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# A Serre weight conjecture for geometric Hilbert modular forms in characteristic $p$

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**Abstract.** Let  $p$  be a prime and  $F$  a totally real field in which  $p$  is unramified. We consider mod  $p$  Hilbert modular forms for  $F$ , defined as sections of automorphic line bundles on Hilbert modular varieties of level prime to  $p$  in characteristic  $p$ . For a mod  $p$  Hilbert modular Hecke eigenform of arbitrary weight (without parity hypotheses), we associate a two-dimensional representation of the absolute Galois group of  $F$ , and we give a conjectural description of the set of weights of all eigenforms from which it arises. This conjecture can be viewed as a “geometric” variant of the “algebraic” Serre weight conjecture of Buzzard–Diamond–Jarvis, in the spirit of Edixhoven’s variant of Serre’s original conjecture in the case  $F = \mathbb{Q}$ . We develop techniques for studying the set of weights giving rise to a fixed Galois representation, and prove results in support of the conjecture, including cases of partial weight 1.

**Keywords.** Hilbert modular forms, Serre’s Conjecture

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## 1. Introduction

### 1.1. The weight part of Serre’s Conjecture

Let  $p$  be a rational prime. Serre’s Conjecture [56], now a theorem of Khare and Wintenberger [43, 44] (completed by a result of Kisin [46]), asserts that every odd, continuous, irreducible representation  $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is *modular* in the sense that it is isomorphic to the mod  $p$  Galois representation associated to a modular eigenform. Furthermore, Serre predicts the minimal weight  $k \geq 2$  such that  $\rho$  arises from an eigenform of weight  $k$  and level prime to  $p$ , the recipe for this minimal weight being in terms of the restriction of  $\rho$  to an inertia subgroup at  $p$ . Under the assumption that  $\rho$  is modular, the fact that it arises from an eigenform of Serre’s predicted weight was known prior to the work of Khare–Wintenberger (assuming  $p > 2$ ), and indeed this plays a crucial role in

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their proof of Serre's Conjecture. This fact, called the *weight part of Serre's Conjecture*, was proved by Edixhoven [24] using the results of Gross [37] and Coleman–Voloch [13] on companion forms. Edixhoven also presents (and proves for  $p > 2^1$ ) an alternative formulation, which predicts the minimal weight  $k \geq 1$  such that  $\rho$  arises from a mod  $p$  eigenform of weight  $k$  and level prime to  $p$ , where mod  $p$  modular forms are viewed as sections of certain line bundles on the reduction mod  $p$  of a modular curve. The qualitative difference between the two versions of the conjecture stems from the fact that a mod  $p$  modular form of weight 1 does not necessarily lift to characteristic zero.

There has been a significant amount of work towards generalising the original formulation of the weight part of Serre's Conjecture to other contexts where one has (or expects) Galois representations associated to automorphic forms. This line of research was first developed by Ash and collaborators in the context of  $\mathrm{GL}_n$  over  $\mathbb{Q}$  (in particular [2]), and the most general formulation to date is due to Gee, Herzig and Savitt [31]. We refer the reader to the introduction of [31] for a discussion of this history and valuable perspectives provided by representation theory,  $p$ -adic Hodge theory and the Breuil–Mézard Conjecture.

An important setting for the development of generalisations of the weight part of Serre's Conjecture has been that of Hilbert modular forms, i.e., automorphic forms for  $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$  where  $F$  is a totally real field. Work in this direction was initiated by Buzzard, Jarvis and one of the authors in [5], where a Serre weight conjecture is formulated under the assumption that  $p$  is unramified in  $F$ . For a totally odd, continuous, irreducible representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p), \quad (1.1)$$

there is a notion of  $\rho$  being *modular of weight*  $V$ , where  $V$  is an irreducible  $\overline{\mathbb{F}}_p$ -representation of  $G(\mathbb{F}_p) = \mathrm{GL}_2(\mathcal{O}_F/p\mathcal{O}_F)$ , where  $\mathcal{O}_F$  denotes the ring of integers of  $F$ . In this context, the generalisation of the weight part of Serre's Conjecture assumes that  $\rho$  is modular of *some* weight, and predicts the set of *all* such weights in terms of the restriction of  $\rho$  to inertia groups at primes over  $p$ . This prediction can be viewed as a conjectural description of all pairs  $(\tau_\infty, \tau_p)$  where  $\tau_\infty$  (resp.  $\tau_p$ ) is a cohomological type at  $\infty$  (resp.  $K$ -type at  $p$ ) of an automorphic representation giving rise to  $\rho$  (see [5, Prop. 2.10]). The conjecture was subsequently generalised in [30, 55] to include the case where  $p$  is ramified in  $F$ , and indeed proved under mild technical hypotheses (for  $p > 2$ ) in a series of papers by Gee and collaborators culminating in [33, 34], with an alternative endgame provided by Newton [50].

It is also natural to consider the problem of generalising Edixhoven's variant of the weight part of Serre's Conjecture, especially in view of the innovation due to Calegari–Geraghty [8] on the Taylor–Wiles method for proving automorphy lifting theorems. By contrast with the original formulation of the weight part of Serre's Conjecture, there has

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<sup>1</sup>Both versions in the case  $p = 2$  should ultimately follow from the results of Khare–Wintenberger and Kisin, as explained in [6].

been relatively little work in this direction. The main aim of this paper is to formulate such a variant in the setting of Hilbert modular forms associated to a totally real field  $F$  in which  $p$  is unramified. More precisely, for  $\rho$  as in (1.1), we give a conjectural description of the set of all weights of mod  $p$  Hilbert modular eigenforms giving rise to  $\rho$ , where we view mod  $p$  Hilbert modular forms as sections of certain line bundles on the special fibre of a Hilbert modular variety. Furthermore, we develop some tools for studying the set of possible weights, and prove results towards the conjecture in the first case that exhibits genuinely new phenomena relative to the settings of [24] and [5]. A forthcoming paper of the authors generalises all the conjectures and results of this article to the case where  $p$  is ramified in  $F$ . Another direction that warrants further research is consideration of higher degree cohomology of automorphic bundles, and it would be natural to pursue the problem in the context of more general Shimura varieties.

A key point we should make is that in the setting of classical modular forms, the enhancement provided by Edixhoven's variant of Serre's Conjecture pertains essentially just to unramified-at- $p$  Galois representations and weight 1 modular forms, but already in the Hilbert modular setting, a much richer tableau emerges from the geometric variant, in terms of both the related  $p$ -adic Hodge theory and arithmetic of automorphic forms. We touch on this further in the course of outlining the contents of the paper below.

## 1.2. Mod $p$ Hilbert modular forms and Galois representations

The foundations for this paper have their roots in the work of Andreatta–Goren [1], which develops the theory of mod  $p$  Hilbert modular forms and partial Hasse invariants. In particular, they use the partial Hasse invariants to define the filtration, which we refer to instead as the *minimal weight*, of a mod  $p$  Hilbert modular form. However, the framework for [1] is based on an alternative notion of Hilbert modular forms, defined using Shimura varieties and automorphic forms associated to the reductive group  $G^*$ , the preimage of  $\mathbb{G}_m$  under  $\det : G \rightarrow \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$  where  $G = \text{Res}_{F/\mathbb{Q}} \text{GL}_2$ . We wish to work throughout with automorphic forms with respect to  $G$  itself, which are more amenable to the theory of Hecke operators and associated Galois representations. To this end we need to adapt the setup of [1].

