and $\mathrm{ch}(V/S) = \mathrm{tr}^{V/S}(e^{-F^{V/S}})$ is the relative Chern character, i.e., Chern character of the twisted curvature $F^{V/S}$ of the bundle S.

6.2. Conclusion. In this subsection we will figure out ind $D_{V(\sigma_P)}$ where D is the Dolbeault operator on ΣX, and where $V(\sigma_P)$ is a bundle over ΣX. $D_{V(\sigma_P)}$ is a generalized Dirac operator and we calculate the case when $\mathcal{D} = D_{V(\sigma_P)}$, $M = \Sigma X$ in the previous subsections. Firstly we have the following proposition, as a corollary to Theorem 6.10.

Proposition 6.11. *Let G be a locally compact unimodular group and let M be proper cocompact G-manifold of dimension n having an almost complex structure, curvature R, a cutoff function $c \in C_c^\infty(M)$ and a G-bundle E with curvature F. Let $D\colon L^2(M, \bigwedge^{0,*} T^*M) \to L^2(M, \bigwedge^{0,*} T^*M)$ be the Dolbeault operator on M. Then the L^2-index of the twisted Dirac operator D_E is*

$$\operatorname{ind} D_E = \int_M c\,\mathrm{Td}(M)\,\mathrm{ch}(E),$$

where $\mathrm{Td}(M) = \det(\frac{R}{1-e^R})$ *and* $\mathrm{ch}(E) = \mathrm{tr}_s(e^{-F})$.

Both $\mathrm{Td}(M)$ and $\mathrm{ch}(E)$ are G-invariant forms. So the integral does not depend on the choice of the cutoff function. If $M = \Sigma X$, then the cutoff function on M can be obtained from the cutoff function on X by setting the values of the elements in the same fiber to be the same. The following index formula is immediate assuming the proposition.

Theorem 6.12. *Let X be a complete Riemannian manifold where a locally compact unimodular group G acts properly, cocompactly and isometrically. If P is a zero order properly supported elliptic pseudo-differential operator, then the L^2 index of P is given by the formula*

$$\operatorname{ind} P = \int_{TX} c(x)(\hat{A}(X))^2\,\mathrm{ch}(\sigma_P).$$

Proof. Set $M = \Sigma X$, $V = V(\sigma_P)$. Clearly, M has an almost complex structure. By Proposition 5.9 and Proposition 6.11,

$$\operatorname{ind} P = \int_{\Sigma X} c(x)\,\mathrm{Td}(\Sigma M)\,\mathrm{ch}(V_{\sigma_P}) = \int_{TX} c(x)\,\mathrm{Td}(TX \otimes \mathbb{C})\,\mathrm{ch}(\sigma_P).$$

The statement follows from $\mathrm{Td}(TX \otimes \mathbb{C}) = (\hat{A}(X))^2$. □

Proof of Proposition 6.11. Let J be an almost complex structure on M. Say x_i, y_i, $1 \le i \le m$, form a local frame of TM and $J(x_i) = y_i$, $J(y_i) = -x_i$. Now J extends $\mathbb{C}$-linearly to $TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$, where $TM^{1,0} = \{v - iJv \mid v \in TM\}$ is the set of holomorphic tangent vectors of the form $z_j \doteq x_j - iy_j$ and $TM^{0,1} = \{v + iJv, v \in TM\}$ is the set of anti-holomorphic tangent vectors of form $\bar{z_j} \doteq x_j + iy_j$. We have real isomorphisms $\pi^{1,0}\colon TM \to TM^{1,0}$, $v \mapsto$

$v^{1,0} = \frac{1}{2}(v - iJv)$, and $\pi^{0,1}\colon TM \to TM^{0,1}$, $v \mapsto v^{0,1} = \frac{1}{2}(v + iJv)$. Therefore $(TM, J) \simeq TM^{1,0} \simeq \overline{TM^{0,1}}$ as an almost complex bundle.

