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Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations

Abstract. We study uniformly elliptic fully nonlinear equations

$$F(D^2u, Du, u, x) = 0,$$

and prove results of Gidas–Ni–Nirenberg type for positive viscosity solutions of such equations. We show that symmetries of the equation and the domain are reflected by the solution, both in bounded and unbounded domains.

Keywords. Symmetry, fully nonlinear equations, viscosity solutions

1. Introduction

In this note we establish symmetry results of Gidas–Ni–Nirenberg type for viscosity solutions of the equation

$$\begin{cases} F(D^2u, Du, u, x) = 0 & \text{in } \mathcal{O}, \\ u > 0 & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O}$ is a domain in $\mathbb{R}^n$, F is a continuous function defined on $\mathcal{S}^n(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \times \overline{\mathcal{O}}$ with values in $\mathbb{R}$ and $\mathcal{S}^n(\mathbb{R})$ denotes the space of real, $n \times n$, symmetric matrices. More precise assumptions on F are given later on. The solution u of this nonlinear problem is scalar and Du, D^2u denote respectively the gradient and the Hessian matrix of u.

A model problem for (1.1) will be the equations

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + f(u) = 0, \qquad \mathcal{M}^-_{\lambda,\Lambda}(D^2u) + f(u) = 0, \tag{1.2}$$

where f is a locally Lipschitz continuous function and $\mathcal{M}^+_{\lambda,\Lambda}, \mathcal{M}^-_{\lambda,\Lambda}$ are the extremal Pucci operators ([17], [10]), with parameters $0 < \lambda \leq \Lambda$, defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \qquad \mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

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for any symmetric $n \times n$ matrix M. Here $e_i = e_i(M), i = 1, \ldots, n$, denote the eigenvalues of M. Pucci's operators are extremal in the sense that $\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM)$, $\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM)$, where $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of all symmetric matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$.

Let us recall the classical result of Gidas, Ni and Nirenberg [15], which states that positive C^2-solutions of the Dirichlet problem for the equation

$$\Delta u + f(u) = 0, \qquad f \in C^{0,1}(\mathbb{R}), \tag{1.3}$$

(this is equation (1.2) corresponding to $\lambda = \Lambda = 1$) in a ball are necessarily radial, or more generally, if the domain is symmetric and convex with respect to a hyperplane then the solutions have the same symmetry. Related results for (1.3) in the whole space and exterior domains were obtained by C. Li [16], W. Reichel [22], and B. Sirakov [24], under the supplementary hypothesis that f is nonincreasing in a right neighbourhood of zero.

Symmetry results in the spirit of Gidas, Ni and Nirenberg for classical solutions of fully nonlinear equations of type (1.1) were obtained by C. Li [16]. Extensions and simple proofs of these results are due to Berestycki and Nirenberg [7].

An essential hypothesis in [7] is that the operator F is C^1 in the matrix of the second derivatives of $u \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$. This prevents applying these results to important classes of equations, such as equations involving Pucci's operators, Bellman or Isaacs equations. On the other hand, the symmetry result was proved for viscosity solutions of (1.1), without differentiability assumption on F, by Badiale [2] (see also Badiale–Bardi [3] for results on general first order equations), under the hypothesis that the operator F satisfies a comparison principle. This is a quite strong assumption, which essentially requires that the operator F is nonincreasing with respect to the u variable or at least convex in the (Du, D^2u) variables (cf. Section 5 in [13]).

It is our purpose here to join together and extend the above quoted results. We are going to show that the moving planes method of Alexandrov [1] and Serrin [23], in its version developed in [7], can actually be adapted to work in the setting of viscosity solutions and general equations (1.1).

Before proceeding to the precise statements, let us recall that existence and uniqueness of viscosity solutions of boundary value problems of type (1.1) has been very extensively studied for proper operators, that is, when F is nonincreasing in u: see for example [13], [11], [12]. Quite recently existence of solutions of nonproper equations of type (1.2) was established by Felmer–Quaas [14] and Quaas–Sirakov [19]. In particular, in [14] it was shown that the Dirichlet problem for (1.2) in a ball has a positive radial solution when f has some power growth at infinity. It follows from our result that actually any positive solution is radial. It is well known that proving symmetry for solutions is an important step towards proving uniqueness in the nonproper case.

Next we list our assumptions on the nonlinearity F.

(H1) *(Regularity)* For all $R > 0$ there exists a constant $K_R > 0$ and a function $\omega_R : \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega_R(0^+) = 0$ such that, for any $x, y \in \overline{\mathcal{O}}, p, q \in \mathbb{R}^n, M, N \in \mathcal{S}^n(\mathbb{R}), u_1, u_2 \in [-R, R]$,

$$|F(M, p, u_1, x) - F(N, q, u_2, y)| \leq K_R\{|p - q| + \|M - N\|$$
$$+ |u_1 - u_2| + |x - y|(\|M\| + \|N\|)\}$$
$$+ \omega_R(|x - y|(1 + |p| + |q|)).$$

(H2) *(Uniform ellipticity)* There exists $\kappa > 0$ such that, for any $x \in \overline{\mathcal{O}}, u \in \mathbb{R}, p \in \mathbb{R}^n,$ $M, N \in \mathcal{S}^n(\mathbb{R})$ with $N \geq 0$,

$$F(M + N, p, u, x) - F(M, p, u, x) \geq \kappa \operatorname{tr}(N) .$$

It is standard to show that $(H1)$ and $(H2)$ imply the following assumption (actually, $(H1)$–$(H2)$ reduce to $(H3)$ when F is independent of x):

(H3) For any $R > 0$ there exists a constant $K_R > 0$ such that for each $M, N \in \mathcal{S}^n(\mathbb{R})$, $p, q \in \mathbb{R}^n, x \in \overline{\mathcal{O}}, u, v \in [-R, R]$, we have

$$F(M, p, u, x) - F(N, q, v, x) \geq \mathcal{M}^-_{\lambda,\Lambda}(M - N) - K_R(|p - q| + |u - v|),$$
$$F(M, p, u, x) - F(N, q, v, x) \leq \mathcal{M}^+_{\lambda,\Lambda}(M - N) + K_R(|p - q| + |u - v|),$$

with λ, Λ depending on K_R and κ.