We begin by recalling the definition of Hilbert modular varieties in §2 and Hilbert modular forms in §3 in our context. For us, a weight will be a pair  $(k, l) \in \mathbb{Z}^\Sigma \times \mathbb{Z}^\Sigma$ , where  $\Sigma$  is the set of embeddings  $F \rightarrow \mathbb{Q}$ . A fundamental observation is the absence in characteristic  $p$  of the parity condition on  $k$  that appears in the usual definition of weights of Hilbert modular forms (with respect to  $G$ , as opposed to  $G^*$ ) in characteristic zero; our  $(k, l)$  is arbitrary. In §4 we explain the construction of Hecke operators in our setting, and in §5 we recall (and adapt) the definition of partial Hasse invariants from [1].

In §6 we establish the existence of Galois representations associated to mod  $p$  Hilbert modular eigenforms of arbitrary weight. More precisely, we prove (see Theorem 6.1.1):

**Theorem.** *If  $f$  is a mod  $p$  Hilbert modular eigenform of weight  $(k, l)$  and level  $U(\mathfrak{n})$  with  $\mathfrak{n}$  prime to  $p$ , then there is a Galois representation  $\rho_f : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  such that*

if  $v \nmid np$ , then  $\rho_f$  is unramified at  $v$  and the characteristic polynomial of  $\rho_f(\text{Frob}_v)$  is  $X^2 - a_v X + d_v \text{Nm}_{F/\mathbb{Q}}(v)$ , where  $T_v f = a_v f$  and  $S_v f = d_v f$ .

Under parity hypotheses on  $k$ , this was proved by Emerton–Reduzzi–Xiao [26] and Goldring–Koskivirta [36] (independently). The new ingredient allowing us to treat arbitrary  $(k, l)$  is to use congruences to forms of level divisible by primes over  $p$ . This introduces a number of technical difficulties, the most critical of which is overcome using a cohomological vanishing result proved in joint work with Kassaei [20].

### 1.3. A geometric Serre weight conjecture

In §7, we introduce the notion of *geometric modularity* and formulate a conjecture that specifies the set of weights for which a given  $\rho$  is geometrically modular. A key point is that the geometric setting allows for the notion of a minimal weight (among the possible  $k$  for a fixed  $l$ ) of eigenforms giving rise to  $\rho$ , something not apparent in the framework of [5]. An investigation of this phenomenon and its interaction with properties of  $\Theta$ -cycles (under which  $l$  varies) in this context led us to the expectation that this minimal weight should lie in the cone  $\Xi_{\min}^+ = \Xi_{\min} \cap \mathbb{Z}_{>0}^\Sigma$ , where

$$\Xi_{\min} := \{k \in \mathbb{Z}^\Sigma \mid pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau} \text{ for all } \tau \in \Sigma\},$$

and that geometric modularity of  $\rho$  for weights in  $\Xi_{\min}^+$  can be characterised using  $p$ -adic Hodge theory. In particular, we make the following conjecture (see Conjecture 7.3.1 for a stronger version, and Definitions 7.2.1 and 7.2.2 for conventions on Hodge–Tate weights):

**Conjecture.** *Suppose that  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible and geometrically modular (of some weight), and let  $l \in \mathbb{Z}^\Sigma$ . Then there exists  $k_{\min} = k_{\min}(\rho, l) \in \Xi_{\min}^+$  such that the following holds for all  $k \in \Xi_{\min}^+$ :*

$$\begin{aligned} \rho \text{ is geometrically modular of weight } (k, l) &\iff k \geq_{\text{Ha}} k_{\min} \\ &\iff \rho|_{G_{F_v}} \text{ has a crystalline lift of weight } (k_\tau, l_\tau)_{\tau \in \Sigma_v} \text{ for all } v \mid p. \end{aligned}$$

The inequality in the statement means that  $k - k_{\min}$  is a non-negative integral linear combination of weights of partial Hasse invariants, and the existence of a weight  $k_{\min}$  such that the second equivalence holds already has deep consequences in  $p$ -adic Hodge theory which are not apparent from the framework of algebraic Serre weight conjectures. We also stress that one might have expected that, by analogy with the algebraic setting in [5], the preceding conjecture held with  $\Xi_{\min}^+$  replaced by the set of totally positive weights. Indeed, we had been positing such a formulation to experts on Serre weight conjectures, until R. Bartlett showed us a counterexample to the resulting  $p$ -adic Hodge-theoretic implications; this led us to examine the possible  $\Theta$ -cycles more closely and arrive at the above version. The role of  $\Xi_{\min}$  in the conjecture in turn inspired the main result of [19], thus answering the basic question posed in [1] of whether the minimal weight of a mod  $p$  Hilbert modular form is totally non-negative; in fact [19, Cor. 1.2] establishes the stronger result that it lies in  $\Xi_{\min}$ .

In §7 we also explain the relation with the Serre weight conjectures of [5], which can be viewed as specifying the weights  $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$  (i.e., *algebraic weights*) for which  $\rho$  is *algebraically* modular. In particular, Conjecture 7.5.2 predicts that geometric and algebraic modularity of  $\rho$  for a weight  $(k, l)$  are equivalent if  $k \in \mathbb{Z}_{\geq 2}^{\Sigma} \cap \mathfrak{E}_{\min}$ . The hypothesis  $k \in \mathbb{Z}_{\geq 2}^{\Sigma}$  is needed for the notion of algebraic modularity, and it is not hard to see that it implies geometric modularity even without the assumption  $k \in \mathfrak{E}_{\min}$ , at least if all  $k_{\tau}$  have the same parity (see Proposition 7.5.4). On the other hand, the opposite implication seems much more difficult, and its failure for  $k \notin \mathfrak{E}_{\min}$  can be observed through the optic of modular representation theory (see Remark 7.5.3). Furthermore, for *algebraic* weights in  $\mathfrak{E}_{\min}$ , the Breuil–Mézard Conjecture [4] (or more precisely its generalisation in [33, Conj. 1.1.5]) converts the  $p$ -adic Hodge-theoretic implications of Conjecture 7.3.1 into non-trivial results on the mod  $p$  representation theory of  $\mathrm{GL}_2(\mathbb{F}_{p^r})$  (see Wiersema’s PhD thesis [62]). We remark also that the interplay between algebraic and geometric Serre weights is reflected in the geometry of Hilbert modular varieties; this theme motivated the construction in [20] of a filtration and Jacquet–Langlands relation for mod  $p$  Hilbert modular forms of pro- $p$ -Iwahori level at  $p$ .