In the same way, the complexified cotangent bundle decomposes into $T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{1,0}$, where $T^*M^{1,0} = \{\eta \in T^*M \otimes \mathbb{C} \mid \eta(Jv) = i\eta(v)\}$, consisting of covectors of form $z^j \doteq x^j + iy^j$, is the $\mathbb{C}$-dual of $TM^{1,0}$ (notation: $x^j(x_i) = \delta_{ij}, y^j(y_i) = \delta_{ij}$) and $T^*M^{0,1} = \{\eta \in T^*M \otimes \mathbb{C} \mid \eta(Jv) = -i\eta(v)\}$, consisting of covectors of form $\bar{z^j} \doteq x^j - iy^j$, is the $\mathbb{C}$-dual of $TM^{0,1}$.

Let Ω^*M be the set of smooth sections of $\bigwedge^* M$, which splits into types (p,q) with $\bigwedge^{p,q} T^*M = (\bigwedge^p T^*M^{1,0}) \otimes (\bigwedge^q T^*M^{0,1})$. If $\alpha \in \Omega^{p,q}(M)$, then the differential decomposes into $\mathrm{d}\alpha = \sum_{i=0}^{p+q+1} (\mathrm{d}\alpha)^{i,p+q+1-i}$ and set $\partial\alpha = (\mathrm{d}\alpha)^{p+1,q}$, $\bar{\partial}\alpha = (\mathrm{d}\alpha)^{p,q+1}$. The Dolbeault operator $\bar{\partial}\colon \Omega^{0,q} \to \Omega^{0,q+1}$ is the order 1 differential operator given by $\bar{\partial} = \frac{\partial}{\partial y} + i\frac{\partial}{\partial x}$ in the local coordinate $(x,y) \in M$. If we incorporate the grading, the Dolbeault operator is $\bar{\partial} + \bar{\partial}^*$ on $\Omega^{0,*}M$.

The Dolbeault operator "is" the canonical Dirac operator on M in the sense that they have the same symbol. The canonical Dirac operator on M is defined as follows. The bundle $S = \bigwedge^{0,*} T^*M$ has an action of the cotangent vectors via Clifford multiplication:

$$\mathrm{c}(\eta)s = \sqrt{2}(\epsilon(\eta^{0,1})(s) - \iota(\eta^{1,0})s), \quad \eta \in T^*M,\ s \in \textstyle\bigwedge^{0,*} T^*M.$$

Here, $\mathrm{c}(x^i) = \frac{1}{\sqrt{2}}(\epsilon(\bar{z}) - \iota(z)), \mathrm{c}(x^i)\mathrm{c}(x^j) + \mathrm{c}(x^j)\mathrm{c}(x^i) = -2\delta_{ij}$ and ϵ is the exterior multiplication and ι is the $\mathbb{C}$-linear compression by a vector.

The canonical Dirac operator is defined to be $D = \sum \mathrm{c}(e^i)\nabla^L_{e_i}$ where $\{e_i\}$ forms a local orthonormal basis of TM and ∇^L is the Levi-Civita connection on S. Now if there is an auxiliary complex G-vector bundle $E \to M$, with a G-invariant Hermitian metric and G-invariant connection ∇^E, the Dolbeault operator D_E acting on $V = S \otimes E$ with coefficients in E can be represented by (up to a lower order term):

$$D_E = \sum \mathrm{c}(e_i)\nabla^V_{e_i}, \quad \text{where } \nabla^V = \nabla^L \otimes 1 + 1 \otimes \nabla^E.$$

Let ∇ be the Levi-Civita connection on M (on $(TM)^{0,1}$, being more precise) and let $R = \nabla^2 \in \bigwedge^2(M, \mathfrak{so}(TM))$ be the Riemannian curvature, the matrix with coefficients of two-forms representing the curvature of M,

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \quad X, Y \in C^\infty(M, TM).$$