Another example we have in mind is the standard quasilinear equation

$$\operatorname{tr}[B(x)D^2u] + H(x, u, Du) = 0 \quad \text{in } \mathcal{O}, \tag{1.4}$$

where B is an $n \times n$ real symmetric matrix and H a continuous function. In this case $(H2)$ is satisfied if $B(x) \geq \kappa \operatorname{Id}$ for all $x \in \mathcal{O}$, and $(H1)$ is satisfied if

 (i) B is a bounded and locally Lipschitz continuous function of x;
(ii) the function H satisfies: for any $R > 0$, there are a constant $K_R > 0$ and a function $\omega_R : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega_R(0^+) = 0$ such that, for any $x, y \in \overline{\mathcal{O}}, p, q \in \mathbb{R}^n, u, v \in [-R, R]$,

$$|H(x, u_1, p) - H(y, u_2, q)| \leq \omega_R(|x - y|(1 + |p| + |q|)) + K_R(|p - q| + |u - v|).$$

For any matrix $M = (m_{ij}) \in \mathcal{S}^n(\mathbb{R})$ we denote by $M^{(k)}$ the matrix obtained from M by replacing m_{ik} and m_{kj} by $-m_{ik}$ and $-m_{kj}$ respectively, for any $i \neq k, j \neq k$. Note that M and $M^{(k)}$ always have the same eigenvalues.

For any vector $p \in \mathbb{R}^n$ we write $p^{(k)} = (p_1, \ldots, p_{k-1}, -p_k, p_{k+1}, \ldots, p_N)$. We consider the following hypothesis.

(O_k) $\mathcal{O}$ is convex in the direction x_k, symmetric with respect to the hyperplane $\{x_k = 0\}$, and for all $M \in \mathcal{S}^n(\mathbb{R}), p \in \mathbb{R}^n, u \in (0, \infty), x \in \mathcal{O}$,

$$F(M, p, u, x) = F(M^{(k)}, p^{(k)}, u, x^{(k)}),$$

and F is nonincreasing in x_k for $x_k > 0$.

Then we have the following theorem.

Theorem 1.1. *Suppose $\mathcal{O} \subset \mathbb{R}^n$ is bounded, and assume (H1), (H2), and (O_k) for some $k \in \{1, \ldots, N\}$. Let $u \in C(\overline{\mathcal{O}})$ be a viscosity solution of (1.1). Then u is symmetric in x_k, that is, $u(x) = u(x^{(k)})$ for all $x \in \mathcal{O}$. In addition, u is strictly decreasing in $x_k > 0$.*

Corollary 1.1. *Suppose $\mathcal{O}$ is a ball centred at the origin, F is radial in x and nonincreasing in $|x|$, and satisfies (H1), (H2), and (O_k) for all $k \in \{1, \ldots, N\}$. Then every viscosity solution of (1.1) is radial and strictly decreasing in $|x|$.*

Next, we turn to symmetry in unbounded domains for autonomous equations.

In [4] Badiale and Bardi showed that positive solutions of a large class of (not necessarily uniformly) elliptic equations in $\mathbb{R}^n$ or exterior domains are asymptotically radial, that is, level sets of the solutions approach spheres as $|x|$ goes to infinity. The following theorems can be seen as completion of these results for uniformly elliptic equations and symmetric domains, for which we can show that all level sets are spheres.

Theorem 1.2. *Assume (O_k) for all $k \in \{1, \ldots, N\}$ and suppose F does not depend on x, satisfies (H1)–(H2) and is nonincreasing in $u \in [0, \delta)$ for some $\delta > 0$. Let $u \in C(\mathbb{R}^n)$ be a viscosity solution of*

$$
\begin{cases}
F(D^2u, Du, u) = 0 & in\ \mathbb{R}^n, \\
u > 0 & in\ \mathbb{R}^n, \\
u \to 0 & as\ |x| \to \infty.
\end{cases}
\tag{1.5}
$$

Then u is radial and strictly decreasing in $|x|$.

The proof of Theorem 1.2 is a particular case of the proof of the following more general result on symmetry in exterior domains.

Theorem 1.3. *Suppose $\mathcal{O} = \mathbb{R}^n \setminus B$ for some ball B, F does not depend on x, satisfies (H1)–(H2) and (O_k) for all $k \in \{1, \ldots, N\}$, and F is nonincreasing in $u \in [0, \delta)$ for some $\delta > 0$. Let $u \in C(\overline{\mathcal{O}})$ be a viscosity solution of*

$$
\begin{cases}
F(D^2u, Du, u) = 0 & in\ \mathcal{O}, \\
u > 0 & in\ \mathcal{O}, \\
u = a & on\ \partial B, \\
u \to 0 & as\ |x| \to \infty,
\end{cases}
\tag{1.6}
$$

for some $a > 0$. Suppose in addition that for all $x \in \partial B$ and for all directions $v \in \mathbb{R}^n \setminus \{0\}$ such that $v \cdot n(x) > 0$, where $n(x) = x/|x|$, we have

$$
\limsup_{t \searrow 0} \frac{u(x + tv) - u(x)}{t} \leq 0.
\tag{1.7}
$$

Then u is radial and strictly decreasing in $|x|$.