In the case  $F = \mathbb{Q}$ , the only non-algebraic weights with  $k \in \mathfrak{E}_{\min}^+$  have the form  $(1, l)$ , for which Conjecture 7.3.2<sup>2</sup> reduces to the statement that  $\rho$  is unramified at  $p$  if and only if it arises from a mod  $p$  eigenform of weight 1. For general  $F$ , the analogous statement relating (modular)  $\rho$  unramified at  $p$  to parallel weight 1 forms is established (under technical hypotheses) by work of Gee–Kassaei [32] and Dimitrov–Wiese [23]; however, there is a much richer range of possibilities for  $\rho$  to arise (minimally) from forms with non-algebraic minimal weights. The first instance where this is apparent is for real quadratic fields  $F$  in which  $p$  is inert, and we investigate this in detail in §11.

#### 1.4. Partial $\Theta$ -operators, $q$ -expansions and the inert quadratic case

We have already indicated how the perspective afforded by Conjectures 7.3.1 and 7.5.2 leads to new results on the geometry of Shimura varieties and mod  $p$  automorphic forms, as in [19] and [20]. On the other hand, progress on these conjectures evidently requires more geometric input than was needed for the proof of the algebraic Serre weight conjecture of [5]. To that end, we review and develop several useful general tools in the context of Hilbert modular varieties, beginning with the theory of  $\Theta$ -operators in §8. We again proceed by adapting the treatment in [1], but in doing so we introduce some new perspectives which we feel simplify and clarify some aspects of their construction. (See Remark 8.1.3 and the proof of Theorem 8.2.2.) Furthermore, the approach taken here leads

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<sup>2</sup>Strictly speaking, we assume  $F \neq \mathbb{Q}$  throughout the paper to allow for a more uniform exposition, and because nothing new would be presented in the case  $F = \mathbb{Q}$ . For  $F = \mathbb{Q}$ , the equivalence between algebraic and geometric modularity for  $k \geq 2$  is standard, and the analogue of Conjecture 7.3.2 reduces via the Breuil–Mézard Conjecture to Edixhoven’s variant of the weight part of Serre’s Conjecture, hence is known (at least if  $p > 2$ ).

to a substantial improvement on the results in [1] on  $\Theta$ -operators when  $p$  is ramified in  $F$ ; see [18].

In the last few sections, we make critical use of  $q$ -expansions. Most of §9 is a straightforward application of standard methods and results describing  $q$ -expansions and the effect on them of Hecke operators. In addition to this, we construct partial Frobenius operators, whose image we relate to the kernel of partial  $\Theta$ -operators in Theorem 9.8.2; this argument is (to our knowledge) new, and the result generalises a theorem of Katz [40]. These results are further generalised in [18] to the case where  $p$  is ramified in  $F$ .

In §10 we first prove various technical results on eigenforms and their  $q$ -expansions. We then study the behaviour of the minimal weight for  $\rho$  as  $l$  varies (see Theorem 10.4.2), and prove that if an eigenform of algebraic weight is ordinary at a prime over  $p$ , then so is the associated Galois representation (Theorem 10.7.1).

Finally, in §11 we specialise to the inert quadratic case. We first use results from integral  $p$ -adic Hodge theory to describe those  $\rho$  for which the (conjectural) minimal weight is not algebraic (i.e., has  $k_\tau = 1$  for some  $\tau$ ). We then use the tools developed in the preceding sections to transfer modularity results between algebraic and non-algebraic weights. In particular, we prove cases of Conjecture 7.3.2 in the setting of partial weight 1, conditional on our conjectured equivalence between algebraic and geometric modularity (see Theorem 11.4.1). Since one direction of this equivalence is easy under a parity hypothesis, we also obtain the following unconditional result (Theorem 11.4.3):

**Theorem.** *Suppose that  $[F : \mathbb{Q}] = 2$ ,  $p$  is inert in  $F$ ,  $3 \leq k_0 \leq p$ ,  $k_0$  is odd and  $\rho : G_F \rightarrow \mathrm{GL}_2(\mathbb{F}_p)$  is irreducible and modular. If  $\rho|_{G_{F_p}}$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ , then  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ .*

Our method is indicative of a general strategy for transferring results from the setting of algebraic Serre weight conjectures to their geometric variants; further work in this direction is carried out in Wiersema’s thesis [62].

## 2. Hilbert modular varieties

In this section we recall the definitions and basic properties of the models for Hilbert modular varieties used throughout the paper.

### 2.1. General notation

Let  $p$  be a fixed rational prime. Let  $F$  be a totally real field in which  $p$  is unramified. We let  $O_F$  denote the ring of integers of  $F$ , and  $O_{F,\ell} = O_F \otimes \mathbb{Z}_\ell$  for any prime  $\ell$ .

Since this paper offers nothing new in the case  $F = \mathbb{Q}$  (relative to [24]), we will assume throughout that  $F \neq \mathbb{Q}$  in order to avoid complications arising from consideration of the cusps.

Let  $\mathfrak{d} = \mathfrak{d}_{F/\mathbb{Q}}$  denote the different of  $F$  over  $\mathbb{Q}$ . Fix algebraic closures  $\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_p$  of  $\mathbb{Q}$  and  $\mathbb{Q}_p$  respectively, and fix embeddings of  $\overline{\mathbb{Q}}$  into  $\overline{\mathbb{Q}}_p$  and  $\mathbb{C}$ .

Let  $\Sigma$  denote the set embeddings of  $F$  into  $\overline{\mathbb{Q}}$ . Let  $L$  denote a finite extension of  $\mathbb{Q}_p$  in  $\overline{\mathbb{Q}_p}$  containing the image of every embedding in  $\Sigma$ ,  $\mathcal{O}$  its ring of integers,  $\pi$  a uniformiser and  $E = \mathcal{O}/\pi$  its residue field. We identify  $\Sigma$  with the set of embeddings of  $F$  into  $L$  (and hence of  $O_F$  into  $\mathcal{O}$ ), as well as the set of embeddings of  $F$  into  $\mathbb{R}$ .

If  $T$  is a subset of  $F_\infty = F \otimes \mathbb{R} \simeq \prod_{\tau \in \Sigma} \mathbb{R}$ , we let  $T_+$  be the set of totally positive elements in  $T$ .

### 2.2. Hilbert modular varieties of level $N$

**Definition 2.2.1.** For a fractional ideal  $J$  of  $F$  and an integer  $N \geq 3$ , let  $\mathcal{M}_{J,N}$  denote the functor which sends an  $\mathcal{O}$ -scheme  $S$  to the set of isomorphism classes of data  $(A, i, \lambda, \eta)$  comprising

- an abelian scheme  $A/S$  of relative dimension  $[F : \mathbb{Q}]$ ,
- a ring homomorphism  $\iota : O_F \rightarrow \text{End}(A/S)$ ,
- an  $O_F$ -linear isomorphism  $\lambda : (J, J_+) \simeq (\text{Sym}(A/S), \text{Pol}(A/S))$  such that the induced map  $A \otimes_{O_F} J \rightarrow A^\vee$  is an isomorphism, where  $\text{Sym}(A/S)$  (resp.  $\text{Pol}(A/S)$ ) denotes the étale sheaf whose sections are symmetric  $O_F$ -linear morphisms (resp. polarisations)  $A \rightarrow A^\vee$ ,
- an  $O_F$ -linear isomorphism  $\eta : (O_F/N)^2 \simeq A[N]$ .