In the orthonormal frame e_i of TM, $R(e_i, e_j) = -\sum_{k<l}(R(e_i,e_j)e_l, e_k)e^k \wedge e^l$, where we identify $\mathfrak{so}(TM)$ with the bundle of two-forms on M. Now we have a Clifford module S, where $\mathbb{C}\mathrm{l}(T^*M) = \mathrm{Hom}(S)$, on which T^*M acts by Clifford multiplication. On S there is a Clifford connection ∇^S such that the Clifford multiplication by unit vectors preserves the metric and ∇^S is compatible with the connection on M. Let $R^S = (\nabla^S)^2$ be the curvature associated to ∇^S. It is well known that the

Lie algebra isomorphism $\mathfrak{spin}_n \simeq \mathfrak{so}_n$ given by $\frac{1}{4}[v,w] \mapsto v \wedge w$ implies that

$$R^S(e_i,e_j) = \frac{1}{2}\sum_{k<l}(R(e_i,e_j)e_k,e_l)\mathrm{c}(e_k)\mathrm{c}(e_l) = \frac{1}{4}\sum_{k,l}(R(e_i,e_j)e_k,e_l)\mathrm{c}(e_k)\mathrm{c}(e_l).$$

On S, there is also a Levi-Civita connection, denoted by ∇^L. The associated curvature $R^L = (\nabla^L)^2 \in \mathrm{Hom}\, S$ is written in the form

$$R^L = R^S + F,$$

where $R^S(\cdot,\cdot) = \frac{1}{4}\sum_{k,l}(R(\cdot,\cdot)\bar{z_k},z_l)\mathrm{c}(\bar{z_k})\mathrm{c}(z_l) + \frac{1}{4}\sum_{k,l}(R(\cdot,\cdot)z_k,\bar{z_l})\mathrm{c}(z_k)\mathrm{c}(\bar{z_l}) \in \mathrm{Cl}(TM)$ and $F \in \mathrm{Hom}_{Cl}\, V$ is the twisting curvature.

Recall that the curvature of the Levi-Civita connection on ΛV^* is the derivation of the algebra ΛV^* which coincides with $R(e_i,e_j)$ on V and is given by the formula

$$\sum_{k,l}\langle e^k, R(e_i,e_j)e_l\rangle \epsilon(e_k)\iota(e^l) = \sum_{k,l}(R(e_i,e_j)e_l,e_k)\epsilon(e^k)\iota(e^l).$$

Let R^- be the curvature of the Levi-Civita connection on $T^{0,1}M$. Note that $R = R^-$. Then the curvature of ∇^L on S is given by

$$R^L(\cdot,\cdot) = \frac{1}{4}\sum_{i,j}(R^-(\cdot,\cdot)z_i,\bar{z_j})\epsilon(\bar{z_j})\iota(z_i) = -\frac{1}{8}\sum_{i,j}(R^-(\cdot,\cdot)z_i,\bar{z_j})\mathrm{c}(\bar{z_j})\mathrm{c}(z_i).$$

Using the fact that $\mathrm{c}(z_i)^2 = 0, \mathrm{c}(\bar{z_i})^2 = 0, \mathrm{c}(z_i)\mathrm{c}(\bar{z_j}) + \mathrm{c}(\bar{z_j})\mathrm{c}(z_i) = -4\delta_{ij}$, where $\mathrm{c}(z) = \mathrm{c}(x) + i\,\mathrm{c}(y), \mathrm{c}(\bar{z}) = \mathrm{c}(x) - i\,\mathrm{c}(y)$. We have

$$F^{V/S} = R^L - R^S + F^E = \frac{1}{2}\sum_k (Rz_k,\bar{z_k}) + F^E = \frac{1}{2}\mathrm{Tr}\, R + F^E$$

and a direct calculation shows that

$$\begin{aligned}\hat{A}(M)e^{F^{V/S}} &= \det \frac{R/2}{\sinh R/2} e^{\frac{1}{2}\mathrm{Tr}\, R}(e^{F^E}) = \det \frac{R}{e^R - 1}(e^{F^E}) \\ &= \mathrm{Td}(M)\,\mathrm{Tr}(e^{-F^E}). \qquad \square\end{aligned}$$

The following theorem is an immediate corollary to Theorem 6.12.