2. Proofs

We first show that under $(H1)$–$(H2)$ the difference of a lower semicontinuous superso-lution and an upper semicontinuous subsolution of (1.1) is a supersolution of an equation involving a positively homogeneous uniformly elliptic operator.

In the following we will denote by $BUSC(\overline{\mathcal{O}})$ and $BLSC(\overline{\mathcal{O}})$ respectively the set of bounded upper and lower semicontinuous functions in $\overline{\mathcal{O}}$.

Proposition 2.1. *Assume that F satisfies $(H1)$–$(H2)$. Let $u_1 \in BUSC(\overline{\mathcal{O}})$ and $u_2 \in BLSC(\overline{\mathcal{O}})$ be respectively a viscosity subsolution and a viscosity supersolution of $F(D^2u, Du, u, x) = 0$ in $\mathcal{O}$. Then there exist positive constants λ, Λ, b (depending on the L^∞-norms of u_1 and u_2) and a bounded function $c(x)$ (whose L^∞-norm depends only on the L^∞-norms of u_1 and u_2, and on the local Lipschitz norm of F with respect to u on the ranges of u_1, u_2) such that the function $w = u_2 - u_1$ is a viscosity* supersolution *of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2w) - b|Dw| + c(x)w = 0 \quad in \ \mathcal{O}. \tag{2.8}$$

If in addition F is nonincreasing with respect to u then $c \leq 0$ in $\mathcal{O}$.

We first make some comments. The strategy of the proof of Proposition 2.1 is similar to the one of the comparison principle for fully nonlinear operators (see [13]). A difficulty comes from $(H1)$–$(H2)$ and can be seen on a term like $\mathrm{tr}(B(x)D^2u)$ (in the case of quasilinear equations): in general, one assumes that B has the form $B = \sigma\sigma^T$ for some Lipschitz continuous matrix σ and the uniqueness proof uses σ in an essential way, both in the degenerate and nondegenerate case. Here we want just to assume B to be elliptic and Lipschitz continuous and we do not want to use σ. To this end we use Lemma 2.2 of Barles and Ramaswamy [6], which we recall within the proof of Proposition 2.1.

Proof of Proposition 2.1. Let $\phi \in C^2(\overline{\mathcal{O}})$ and $\bar{x} \in \mathcal{O}$ be a local minimum of $w - \phi$, say in $\overline{B(\bar{x}, r)}$ for some $r > 0$.

For all $\varepsilon > 0$, we introduce the auxiliary function

$$\Phi_\varepsilon(x, y) = u_2(x) - u_1(y) - \phi(x) + |x - y|^2/\varepsilon^2 + |x - \bar{x}|^4. \tag{2.9}$$

Let $(x_\varepsilon, y_\varepsilon)$ be the minimum point of $\Phi_\varepsilon(x, y)$ in $\overline{B(\bar{x}, r)} \times \overline{B(\bar{x}, r)}$. Since $\bar{x}$ is a strict local minimum point of $x \mapsto w(x) - \phi(x) + |x - \bar{x}|^4$, standard arguments show that

$$(x_\varepsilon, y_\varepsilon) \to (\bar{x}, \bar{x}) \quad \text{and} \quad |x_\varepsilon - y_\varepsilon|^2/\varepsilon^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, if $\zeta_\varepsilon(x, y) := \phi(x) - |x - y|^2/\varepsilon^2 - |x - \bar{x}|^4$, we know that (cf. [13]), for every $\alpha > 0$, there exist $X, Y \in \mathcal{S}^n(\mathbb{R})$ such that

$$(D_x\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), X) \in \overline{J}^{2,-}_{\overline{\mathcal{O}}} u_2(x_\varepsilon), \quad (-D_y\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), Y) \in \overline{J}^{2,+}_{\overline{\mathcal{O}}} u_1(y_\varepsilon),$$

$$-(1/\alpha + \|D^2\zeta_\varepsilon(x_\varepsilon, y_\varepsilon)\|)\,\mathrm{Id} \leq \begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \leq -(\mathrm{Id} - \alpha D^2\zeta_\varepsilon(x_\varepsilon, y_\varepsilon))D^2\zeta_\varepsilon(x_\varepsilon, y_\varepsilon),$$

and

$$F(X, D_x\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), u_2(x_\varepsilon), x_\varepsilon) \leq 0,$$
$$F(Y, -D_y\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), u_1(y_\varepsilon), y_\varepsilon) \geq 0.$$

The key point is to estimate

$$F(X, D_x\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), u_2(x_\varepsilon), x_\varepsilon) - F(Y, -D_y\zeta_\varepsilon(x_\varepsilon, y_\varepsilon), u_1(y_\varepsilon), y_\varepsilon).$$

To this end we first choose $\alpha = \varepsilon^2$ and we use Lemma 2.2 in [6] which says the following : if the matrices X, Y satisfy

$$-\frac{\tilde{K}}{\varepsilon^2} \, \mathrm{Id} \le \begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \le \frac{\tilde{K}}{\varepsilon^2} \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} \qquad (2.10)$$

then

$$Y - X \le -\frac{\tilde{K}\varepsilon^2}{6}(tX + (1-t)Y)^2 \quad \text{for all } t \in [0, 1].$$

A slight modification of the arguments in [6] allows one to take into account the $D^2\phi$ and $D^2(|x - \bar{x}|^4)$ terms and yields

$$Y - X' \le -\frac{\tilde{K}\varepsilon^2}{6}(tX' + (1-t)Y)^2 + O(\varepsilon) \quad \text{as } \varepsilon \to 0, \qquad (2.11)$$

for all $t \in [0, 1]$, where $X' = X + \|D^2\phi\| \, \mathrm{Id} + O(|x - \bar{x}|^2)$.