We call such a quadruple a  $J$ -polarised Hilbert–Blumenthal abelian variety with level  $N$  structure (or simply an HBAV when  $J$  and  $N$  are fixed) over  $S$ .

The functor  $\mathcal{M}_{J,N}$  is representable by a smooth  $\mathcal{O}$ -scheme, which we shall denote  $Y_{J,N}$ ; see [16, Thm. 2.2] and the discussion before it, from which it also follows (using for example [10, Thm. 1.4]) that  $Y_{J,N}$  is quasi-projective over  $\mathcal{O}$ .

Let  $Z_{J,N}$  denote the finite  $\mathcal{O}$ -scheme representing  $O_F$ -linear isomorphisms  $J/NJ \simeq \mathfrak{d}^{-1} \otimes \mu_N$ . If  $(A, \iota, \lambda, \eta)$  is an HBAV over  $S$ , then  $\lambda \otimes \wedge^2 \eta$  defines an isomorphism

$$J/NJ = J \otimes_{O_F} \wedge_{O_F}^2 (O_F/N)^2 \simeq \text{Sym}(A/S) \otimes_{O_F} \wedge_{O_F}^2 A[N],$$

where  $A[N]$  is viewed as an étale sheaf on  $S$ . Composing with the isomorphisms

$$\text{Sym}(A/S) \otimes_{O_F} \wedge_{O_F}^2 A[N] \simeq \text{Hom}(O_F, \mu_N) \simeq \mathfrak{d}^{-1} \otimes \mu_N$$

induced by the Weil pairing and the trace pairing thus gives an element of  $Z_{J,N}(S)$ . In particular, taking  $S = Y_{J,N}$  and the universal HBAV over it, we obtain a canonical morphism  $Y_{J,N} \rightarrow Z_{J,N}$  with geometrically connected fibres.

### 2.3. Unit action on polarisations

The group  $O_{F,+}^\times$  of totally positive units in  $O_F$  acts on  $Y_{J,N}$  by  $v$  in  $O_{F,+}^\times$  sending  $(A, \iota, \lambda, \eta) \in Y_{J,N}(S)$  for every  $\mathcal{O}$ -scheme  $S$  to  $(A, \iota, v\lambda, \eta) \in Y_{J,N}(S)$ . Similarly  $u \in \text{GL}_2(O_F/NO_F)$  acts by sending  $(A, \iota, \lambda, \eta)$  to  $(A, \iota, \lambda, \eta \circ r_{u^{-1}})$  where  $r_{u^{-1}}$  denotes

right multiplication by  $u^{-1}$ , thus defining a right action of  $\mathrm{GL}_2(O_F/NO_F)$ , and hence of  $\mathrm{GL}_2(\widehat{O}_F)$  on  $Y_{J,N}$  through the projection  $\mathrm{GL}_2(\widehat{O}_F) \rightarrow \mathrm{GL}_2(O_F/NO_F)$  where  $\widehat{O}_F$  denotes the profinite completion of  $O_F$ . If  $\mu \in O_F^\times$ , then the action of  $\mu^2 \in O_{F,+}^\times$  on  $Y_{J,N}$  coincides with that of  $\mu^{-1}I_2 \in \mathrm{GL}_2(\widehat{O}_F)$  (where  $I_2$  denotes the 2-by-2 identity matrix).

2.4. Adelic action on level structures

Let  $U$  be an open compact subgroup of  $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2(\widehat{\mathbb{Z}}) \simeq \mathrm{GL}_2(\widehat{O}_F)$  containing  $\mathrm{GL}_2(O_{F,p})$ . Choose an integer  $N \geq 3$  such that  $N$  is not divisible by  $p$  and  $U(N) \subset U$ , where  $U(N) := \ker(\mathrm{GL}_2(\widehat{O}_F) \rightarrow \mathrm{GL}_2(O_F/NO_F))$ . Then the action of  $O_{F,+}^\times \times \mathrm{GL}_2(\widehat{O}_F)$  induces one on  $Y_{J,N}$  of the finite group

$$G_{U,N} := (O_{F,+}^\times \times U) / \{(\mu^2, u) \mid \mu \in O_F^\times, u \in U, u \equiv \mu I \pmod{N}\}.$$

Note that the action of  $(v, u) \in O_{F,+}^\times \times \mathrm{GL}_2(\widehat{O}_F)$  on  $Y_{J,N}$  is compatible with the natural action on  $Z_{J,N}$  defined by multiplication by  $v \det(u)^{-1}$ .

We will show that if  $U$  is sufficiently small, then  $G_{U,N}$  acts freely on  $Y_{J,N}$ . To make this precise, let  $\mathcal{P}_F$  denote the set of primes  $r$  in  $\mathbb{Q}$  such that the maximal totally real subfield  $\mathbb{Q}(\mu_r)^+$  of  $\mathbb{Q}(\mu_r)$  is contained in  $F$ , and let  $\mathcal{C}_F$  denote the set of quadratic CM-extensions  $K/F$  (in a fixed algebraic closure of  $F$ ) such that either

- $K = F(\mu_r)$  for some odd prime  $r \in \mathcal{P}_F$ , or
- $K = F(\sqrt{\beta})$  for some  $\beta \in O_F^\times$ .

Note that the sets  $\mathcal{P}_F$  and  $\mathcal{C}_F$  are finite.

For an ideal  $\mathfrak{n}$  of  $O_F$ , we define the following open compact subgroups of  $\mathrm{GL}_2(\widehat{O}_F)$ :

$$\begin{aligned} U_0(\mathfrak{n}) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{O}_F) \mid c \in \mathfrak{n}\widehat{O}_F \right\}, \\ U_1(\mathfrak{n}) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{n}) \mid d - 1 \in \mathfrak{n}\widehat{O}_F \right\}, \\ {}^1U_1(\mathfrak{n}) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(\mathfrak{n}) \mid a - 1 \in \mathfrak{n}\widehat{O}_F \right\}. \end{aligned}$$

**Lemma 2.4.1.** *Suppose that one of the following holds:*

- $U \subset {}^1U_1(\mathfrak{n})$  for some  $\mathfrak{n}$  such that if  $r \in \mathcal{P}_F$ , then  $\mathfrak{n}$  does not contain  $\mathfrak{r}O_F$  where  $\mathfrak{r}$  is the prime over  $r$  in  $\mathbb{Q}(\mu_r)^+$ , or
- $U \subset U_0(\mathfrak{n})$  for some  $\mathfrak{n}$  such that if  $\mu_r \subset K$  and  $K \in \mathcal{C}_F$ , then  $\mathfrak{n} \subset \mathfrak{q}$  for some prime  $\mathfrak{q}$  of  $F$  inert in  $K$  and not dividing  $r$ .

Then  $G_{U,N}$  acts freely on  $Y_{J,N}$ .