Theorem 6.13 (Atiyah's L^2-index theorem). *Let D be an elliptic operator on a compact manifold X and $\widetilde{D}$ be the $\pi_1(X)$-invariant operator defined on the universal cover space $\widetilde{X}$ as the lift of D. Then* $\mathrm{ind}\, \widetilde{D} = \mathrm{ind}\, D$.

6.3. L^2-index theorem for homogeneous spaces of Lie groups. Let G be a unimodular Lie group and H be a compact subgroup. Consider the homogenous space $M = G/H$ of left cosets of H in G, a G-bundle $\bar{E}$ over M and a G-invariant elliptic operator D on $\bar{E}$. The fiber of $\bar{E}$ at eH, denoted by $E = \bar{E}|_{eH}$, is an H-space, so that $\bar{E} = G \times_H E$. Similarly, set $V = T_{eH}M$, then $TM = G \times_H V$. Let $\Omega \in \bigwedge^2(TM)^* \otimes \mathfrak{gl}(TM)$ be the curvature of M, associated to the G-invariant Levi-Civita connection on TM. Then we have the G-invariant $\hat{A}$-class

$$\hat{A}(M) = \det{}^{\frac{1}{2}} \frac{\Omega/4\pi i}{\sinh \Omega/4\pi i}.$$

Let $\Omega^E \in \bigwedge^2(\Sigma M)^* \otimes \mathfrak{gl}(V(\sigma_D))$ be a curvature form associated to some G-invariant connection on $V(\sigma_D)$ over ΣM. Then

$$\operatorname{ch}(\sigma_D) = \operatorname{Tr} e^{\Omega^E}|_{TM}$$

is the Chern character of $V(\sigma_A)$ restricted to TM. Let Ω_V be the curvature tensor Ω restricted to $V = T_{eH}M$ and Ω_V^E be the curvature tensor Ω^E restricted to V. Then we define the corresponding $\hat{A}$-class and Chern character by

$$\hat{A}(M)_V \doteq \det{}^{\frac{1}{2}} \frac{\Omega_V/2}{\sinh \Omega_V/2} \quad \text{and} \quad \operatorname{ch}(\sigma_D)_V \doteq \operatorname{Tr} e^{\Omega_V^E}.$$

We have as a corollary the L^2-index theorem for homogeneous spaces.

Corollary 6.14. *The L^2-index of a G-invariant elliptic operator $D\colon L^2(M, \bar{E}) \to L^2(M, \bar{E})$ is*

$$\operatorname{ind} D = \int_V \hat{A}^2(M)_V \operatorname{ch}(\sigma_D)_V. \tag{6.16}$$

Proof. The L^2-index theorem of D says that

$$\operatorname{ind} D = \int_{TM} c\hat{A}^2(M) \operatorname{ch}(\sigma_D).$$

Since $TM = G \times_H V$, the integration of the form $c\hat{A}^2(M)\operatorname{ch}(\sigma_A)$ on TM can be computed by lifting to an H-invariant form on $G \times V$, by integrating over the group part and then the tangent space at eH. Since $\hat{A}^2(M)\operatorname{ch}(\sigma_D)$ is G-invariant, at any $g \in G$ the form will be the same as its value at the unit e of G: $\hat{A}^2(M)_V \operatorname{ch}(\sigma_A)_V$. Hence,

$$\begin{aligned}\int_{TM} c\hat{A}^2(M)\operatorname{ch}(\sigma_D) &= \int_V \hat{A}^2(M)_V \operatorname{ch}(\sigma_D)_V \int_G c(g^{-1}v)\mathrm{vol} \\ &= \int_V \hat{A}^2(M)_V \operatorname{ch}(\sigma_D)_V,\end{aligned}$$