Now we are ready to estimate $F(X, p, u_2, x) - F(Y, q, u_1, y)$ with $p = D_x\zeta_\varepsilon(x_\varepsilon, y_\varepsilon)$ and $q = -D_y\zeta_\varepsilon(x_\varepsilon, y_\varepsilon)$. By using $(H1)$–$(H2)$ together with the inequality (2.11) for $t = 0$, we get

$$
\begin{aligned}
0 \ge\ & F(X, p, u_2, x) - F(Y, q, u_1, y) \\
\ge\ & F(X', p, u_1, x) - F(Y, p, u_1, x) \\
& - K_R(|p - q| + |x - y|(|p| + |q| + \|Y\|)) + \omega_R(|x - y|(1 + |p| + |q|)) \\
& + \mathcal{M}^-_{\lambda,\Lambda}(\|D^2\phi\| \, \mathrm{Id} + O(|x - \bar{x}|^2)) + F(X, p, u_2, x) - F(X, p, u_1, x) \\
\ge\ & \kappa \frac{\tilde{K}\varepsilon^2}{6} \mathrm{tr}(Y^2) + O(\varepsilon) + \mathcal{M}^-_{\lambda,\Lambda}(\|D^2\phi\| \, \mathrm{Id} + O(|x - \bar{x}|^2)) \\
& - K_R(|p - q| + |x - y|(|p| + |q| + \|Y\|)) + \omega_R(|x - y|(1 + |p| + |q|)) \\
& + F(X, p, u_2, x) - F(X, p, u_1, x) \,,
\end{aligned}
$$

where $R = \max(\|u_1\|_\infty, \|u_2\|_\infty)$. In the last inequality, the "bad" term is $K_R|x - y| \, \|Y\|$ since the estimates on the test function ϕ do not ensure that this term converges to 0. However, this term is controlled by the "good" term $\mathrm{tr}(Y^2)$ in the following way: by Cauchy–Schwarz's inequality,

$$K|x - y| \, \|Y\| \ge -\kappa \frac{\tilde{K}\varepsilon^2}{6} \mathrm{tr}(Y^2) - O(|x - y|^2/\varepsilon^2).$$

And this estimate is now sufficient since we know that $|x - y|^2/\varepsilon^2 \to 0$ as $\varepsilon \to 0$.

Thus by letting $\varepsilon \to 0$ we are led to

$$\mathcal{M}^-(D^2\phi(\bar{x})) - b|D\phi(\bar{x})| + c(\bar{x})w(\bar{x}) \le 0 \,,$$

where $b = K_R$ and

$$c(\bar{x}) = \begin{cases} \dfrac{F(X, p, u_2(\bar{x}), \bar{x}) - F(X, p, u_1(\bar{x}), \bar{x})}{u_2(\bar{x}) - u_1(\bar{x})} & \text{if } u_2(\bar{x}) \neq u_1(\bar{x}), \\ 0 & \text{otherwise,} \end{cases}$$

so the conclusion follows. □

A fundamental tool in the theory of strong and viscosity solutions of elliptic equations is the Alexandrov–Bakelman–Pucci (ABP) estimate, a version of which we quote next.

Proposition 2.2. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain and let $w \in C(\overline{\mathcal{O}})$ be a viscosity solution of*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - b|Dw| + c(x)w \leq f(x), \tag{2.12}$$

where $b \in \mathbb{R}$ and $c, f \in L^\infty(\overline{\mathcal{O}})$. Suppose $c(x) \leq 0$ in $\mathcal{O}$. Then there exists a constant C_*, depending only on λ, Λ, $|b|$, and $\mathrm{diam}(\mathcal{O})$, such that

$$\sup_{\mathcal{O}}(-w) \leq \sup_{\partial\mathcal{O}} w^- + C_* \|f\|_{L^n(\mathcal{O})}.$$

Proof. This is a consequence of Proposition 2.12 in [11] (the proof of which is due to Trudinger). To link the notations in this paper with those in [11] we note that $\mathcal{M}^+_{\lambda,\Lambda}(X) = -\mathcal{P}^-(X)$ and $\mathcal{M}^-_{\lambda,\Lambda}(X) = -\mathcal{P}^+(X)$, where

$$\mathcal{P}^-(X) = -\Lambda \, \mathrm{tr}(X^+) + \lambda \, \mathrm{tr}(X^-),$$
$$\mathcal{P}^+(X) = -\lambda \, \mathrm{tr}(X^+) + \lambda \, \mathrm{tr}(X^-),$$

and X^+, X^- denote the positive and negative parts of $X \in \mathcal{S}^n(\mathbb{R})$. □

It is trivial to deduce a maximum principle in small domains from the ABP inequality.

Proposition 2.3. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain and suppose $b \geq 0$ and $c(x) \in L^\infty(\overline{\mathcal{O}})$. There exists a constant $r > 0$, depending on λ, Λ, b, $\mathrm{diam}(\mathcal{O})$, $\|c\|_{L^\infty(\overline{\mathcal{O}})}$, such that any viscosity solution $w \in C(\overline{\mathcal{O}})$ of*

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - b|Dw| + c(x)w \leq 0 & \text{in } \mathcal{O}, \\ w \geq 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{2.13}$$

is nonnegative in $\mathcal{O}$, provided $|\mathcal{O}| \leq r$.