*Proof.* For  $G_{U,N}$  to act freely on  $Y_{U,N}$  means that the morphism  $G_{U,N} \times Y_{U,N} \rightarrow Y_{U,N} \times_{\mathcal{O}} Y_{U,N}$  defined by  $(g, x) \mapsto (gx, x)$  is a closed immersion. Since this morphism is finite and the fibre over every closed point is reduced, it suffices to prove that for every



geometric point  $x \in Y_{U,N}(S)$ , the map  $G_{U,N} \rightarrow Y_{U,N}(S)$  defined by  $g \mapsto gx$  is injective, i.e., the stabiliser of  $x$  in  $G_{U,N}$  is trivial.

Suppose then that  $(A, \iota, \lambda, \eta)$  is an HBAV over an algebraically closed field, and that  $(v, u) \in O_{F,+}^\times \times U$  is such that  $(A, \iota, v\lambda, \eta \circ r_{u^{-1}})$  is isomorphic to  $(A, \iota, \lambda, \eta)$ . This means that there is an automorphism  $\alpha$  of  $A$  such that  $\alpha$  commutes with the action of  $O_F$  and satisfies  $\alpha \circ \eta = \eta \circ r_{u^{-1}}$  and  $\lambda(j) = \alpha^\vee \circ \lambda(vj) \circ \alpha$  for  $j \in J$ .

We wish to prove that  $\alpha = \iota(\mu)$  for some  $\mu \in O_F^\times$ . Suppose this is not the case. Viewing  $F$  as a subfield of  $\text{End}^0(A) = \mathbb{Q} \otimes \text{End}(A)$  via  $\iota$ , it follows from the classification of endomorphism algebras of abelian varieties that  $F(\alpha)$  is a quadratic CM-extension  $K/F$ . Since  $\alpha$  is an automorphism, it is a unit in an order in  $K$ , so  $\alpha \in O_K^\times$ . Since  $O_F^\times$  and  $O_K^\times$  have the same rank and  $\alpha \notin O_F^\times$ , we have  $\alpha^n \in O_F^\times$  for some  $n > 0$ ; replacing  $\alpha$  by a power, we may assume  $n$  is a prime  $r$ . Since  $\alpha^r \in F$  and  $K = F[\alpha]$  is Galois over  $F$ , it follows that  $\zeta_r \in K$ , and hence either  $K = F(\mu_r)$  or  $r = 2$ . In either case we conclude that  $r \in \mathcal{P}_F$ ,  $\mu_r \subset K$  and  $K \in \mathcal{C}_F$ .

Now let  $f(X)$  denote the minimal polynomial of  $\alpha$  over  $F$ . Note that since  $\alpha^r \in O_F^\times$ , we have

$$f(X) = (X - \alpha)(X - \zeta_r \alpha) = X^2 - (1 + \zeta_r)\alpha X + \zeta_r \alpha^2 \tag{2.1}$$

for some  $\zeta_r \in \mu_r$ . For each prime  $\ell$  of  $F$  not dividing  $p$ , the  $\ell$ -adic Tate module  $T_\ell(A)$  is free of rank 2 over  $O_{F,\ell}$  and is annihilated by  $f(\alpha)$ , so  $f(X)$  is in fact the characteristic polynomial of  $\alpha$  on  $T_\ell(A)$ . It follows that  $f(X)$  is the characteristic polynomial of  $\alpha$  on  $A[N]$ , and hence also the characteristic polynomial of  $u$  on  $(O_F/N)^2$ .

Suppose now that  $U$  is as in the first bullet in the statement of the lemma. Since  $U \subset {}^1U_1(\mathfrak{n})$ , the characteristic polynomial of  $u$  is  $(X - 1)^2 \pmod{\mathfrak{n}}$ . Comparing with (2.1), we see that  $(1 + \zeta_r)\alpha \equiv 2 \pmod{\mathfrak{n}}$  and  $\zeta_r \alpha^2 \equiv 1 \pmod{\mathfrak{n}}$ . If  $r = 2$ , this implies  $2 \in \mathfrak{n}$ , contradicting the hypothesis on  $\mathfrak{n}$ . If  $r$  is odd, this implies  $\zeta_r \alpha^2 (\zeta_r - \zeta_r^{-1})^2 \in \mathfrak{n}$ ; since  $\zeta_r \alpha^2 \in O_F^\times$  and  $(\zeta_r - \zeta_r^{-1})^2$  generates  $\mathfrak{r}$ , this also contradicts the hypothesis on  $\mathfrak{n}$ .

Suppose now that  $U$  is as in the second bullet of the statement. Then there is a prime  $\mathfrak{q}$  dividing  $\mathfrak{n}$  such that  $\mathfrak{q}$  is inert in  $K$  and does not divide  $r$ . Since the discriminant of  $f(X)$  is only divisible by primes over  $r$ , we have  $O_{K,\mathfrak{q}} = O_{F,\mathfrak{q}}[\alpha]$ , so  $f(X)$  is irreducible modulo  $\mathfrak{q}$ . On the other hand, since  $u \in U_0(\mathfrak{q})$  its characteristic polynomial factors over  $O_F/\mathfrak{q}$ , and we again obtain a contradiction.

We have now shown that  $\alpha = \iota(\mu)$  for some  $\mu \in O_F^\times$ . It follows that  $u^{-1} \equiv \mu I \pmod{N}$ , and  $v = \mu^{-2}$ . Therefore the image of  $(v, u)$  in  $G_{U,N}$  is trivial, as required. ■

**Caveat 2.4.2.** Unless otherwise indicated, we assume throughout the paper that the open compact subgroup  $U$  of  $\text{GL}_2(\widehat{O}_F)$  contains  $\text{GL}_2(O_{F,p})$  and is sufficiently small that the conclusion of Lemma 2.4.1 holds for some, hence all,  $N \geq 3$  such that  $U(N) \subset U$ .

### 2.5. Hilbert modular varieties of level $U$

We fix a set  $T$  of representatives  $t$  in  $(\mathbb{A}_F^\infty)^\times$  for the strict ideal class group  $(\mathbb{A}_F^\infty)^\times / F_+^\times \widehat{O}_F^\times \simeq \mathbb{A}_F^\times / F^\times \widehat{O}_F^\times F_{\infty,+}^\times$ , and let  $J_t$  denote the corresponding fractional ideal of  $F$ . We

assume the representatives  $t$  are chosen so that the  $J_t$  are prime to  $p$  i.e.,  $t_p \in O_{F,p}^\times$  for each  $t \in T$ .

Since  $Y_{J_t,N}$  is quasi-projective over  $\mathcal{O}$ , the quotient  $Y_{J_t,N}/G_{U,N}$  is representable by a scheme over  $\mathcal{O}$  (by [38, Prop.V.1.8]), and we define

$$Y_U = \coprod_{t \in T} Y_{J_t,N}/G_{U,N}.$$

Then  $Y_U$  is smooth over  $\mathcal{O}$  and the projection  $\coprod_{t \in T} Y_{J_t,N} \rightarrow Y_U$  is Galois and étale with Galois group  $G_{U,N}$  (in view of Lemma 2.4.1 and Caveat 2.4.2). Moreover,  $Y_U$  is defined over  $\mathcal{O} \cap \overline{\mathbb{Q}}$  and is independent of the choices of  $N$  and  $T$ .