where vol is the volume form on G. $\square$

Remark 6.15. The formula (6.16) is essentially the L^2-index formula in [9]. We shall compare the two formulas as follows. On the Lie algebra $\mathfrak{g}$ of G there is an H-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an H-invariant complement. $V = T_{eH}(G/H)$ is a candidate for $\mathfrak{m}$. There is a curvature form on $\mathfrak{m}$ defined by $\Theta(X,Y) = -\frac{1}{2}\theta([X,Y]), X, Y \in \mathfrak{m}$ where θ is the connection form given by the projection $\theta\colon \mathfrak{g} \to \mathfrak{h}$. Θ composed with $r\colon \mathfrak{h} \to \mathfrak{gl}(E)$, the differential of a unitary representation of H on some vector space E, is an H-invariant curvature form $\Theta_r(X,Y) = r(\Theta(X,Y))$, $X, Y \in \mathfrak{m}$. Then

$$\mathrm{ch}\colon R(H) \to H^*(g, H), \quad r \mapsto \mathrm{Tr}\, e^{\Theta_r},$$

is a well-defined Chern character ([9] page 309). Also, compose the curvature form (6.15), with $\mathfrak{h} \to \mathfrak{gl}(V)$, the differential of the H-module structure of V. And a curvature form $\Theta_V \in \bigwedge^2 \mathfrak{m}^* \otimes \mathfrak{gl}(V)$ on V is constructed and the $\hat{A}$-class is defined as

$$\hat{A}(\mathfrak{g}, H) = \det{}^{\frac{1}{2}} \frac{\Theta_V/2}{\sinh \Theta_V/2}.$$

The L^2-index formula of D in [9] is

$$\mathrm{ind}\, D = \int_V \mathrm{ch}(a)\hat{A}(\mathfrak{g}, H), \tag{6.17}$$

where a is an element of the representation ring $R(H)$, specifically a is the pre-image of $V(\sigma_D)|_{V^+}$ under the Thom isomorphism $R(H) \to K_H(V)$. Here, V^+ is the space built from V by adding one point at infinity. It is the ball fiber in ΣM at eH. Note that the Thom isomorphism exists only for the case when the action of H on V, lifts to $\mathrm{Spin}(V)$. The general case was done by introducing a double covering of H and by reducing the problem to this situation [9], p. 307.

To see that (6.16) and (6.17) are the same formula, we prove the following assertions.

(1) $\hat{A}(M)_V = \hat{A}(\mathfrak{g}, H)$.

In fact, since $TM = G \times_H V$ is a principal G-bundle over V/H and V is a principal H-bundle over V/H by [19], II, Prop. 6.4, the connection form on TM restricted to V is also a connection form. Also, on G/H the restriction of any G-invariant tensor on TM to V is an H-invariant tensor on V. Therefore Ω_V is an H-invariant curvature form on V and the restriction $\hat{A}(M)_V$ is the $\hat{A}$-class defined by curvature Ω_V. By definition $\hat{A}(\mathfrak{g}, H)$ is the $\hat{A}$-class of the curvature Θ_V on V, $\hat{A}$-class of another connection on the same V. The statement is proved because $\hat{A}$ is a topological invariant and is independent of the choice of connection on V.

(2) $\mathrm{ch}(\sigma_D)_V = \mathrm{ch}(a)$.

Similarly to the last proof, Ω_V^E is an H-invariant curvature form of $V(\sigma_D)|_{V^+}$ restricted to V. Recall that $V(\sigma_D)$ is glued by the G-invariant symbol σ_D and therefore it is determined by its restriction at the ball fiber, V^+. By definition $V(\sigma_D)|_{V^+}$ is glued two copies of $BV \times E$ on the boundary by $\sigma_D|_{SV}$. Note that the evaluation of $\sigma_D|_{SV}$ at $\xi \in SV$ is $\sigma_D(eH, \xi) \in \mathrm{GL}(E)$, $\xi \in V$, $\|\xi\| = 1$. We have an H-bundle $V(\sigma_D)|_V = V \times_H E$ where a is the associated representation of H in E. Hence the curvature Ω_V^E is a composed with some curvature form on V. The statement follows from the fact that $\mathrm{ch}(a)$ is independent of the connection and the choice of the H-invariant splitting of G.

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