Proof. We apply Proposition 2.2 to

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - b|Dw| - c^-(x)w \leq -c^+ w \leq c^+ w^-$$

and get

$$\sup_{\mathcal{O}} w^- \leq C_* \|c^+\|_{L^\infty(\mathcal{O})} \|w^-\|_{L^\infty(\mathcal{O})} |\mathcal{O}|^{1/n},$$

from which the result follows. □

The following regularity result is a variation of the $C^{1,\alpha}$-regularity result proven by Caffarelli [9] in the case when F depends only on D^2u and by Święch [25] and Wang [26] in the general case of a proper equation, i.e. when F is nonincreasing with respect to u.

Proposition 2.4. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain and assume that F satisfies $(H1)$–$(H2)$. Let $u \in C(\overline{\mathcal{O}})$ be a viscosity solution of*

$$F(D^2u, Du, u, x) = 0 \quad in \ \mathcal{O}.$$

Then $u \in C^{1,\alpha}_{\text{loc}}(\mathcal{O})$ for some $\alpha \in (0, 1)$.

Proof. This theorem was proven in [25] under the supplementary assumptions that the constant K_R in $(H1)$–$(H2)$ is independent of R and that F is nonincreasing in u.

In order to reduce to the result of [25], suppose that the solution satisfies $|u| \le K$. We set $F_0 = F$ if $|u| \le K$, $F_0 = F(M, p, K, x)$ if $u > K$, and $F_0 = F(M, p, -K, x)$ if $u < -K$. Then F_0 is globally Lipschitz with respect to u. We notice that u is also a solution of $F_0(D^2u, Du, u, x) - ku = -ku \in C(\overline{\mathcal{O}})$. If k is large enough the operator $F_0 - ku$ is proper, thus we can apply the result in [25] to conclude the proof. $\square$

The next proposition asserts the existence of a principal eigenvalue and a principal eigenfunction for an operator without zero order term. We shall also use the fact that the principal eigenvalue goes to infinity as the measure of the domain goes to zero.

Proposition 2.5. *Suppose $\mathcal{O}$ is a bounded smooth domain. Then there exists a number $\lambda_1 = \lambda_1(\mathcal{O}) > 0$ and a function $\varphi_1 \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ which satisfy*

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2\varphi_1) - b|D\varphi_1| + \lambda_1\varphi_1 = 0 & in \ \mathcal{O}, \\ \varphi_1 > 0 & in \ \mathcal{O}, \\ \varphi_1 = 0 & on \ \partial\mathcal{O}. \end{cases}$$

In addition, we have $\lambda_1(\mathcal{O}) \to \infty$ as $|\mathcal{O}| \to 0$.

Proof. In the case $b = 0$ this result was proved by Quaas in [18] (see also [8]). Essentially the same proof works for any b, since, by the known existence, uniqueness and regularity results for fully nonlinear operators (see [11], [25]) the operator $\mathcal{M}^-_{\lambda,\Lambda}(D^2\cdot) - b|D \cdot|$ has the same properties as $\mathcal{M}^-_{\lambda,\Lambda}(D^2\cdot)$, namely, its inverse exists and is positivity preserving. Thus Krein–Rutman theory applies to the former just as to the latter. $\square$

We note that in a very recent work [20], [21], the authors established the existence of a principal eigenvalue and eigenfunction for any positively homogeneous operator, convex or concave in D^2u and satisfying a condition of type $(H3)$. In addition, in [21] a multitude of properties of these objects are proven, Proposition 2.5 being a very particular case of these.

We shall make use of a simple lemma, concerning products of viscosity solutions and test functions.

Lemma 2.1. *Let $u \in C(\mathcal{O})$, $u \geq 0$, satisfy*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) - b(x)|Du| + c(x)u \leq f(x) \quad \text{in } \mathcal{O}, \tag{2.14}$$

where $b, c, f \in L^\infty(\mathcal{O})$. Suppose $\psi \in C^2(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ is strictly positive in $\overline{\mathcal{O}}$. Then $\overline{u} = u/\psi$ satisfies the inequality

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\overline{u}) - \overline{b}(x)|D\overline{u}| + \overline{c}(x)\overline{u} \leq \overline{f}(x), \tag{2.15}$$

where

$$\overline{b} = b + 2\Lambda\sqrt{n}\psi^{-1}|D\psi|, \quad \overline{c}(x) = \frac{\mathcal{M}^-_{\lambda,\Lambda}(D^2\psi) - b|D\psi|}{\psi} + c(x), \quad \overline{f} = \frac{f}{\psi}.$$

Proof. Suppose $u \in C^2(\mathcal{O})$, so that $\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) - b(x)|Du| + c(x)u \leq f(x)$ is satisfied in the classical sense. We have

$$Du = \psi D\overline{u} + \overline{u}D\psi, \quad D^2 u = \psi D^2\overline{u} + 2D\psi \otimes D\overline{u} + \overline{u}D^2\psi. \tag{2.16}$$

Here and below, $\otimes$ denotes the symmetric tensor product, i.e. if $p, q \in \mathbb{R}^n$ then $p \otimes q = \frac{1}{2}(p_i q_j + p_j q_i)_{i,j}$. By putting (2.16) into (2.14) and by using

$$\mathcal{M}^-_{\lambda,\Lambda}(M + N) \geq \mathcal{M}^-_{\lambda,\Lambda}(M) + \mathcal{M}^-_{\lambda,\Lambda}(N), \quad \mathcal{M}^-_{\lambda,\Lambda}(\eta M) = \eta\mathcal{M}^-_{\lambda,\Lambda}(M),$$

for $\eta \geq 0$, we obtain the statement of the lemma. Note that $\operatorname{tr}(A(p \otimes q)) \leq |A|\,|p \otimes q| \leq \sqrt{n}\Lambda|p|\,|q|$, where A is a matrix whose eigenvalues lie in $[\lambda, \Lambda]$, and $|A| := \sqrt{\operatorname{tr}(A^t A)}$.