### 2.6. Components

Let  $Z_U = \coprod_{t \in T} Z_{J_t,N}/G_{U,N}$  with  $(v, u) \in G_{U,N}$  acting by multiplication by  $v \det(u)^{-1}$ , so if  $\mu_N(\overline{\mathbb{Q}}) \subset \mathcal{O}$ , then  $Z_U(\mathcal{O})$  can be identified with the set of (geometrically) connected components of  $Y_U$ . Fixing a generator  $\zeta_N$  for  $\mathfrak{d}^{-1} \otimes \mu_N(\mathcal{O})$  as an  $O_F$ -module, we obtain a bijection

$$(\mathbb{A}_F^\infty)^\times / F_+^\times \det(U) \simeq Z_U(\mathcal{O}) \tag{2.2}$$

by sending  $x F_+^\times \det(U)$  to the  $G_{U,N}$ -orbit of the isomorphism  $J_t/NJ_t \simeq \mathfrak{d}^{-1} \otimes \mu_N(\mathcal{O})$  sending the class of  $(x\alpha)^{-1} \in J_t \otimes_{O_F} \widehat{O}_F$  to  $\zeta_N$ , where  $t \in T$  and  $\alpha \in F_+^\times$  (unique up to multiplication by an element of  $O_{F,+}^\times$ ) are chosen so that  $x^{-1} \in t\alpha \widehat{O}_F^\times$ .

### 2.7. Complex points

We recall that  $Y_U$  is defined over  $\mathcal{O} \cap \overline{\mathbb{Q}}$ , and a standard construction yields an isomorphism

$$\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F) / UU_\infty \simeq Y_U(\mathbb{C}) \tag{2.3}$$

where  $U_\infty = \prod_{\tau \in \Sigma} \mathrm{SO}_2(\mathbb{R})\mathbb{R}^\times \subset \prod_{\tau \in \Sigma} \mathrm{GL}_2(\mathbb{R}) = \mathrm{GL}_2(F_\infty)$ , allowing us to view  $Y_U$  as a model for the Hilbert modular variety of level  $U$ . More precisely, by the Strong Approximation Theorem, any double coset as in (2.3) can be written in the form

$$\mathrm{GL}_2(F)g_\infty \mathrm{diag}(1, x)UU_\infty$$

for some  $g_\infty \in \mathrm{GL}_2(F_\infty)$ ,  $x \in (\mathbb{A}_F^\infty)^\times$ , such that  $\det(g_\infty) \in F_{\infty,+}^\times$  and  $x \widehat{O}_F = J \mathfrak{d} \widehat{O}_F$  for some  $J$ . Such a double coset corresponds under (2.3) to the  $G_{U,N}$ -orbit of the HBAV over  $\mathbb{C}$  defined by

$$\mathbb{C} \otimes O_F / (g_\infty(z_0)O_F \oplus (J \mathfrak{d})^{-1})$$

with the evident  $O_F$ -action, isomorphism  $\lambda : (J, J_+) \simeq (\mathrm{Sym}(A/S), \mathrm{Pol}(A/S))$  defined so that  $\lambda(\alpha)$  corresponds to the Hermitian form  $\mathrm{tr}_{F/\mathbb{Q}}(\alpha s \bar{t} / \mathrm{Im}(g_\infty(z_0)))$ , and level  $N$  structure defined by  $(a, b) \mapsto (ag_\infty(z_0) + bx^{-1})/N$ , where  $z_0 = i \otimes 1 \in \mathbb{C} \otimes O_F$ .

### 3. Hilbert modular forms

In this section we recall the definition of Hilbert modular forms as sections of certain line bundles on Hilbert modular varieties.

#### 3.1. Automorphic line bundles

The condition that  $A \otimes_{O_F} J \rightarrow A^\vee$  is an isomorphism (in the definition of an HBAV) is called the ‘‘Deligne–Pappas’’ condition. Our assumption that  $p$  is unramified in  $F$  ensures its equivalence with the ‘‘Rapoport condition’’ that  $\text{Lie}(A/S)$  is, locally on  $S$ , free of rank 1 over  $O_F \otimes O_S$  [16, Cor. 2.9], and hence so is its  $O_S$ -dual  $e^* \Omega_{A/S}^1 \simeq s_* \Omega_{A/S}^1$ , where  $s : A \rightarrow S$  is the structure morphism and  $e : S \rightarrow A$  is the identity section. Since  $O_F \otimes O_S \simeq \bigoplus_{\tau \in \Sigma} O_S$  as a coherent sheaf of  $O_S$ -algebras, we may accordingly decompose  $s_* \Omega_{A/S}^1$  as a direct sum of line bundles on  $S$ . Applying this to the universal HBAV  $A_{J,N}$  over  $Y_{J,N}$ , we obtain a decomposition  $s_* \Omega_{A_{J,N}/Y_{J,N}}^1 = \bigoplus_{\tau \in \Sigma} \omega_\tau$  where each  $\omega_\tau$  is a line bundle on  $S$ . For a tuple  $k = (k_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ , we let  $\omega^{\otimes k}$  denote the line bundle  $\bigotimes_{\tau} \omega_\tau^{\otimes k_\tau}$  on  $Y_{J,N}$ .

**Remark 3.1.1.** Note that the definition of  $\omega^{\otimes k}$  makes sense ‘‘integrally’’ because  $p$  is assumed to be unramified in  $F$  so that the Rapoport condition is satisfied; in the ramified case, one can instead proceed as in [54] using models for Hilbert modular varieties defined by Pappas and Rapoport [52].

Since  $\mathcal{H}_{\text{DR}}^1(A/S) = R^1 s_* \Omega_{A/S}^\bullet$  is locally free of rank 2 over  $O_F \otimes O_S$  (by [53, Lem. 1.3]) sitting in the exact sequence

$$0 \rightarrow s_* \Omega_{A/S}^1 \rightarrow \mathcal{H}_{\text{DR}}^1(A/S) \rightarrow R^1 s_* \mathcal{O}_A \rightarrow 0$$

of locally free modules over  $O_F \otimes O_S$  (given by the Hodge–de Rham spectral sequence),

$$\wedge_{O_F \otimes O_S}^2 \mathcal{H}_{\text{DR}}^1(A/S) \simeq s_* \Omega_{A/S}^1 \otimes_{O_F \otimes O_S} R^1 s_* \mathcal{O}_A \tag{3.1}$$

is locally free of rank 1 over  $O_F \otimes O_S$  and similarly decomposes as a direct sum of line bundles indexed by  $\tau \in \Sigma$ . We let  $\delta_\tau$  denote the line bundles so obtained from the universal HBAV over  $S = Y_{J,N}$ , and for a tuple  $l = (l_\tau)_{\tau \in \Sigma} \in \mathbb{Z}^\Sigma$ , we let  $\delta^{\otimes l}$  denote the line bundle  $\bigotimes_{\tau} \delta_\tau^{\otimes l_\tau}$ . Finally, we let  $\mathcal{L}_{J,N}^{k,l}$  denote the line bundle  $\omega^{\otimes k} \otimes \delta^{\otimes l}$ .