Let u be only continuous. If (2.15) does not hold then there exist $x_0 \in \mathcal{O}$ and $\phi \in C^2(\mathcal{O})$ such that $\phi(x_0) = \overline{u}(x_0)$, $\phi \leq \overline{u}$ in $\mathcal{O}$ and

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x_0)) - \overline{b}(x_0)|D\phi(x_0)| + \overline{c}(x_0)\phi(x_0) > \overline{f}(x_0). \tag{2.17}$$

An easy computation then shows that this implies

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2(\phi\psi)(x_0)) - b|D(\phi\psi)(x_0)| + c(x_0)\phi(x_0)\psi(x_0) > f(x_0).$$

This contradicts (2.14), since $\phi\psi \in C^2(\mathcal{O})$ is a test function such that $\phi(x_0)\psi(x_0) = u(x_0)$ and $\phi\psi \leq u$ in $\mathcal{O}$. $\qquad\square$

Next, we state a strong maximum principle for nonproper operators. In the literature there are more general results in the proper case, for example, a weak Harnack inequality is proven in [26], while in [5] a strong maximum principle is proven for degenerate operators.

Proposition 2.6. *Let $\mathcal{O} \subset \mathbb{R}^n$ be a smooth domain and let $b, c \in L^\infty(\overline{\mathcal{O}})$. Suppose $w \in C(\overline{\mathcal{O}})$ is a viscosity solution of*

$$\begin{cases} \mathcal{M}^-_{\lambda,\Lambda}(D^2 w) - b(x)|Dw| + c(x)w \leq 0 & \text{in } \mathcal{O}, \\ w \geq 0 & \text{in } \mathcal{O}. \end{cases} \tag{2.18}$$

Then either $w \equiv 0$ in $\mathcal{O}$ or $w > 0$ in $\mathcal{O}$ and at any point $x_0 \in \partial\mathcal{O}$ at which $w(x_0) = 0$ we have

$$\liminf_{t \searrow 0} \frac{w(x_0 + t\nu) - w(x_0)}{t} > 0,$$

where $\nu \in \mathbb{R}^n \setminus 0$ is such that $\nu \cdot n(x) < 0$; here $n(x)$ denotes the exterior normal to $\partial\mathcal{O}$ at x_0.

Proof. If $c(x) \leq 0$ in $\mathcal{O}$ this follows from Theorems 1 and 2 in [5]. Note that in that paper the operator F is supposed to be continuous in x, but the arguments there are very easy to adapt to an operator like the one in (2.18), in which the first order and zero order coefficients are only measurable and bounded.

By using Proposition 2.5 and Lemma 2.1 we can show that we can always reduce the problem to a proper one, that is, to a problem in which $c(x) \leq 0$. Indeed, suppose $x_0 \in \mathcal{O}$ is a point at which $w(x_0) = 0$. By Proposition 2.5 there exists a sufficiently small ball B around x_0 such that the first eigenvalue of $\mathcal{M}_{\lambda,\Lambda}^-(D^2 \cdot) - b|D \cdot|$ in this ball is larger than the L^∞-norm of c. By setting $\psi = \varphi_1$ (φ_1 is the first eigenfunction of this operator in B, see Proposition 2.5) and by applying Lemma 2.1 in a smaller concentric ball $B_1 \subset B$ we obtain a proper equation for u/ψ in B_1. Then from the result in [5] it follows that $u \equiv 0$ in the small ball. This means each point in $\mathcal{O}$ at which u vanishes has a neighbourhood in which u is identically zero, so u vanishes everywhere in $\mathcal{O}$. We argue in a similar way if $x_0 \in \partial\mathcal{O}$ and we conclude the proof. $\square$

We are now ready to show that the arguments of [7] adapt to our setting, and permit us to prove Theorem 1.1, and, in a similar manner, that the arguments from [24] can be used to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.1. The following argument, due to Berestycki and Nirenberg, is given here for completeness. Suppose $\mathcal{O}$ is convex in the direction of the vector $e_1 = (1, 0, \ldots, 0)$ and is symmetric with respect to the hyperplane $T_0 = \{x \mid x_1 = 0\}$. We want to show that

$$u(-x_1, x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N) \quad \text{for any } x \in \mathcal{O}.$$

For any $\lambda \in \mathbb{R}$ we define

$$
\begin{aligned}
T_\lambda &= \{x \mid x_1 = \lambda\}, \quad D_\lambda = \{x \mid x_1 > \lambda\}, \quad \Sigma_\lambda = D_\lambda \cap \mathcal{O}, \\
x^\lambda &= (2\lambda - x_1, x_2, \ldots, x_n), \quad \text{the reflexion of } x \text{ with respect to } T_\lambda, \\
v_\lambda(x) &= u(x^\lambda), \quad w_\lambda(x) = u(x^\lambda) - u(x), \quad \text{for } x \in \Sigma_\lambda, \\
d &= \inf\{\lambda \in \mathbb{R} \mid T_\mu \cap \overline{\mathcal{O}} = \emptyset \text{ for all } \mu > \lambda\}.
\end{aligned}
$$

With this notation, our goal is to show that $w_0 \equiv 0$ in Σ_0.