Recall that we defined the action of  $O_{F,+}^\times \times \text{GL}_2(\widehat{O}_F)$  on  $S = Y_{J,N}$  by requiring the pull-back via  $(v, u)$  of the universal HBAV  $(A, \iota, \lambda, \eta)$  to be isomorphic to

$$(A, \iota, v\lambda, \eta \circ r_{u^{-1}});$$

we let  $\alpha_{v,u} : A \rightarrow (v, u)^* A$  be the unique such isomorphism. Note that

$$((v, u)^* \alpha_{v',u'}) \circ \alpha_{v,u} = \alpha_{vv',uu'}$$

for  $(v, u), (v', u') \in O_{F,+}^\times \times \text{GL}_2(\widehat{O}_F)$  (where we identify  $(v, u)^* \circ (v', u')^*$  with  $(vv', uu')^*$  via the natural isomorphism resulting from the equality  $(v', u') \circ (v, u) = (vv', uu')$ ). It follows that the induced  $O_F \otimes O_S$ -linear isomorphisms

$$\alpha_{v,u}^* : (v, u)^*(s_*\Omega_{A/S}^1) \rightarrow s_*\Omega_{A/S}^1, \quad (v, u)^*(R^1s_*\Omega_{A/S}^\bullet) \rightarrow R^1s_*\Omega_{A/S}^\bullet$$

satisfy the relation  $\alpha_{v,u}^* \circ (v, u)^* \alpha_{v',u'}^* = \alpha_{vv',uu'}^*$ . We thus obtain an action of the group  $O_{F,+}^\times \times \text{GL}_2(\widehat{O}_F)$  on the sheaves  $s_*\Omega_{A/S}^1$  and  $R^1s_*\Omega_{A/S}^\bullet$ , and hence on the line bundles  $\mathcal{L}_{J,N}^{k,l}$ , compatible with its action on  $Y_{J,N}$ .

Recall that if  $\mu \in O_F^\times$ , then  $(\mu^2, \mu I_2)$  acts trivially on  $Y_{J,N}$ . In this case the isomorphism  $\alpha_{\mu^2, \mu I_2}$  is given by  $\iota(\mu)$ , and it follows that the induced action on  $\mathcal{L}_{J,N}^{k,l}$  is multiplication by the element  $\mu^{k+2l} := \prod_\tau \tau(\mu)^{k_\tau+2l_\tau}$ . In particular, if  $k_\tau + 2l_\tau$  is an integer  $w$  independent of  $\tau$ , then  $\mu^{k+2l} = \text{Nm}_{F/\mathbb{Q}}(\mu)^w$ . Thus if  $w$  is even, then the action of  $O_{F,+}^\times \times U$  on  $\mathcal{L}_{J,N}^{k,l}$  factors through  $G_{U,N}$  and hence defines descent data; we let  $\mathcal{L}_U^{k,l}$  be the resulting line bundle on  $Y_U$  (given by [38, Cor. VIII.1.3]). The same holds if  $w$  is odd and  $\text{Nm}_{F/\mathbb{Q}}(\mu) = 1$  for all  $\mu \in O_F^\times \cap U$ . Note that the line bundle  $\mathcal{L}_U^{k,l}$  is independent of the choice of  $N$ .

### 3.2. Hilbert modular forms

**Definition 3.2.1.** For two tuples  $k$  and  $l$  above, we say  $(k, l)$  is *paritious* if  $k_\tau + 2l_\tau$  is independent of  $\tau$ . For such  $(k, l)$ , we call an element of  $H^0(Y_U, \mathcal{L}_U^{k,l})$  a *Hilbert modular form of weight  $(k, l)$  and of level  $U$*  (where in addition to Caveat 2.4.2, we assume that  $\text{Nm}_{F/\mathbb{Q}}(\mu) = 1$  for all  $\mu \in O_F^\times \cap U$  if  $k_\tau + 2l_\tau$  is odd).

We now make an observation critical to our consideration of weights of mod  $p$  Hilbert modular forms. Let  $\overline{Y}_{J,N}$  denote the special fibre of  $Y_{J,N}$ , and similarly let  $\overline{\mathcal{L}}_{J,N}^{k,l}$  denote the pull-back of  $\mathcal{L}_{J,N}^{k,l}$  to  $\overline{Y}_{J,N}$ . If  $\mu^{k+2l} \equiv 1 \pmod{\pi}$  for all  $\mu \in O_F^\times \cap U$ , then the action of  $O_{F,+}^\times \times U$  on  $\overline{\mathcal{L}}_{J,N}^{k,l}$  factors through  $G_{U,N}$ , and hence defines descent data, giving rise to a line bundle  $\overline{\mathcal{L}}_U^{k,l}$  on the special fibre  $\overline{Y}_U$  of  $Y_U$  (again independent of the choice of  $N$ ). If  $(k, l)$  is paritious, then this is simply the pull-back of  $\mathcal{L}_U^{k,l}$  to  $\overline{Y}_U$ , but the line bundles  $\overline{\mathcal{L}}_U^{k,l}$  may be defined even if  $(k, l)$  is not paritious. In particular, if  $O_F^\times \cap U$  is contained in the kernel of reduction modulo  $p$ , then  $\mathcal{L}_U^{k,l}$  is defined for *all* pairs  $(k, l)$ . This holds for example if  $U \subset U_1(\mathfrak{n})$  for some ideal  $\mathfrak{n}$  such that the kernel of  $O_F^\times \rightarrow (O_F/\mathfrak{n})^\times$  is contained in the kernel of  $O_F^\times \rightarrow (O_F/p)^\times$ .

More generally, for any  $\mathcal{O}$ -algebra  $R$  in which the image of  $\mu^{k+2l}$  is trivial for all  $\mu \in O_F^\times \cap U$ , we obtain a line bundle  $\mathcal{L}_{U,R}^{k,l}$  on  $Y_{U,R} = Y_U \times_{\mathcal{O}} R$  by descent from the pull-back of the line bundles  $\mathcal{L}_{J,N}^{k,l}$ .

**Definition 3.2.2.** If  $U, k, l$  and  $R$  are such that the image of  $\mu^{k+2l}$  in  $R$  is trivial for all  $\mu \in O_F^\times \cap U$ , then we call an element of  $H^0(Y_{U,R}, \overline{\mathcal{L}}_{U,R}^{k,l})$  a *Hilbert modular form over  $R$  of weight  $(k, l)$  and level  $U$* , and we write  $M_{k,l}(U; R)$  for the  $R$ -module of such forms.

If  $R = E$ , then we call such a form a *mod  $p$  Hilbert modular form* (of weight  $(k, l)$  and of level  $U$ ).

**Definition 3.2.3.** We say that  $U$  is  *$p$ -neat* if  $O_F^\times \cap U$  is contained in the kernel of reduction modulo  $p$  (in addition to  $U$  being sufficiently small in the sense of Caveat 2.4.2).