By hypothesis (O_1) the function v_λ satisfies the same equation as u. Hence, by Proposition 2.1, w_λ satisfies

$$L_\lambda w_\lambda := \mathcal{M}_{\lambda,\Lambda}^-(D^2 w_\lambda) - b|Dw_\lambda| + c_\lambda(x)w_\lambda \leq 0 \quad \text{in } \Sigma_\lambda, \tag{2.19}$$

where c_λ is a measurable function which is bounded independently of λ.

We say the hyperplane T_λ has reached a position $\lambda < d$ provided w_μ is nonnegative in Σ_μ for all $\mu \in [\lambda, d)$. The plane T_λ "starts" at $\lambda = d$ and "moves" to the left as λ decreases. If we prove that T_λ reaches position zero we are done, since then we can take a hyperplane coming from the other side, that is, starting from $-d$ and moving to the right. The situation is totally symmetric so the second hyperplane would also reach position zero. This means that $w_0 \geq 0$ and $w_0 \leq 0$ in Σ_0, hence $w_0 \equiv 0$ in Σ_0.

We first show that the above procedure can begin, that is, there exists $\bar{\lambda} < d$ such that $w_\mu \geq 0$ in Σ_μ for all $\mu \in [\bar{\lambda}, d)$. By using Proposition 2.3 we can find a number $r > 0$ such that the operator L_λ defined above satisfies the maximum principle in any subdomain $\mathcal{O}' \subset \mathcal{O}$ with $|\mathcal{O}'| < r$. We fix $\bar{\lambda} < d$ so close to d that $|\Sigma_\lambda| < r$ for any $\lambda \in [\bar{\lambda}, d)$. Hence, by Proposition 2.3, inequality (2.19) implies that $w_\mu \geq 0$ in Σ_μ for all $\mu \in [\bar{\lambda}, d)$.

Note that, by the definition of w_λ, we have $w_\lambda > 0$ on $\partial\Sigma_\lambda \cap \partial\mathcal{O}$ for any $\lambda \in (0, d)$ (since u vanishes on $\partial\mathcal{O}$ and is strictly positive in $\mathcal{O}$). Hence, by Hopf's lemma (Proposition 2.6), $w_\lambda > 0$ in Σ_λ for $\lambda \in (\bar{\lambda}, d)$.

We can define the number

$$\lambda_0 = \inf\{\lambda \geq 0 \mid w_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \geq \lambda\}.$$

Note that, by continuity with respect to λ, $w_{\lambda_0} \geq 0$ in Σ_{λ_0}. By Hopf's lemma, if $\lambda_0 > 0$ then $w_{\lambda_0} > 0$ in Σ_{λ_0}. Further, we claim that $\partial u / \partial x_1 < 0$ in Σ_{λ_0} (recall that $u \in C^1$, by Proposition 2.4). Indeed, let x be an arbitrary point in Σ_{λ_0} with $x_1 = \lambda > \lambda_0$. Then, by the preceding remarks, $w_\lambda > 0$ in Σ_λ. Since $w_\lambda = 0$ on T_λ, Proposition 2.6 implies

$$0 < \frac{\partial w_\lambda}{\partial x_1}(x) = -2\frac{\partial u}{\partial x_1}(x)$$

(recall that $w_\lambda(x) = u(x^\lambda) - u(x)$).

Suppose for contradiction $\lambda_0 > 0$. We are going to "push" the moving plane to the left of λ_0. Let K be a compact subset of Σ_{λ_0} such that $|\Sigma_{\lambda_0} \setminus K| < r/2$ (r is the number from Proposition 2.3). Since w_{λ_0} is continuous and strictly positive in Σ_{λ_0}, there exists a number $\varepsilon > 0$ such that $w_{\lambda_0} \geq \varepsilon$ in K. Fix a number λ_1 with $0 < \lambda_1 < \lambda_0$ such that $|\Sigma_\lambda \setminus K| < \delta$ for $\lambda \in [\lambda_1, \lambda_0)$. By continuity, if λ_1 is sufficiently close to λ_0 we have $w_\lambda \geq \varepsilon/2 > 0$ in K for any $\lambda \in [\lambda_1, \lambda_0)$. In the remaining part of Σ_λ the function w_λ, $\lambda \in [\lambda_1, \lambda_0)$, satisfies the equation

$$\begin{cases} L_0 w_\lambda \leq 0 & \text{in } \Sigma_\lambda \setminus K, \\ w_{\lambda_1} \geq 0 & \text{on } \partial(\Sigma_\lambda \setminus K). \end{cases}$$

By Proposition 2.3, $w_\lambda \geq 0$ in $\Sigma_\lambda \setminus K$. Hence $w_\lambda \geq 0$ in Σ_λ for any $\lambda \in [\lambda_1, \lambda_0)$. This contradicts the definition of λ_0. $\qquad\square$

Proof of Theorem 1.3. Now we define $\Sigma_\lambda = \{x_1 > \lambda\} \setminus B^\lambda$, where B^λ is the reflexion of B with respect to T_λ. As before the difference function w_λ is defined in Σ_λ and we need to show that $w_0 \geq 0$.

Let us prove that for sufficiently large λ we have $w_\lambda \geq 0$ in Σ_λ. Suppose this is not true, that is, there exists a sequence $\lambda_m \to \infty$ such that w_{λ_m} takes negative values in Σ_{λ_m}.

Let m be large enough so that $0 < u < a = u|_{\partial B}$ in D_{λ_m}. Take $x^{(m)} \in \Sigma_{\lambda_m}$ to be such that

$$w_{\lambda_m}(x^{(m)}) = \min_{\Sigma_{\lambda_m}} w_{\lambda_m} < 0$$

(the minimum is clearly attained for fixed m since $w_{\lambda_m} \geq 0$ on $\partial \Sigma_{\lambda_m}$ and $w_{\lambda_m} \to 0$ as $|x| \to \infty$, $x \in \Sigma_{\lambda_m}$). Note that $0 < u(x^\lambda) < u(x)$ whenever $w_\lambda(x) < 0$ so that for m large enough each point $x^{(m)}$ has a neighbourhood $\mathcal{U}_m$ in which both $u(x), u(x^\lambda) \in (0, \delta)$ (since $u \to 0$ at infinity), where δ is the number from Theorem 1.3. Hence in this neighbourhood $c_{\lambda_m} \leq 0$, by Proposition 2.1, and

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w_{\lambda_m}) - b|Dw_{\lambda_m}| \leq 0 \quad \text{in } \mathcal{U}_m.$$

Since w_{λ_m} attains an interior minimum in $\mathcal{U}_m$, we obtain a contradiction with Proposition 2.6, applied to the last inequality.

This reasoning shows that we can define the critical position λ_0 as before. We again aim to show that $\lambda_0 = 0$. Note that, as in the bounded domain case, we have

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{in } \Sigma_{\lambda_0}. \tag{2.20}$$

First, we claim that $\lambda_0 \leq R$, where R is the radius of B. Suppose this is not true. Then there exist sequences λ_m and $x^{(m)} \in \Sigma_{\lambda_m}$ such that $R < \lambda_m \leq \lambda_0$, $\lambda_m \to \lambda_0$, and w_{λ_m} attains its negative minimum in $\overline{\Sigma}_{\lambda_m}$ at $x^{(m)}$.

The point is that $x^{(m)}$ cannot be on the boundary of Σ_{λ_m}. Indeed, note that $\partial \Sigma_{\lambda_m}$ is regular, $\partial \Sigma_{\lambda_m} = T_{\lambda_m} \cup \partial B^{\lambda_m}$, and write $\partial B^{\lambda_m} = S_l \cup S_r$, where S_l (the left part of ∂B^{λ_m}) contains the points on ∂B^{λ_m} which are such that starting from these points and moving to the left along the direction $-x_1$ one enters Σ_{λ_m}. Each point $z \in S_r$ has its counterpart $\overline{z} \in S_l$ with $z' = \overline{z}'$, where for all $z \in \mathbb{R}^n$, $z' = (z_2, \ldots, z_n)$. Note that, by (2.20),

$$w_{\lambda_m}(z) = a - u(z) > a - u(\overline{z}) = w_{\lambda_m}(\overline{z}), \quad z \in S_r,$$

so w_{λ_m} does not attain its minimum on S_r. On the other hand, by (2.20) and hypothesis (1.7), for any $z \in S_l$ the difference w_{λ_m} decreases strictly to the left of z, so that w_{λ_m} does not attain its minimum on S_l either.

Further, as above we can show that $x^{(m)}$ cannot tend to infinity, so $x^{(m)} \to x_0 \in \overline{\Sigma}_{\lambda_0}$, where x_0 is such that $w_{\lambda_0}(x_0) = 0$ and $Dw_{\lambda_0}(x_0) = 0$. This contradicts Proposition 2.6, applied to $w_{\lambda_0} \geq 0$.

We have shown that the moving plane "enters" B. In particular, repeating this argument for all directions, it follows from (2.20) that each point $x \in \partial B$ has a neighbourhood $\mathcal{U}_x$ such that u is strictly decreasing in $\mathcal{U}_x$ along all directions which make an acute angle with x. This implies that if T_λ meets ∂B at x, then w_λ is positive in $\mathcal{U}_x \cap \Sigma_\lambda$ for any $\lambda > 0$.

Suppose now $0 < \lambda_0 \leq R$. We again take sequences $\lambda_m \to \lambda_0$ and $x^{(m)}$ at which w_{λ_m} attains its negative minimum. In order to show that $x^{(m)}$ cannot be on the boundary of Σ_{λ_m} we now have to distinguish three types of points on ∂B^{λ_m}: the two types considered above, which are treated in the same way, and the points $z \in \partial B^{\lambda_m}$ which are such that if one starts from z and moves to the left along $-x_1$ one enters B^{λ_m} and meets ∂B before

meeting T_{λ_m}. Since $u = a$ on ∂B, for any such z we have $u(z) < a$, by (2.20), so $w_{\lambda_m}(z) > 0$.

Hence $x^{(m)} \to x_0 \in \overline{\Sigma}_{\lambda_0}$. If x_0 belongs to the regular part of Σ_{λ_0} we have a contradiction with Proposition 2.6. If x_0 is on the singular part of Σ_{λ_0}, that is, x_0 is a point at which T_{λ_0} meets ∂B, we have a contradiction with the positivity of the comparison functions in a neighbourhood of ∂B, for sufficiently large m. $\qquad\square$

Proof of Theorem 1.2. We use exactly the same argument as for Theorem 1.3 (this argument is now considerably simpler) replacing ∂B by a point of local maximum of u. It then follows that u is radially symmetric with respect to this point. $\qquad\square$

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