### 3.3. The Koecher Principle

The Koecher Principle implies that  $M_{k,l}(U; R)$  is a finitely generated  $R$ -module (assuming an  $\mathcal{O}$ -algebra  $R$  is Noetherian), and that  $M_{0,0}(U; R) = H^0(Y_{U,R}, \mathcal{O}_{Y_{U,R}})$  is the set of locally constant functions on  $Y_{U,R}$ . Both of these assertions follow from the analogous ones with  $Y_U$  replaced by  $Y_{J,N}$ , proved by Rapoport. (The case  $J = O_F$  is treated by [53, Prop. 4.9 and discussion preceding Prop. 6.11], and the modifications needed for the case of arbitrary  $J$  are given in [11]; see also [21, Thm. 8.3] and [22, Thm. 7.1] for variants with different level structure and descent data in place.)

### 3.4. Canonical trivialisations

We observe that the sheaves  $\delta^{\otimes l}$  on  $Y_{J,N}$  are in fact free (not just locally so). Indeed, if  $A$  is the universal HBAV over  $S = Y_{J,N}$ , then we have a sequence of canonical isomorphisms

$$\begin{aligned} R^1 s_* \mathcal{O}_A &\simeq \text{Lie}(A^\vee) \simeq \text{Lie}(A) \otimes_{O_F} J \simeq \mathcal{H}om_{\mathcal{O}_S}(s_* \Omega_{A/S}^1, \mathcal{O}_S) \otimes_{O_F} J \\ &\simeq \mathcal{H}om_{O_F \otimes \mathcal{O}_S}(s_* \Omega_{A/S}^1, J \mathfrak{d}^{-1} \otimes \mathcal{O}_S), \end{aligned} \tag{3.2}$$

from which it follows that  $\wedge^2_{O_F \otimes \mathcal{O}_S} \mathcal{H}^1_{\text{DR}}(A/S) \simeq s_* \Omega_{A/S}^1 \otimes_{O_F \otimes \mathcal{O}_S} R^1 s_* \mathcal{O}_A$  is canonically isomorphic to  $J \mathfrak{d}^{-1} \otimes \mathcal{O}_S$ , which is free of rank 1 over  $O_F \otimes \mathcal{O}_S$ . Therefore each  $\delta_\tau$  is free of rank 1 over  $\mathcal{O}_S$ , and hence so are the sheaves  $\delta^{\otimes l}$ .

Under the action of  $(v, u) \in O_{F,+}^\times \times U$  on  $Y_{J,N}$ , one finds that the canonical isomorphism  $\psi$  from  $\wedge^2_{O_F \otimes \mathcal{O}_S} \mathcal{H}^1_{\text{DR}}(A/S)$  to  $J \mathfrak{d}^{-1} \otimes \mathcal{O}_S$  is multiplied by  $v$  (in the sense that  $(v, u)^* \psi = (v \otimes 1) \psi \circ \alpha_{v,u}^*$ ). Therefore the action of  $(v, u)$  on the resulting trivialisation of  $\mathcal{L}_{J,N}^{0,l} = \delta^{\otimes l}$  is multiplication by  $v^l$ . In particular, if  $(0, l)$  is paritious (i.e.,  $l_\tau$  is independent of  $\tau$ ), then  $v^l = 1$  so the trivialisation of  $\mathcal{L}_{J,N}^{0,l}$  on  $Y_{J,N}$  is invariant under  $G_{U,N}$ , hence descends to one on  $Y_U$ . Similarly if  $v^l \equiv 1 \pmod{\pi}$  for all  $v \in O_{F,+}^\times$ , then the canonical trivialisation of  $\overline{\mathcal{L}}_{J,N}^{0,l}$  on  $\overline{Y}_{J,N}$  descends to one on  $\overline{Y}_U$ .

### 3.5. Complex Hilbert modular forms

If  $(k, l)$  is paritious, then under the identification (2.3), the line bundle  $\mathcal{L}_U^{k,l}$  gives the usual automorphic line bundle whose sections are classical Hilbert modular forms of weight  $(k, l)$  and level  $U$ . More precisely,  $\mathcal{L}_U^{k,l}$  is defined over  $\mathcal{O} \cap \overline{\mathbb{Q}}$ , and its fibre at the point  $y_{g_\infty, x} \in Y_U(\mathbb{C})$  corresponding to the double coset  $\text{GL}_2(F)g_\infty \text{diag}(1, x)UU_\infty$  has basis  $e^{k,l} = \bigotimes_\tau (ds_\tau^{\otimes k_\tau} \otimes h_\tau^{\otimes l_\tau})$ , where  $s = (s_\tau)_{\tau \in \Sigma}$  are the coordinates on  $\mathbb{C} \otimes F \simeq \mathbb{C}^\Sigma$  and

$h_\tau$  is the basis for  $\delta_\tau$  given by the trivialisation defined above. For  $\phi \in M_{k,l}(U; \mathbb{C})$ , we define the function  $f_\phi : \text{GL}_2(\mathbb{A}_F) \rightarrow \mathbb{C}$  so that

$$\mu_{g_\infty, x}^* \phi = \|x\|^{-1} \det(g_\infty)^{l-1} j(g_\infty, z_0)^k f_\phi(\gamma g_\infty u) e^{k,l} \quad \text{for all } \gamma \in \text{GL}_2(F), u \in U,$$

where  $j(g_\infty, z) = cz + d$  for  $g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R})$  and  $z$  in the complex upper half-plane  $\mathfrak{H}$ , and the exponents  $k$  and  $l - 1$  denote products over the embeddings  $\tau \in \Sigma$ . Then  $\phi \mapsto f_\phi$  defines an isomorphism  $M_{k,l}(U; \mathbb{C}) \simeq A_{k,l}(U)$ , where  $A_{k,l}(U)$  is the set of functions  $f : \text{GL}_2(\mathbb{A}_F) \rightarrow \mathbb{C}$  such that

- $f(\gamma hu) = \det(u_\infty)^{l-1} j(u_\infty, i)^{-k} f(h)$  for all  $\gamma \in \text{GL}_2(F)$ ,  $h \in \text{GL}_2(\mathbb{A}_F)$  and  $u \in UU_\infty$ ;
- $f_h(g_\infty(z_0)) = \det(g_\infty)^{l-1} j(g_\infty, i)^k f(hg_\infty)$  is holomorphic on  $\mathfrak{S}^\Sigma$  for all  $h \in \text{GL}_2(\mathbb{A}_F^\infty)$ .

Note also that  $f \mapsto (f_{\text{diag}(1,x)})$  defines an isomorphism  $A_{k,l}(U) \simeq \bigoplus_x M_k(\Gamma_{U,x})$ , where  $x$  runs over a set of representatives of  $F^\times \backslash \mathbb{A}_F^\times / \det(U) F_{\infty,+}^\times$ ,

$$\Gamma_{U,x} = \text{GL}_2^+(F) \cap \text{diag}(1, x)U \text{diag}(1, x)^{-1}$$

and  $M_k(\Gamma)$  denotes the set of holomorphic functions  $\varphi : \mathfrak{S}^\Sigma \rightarrow \mathbb{C}$  such that  $\varphi(\gamma(z)) = \det(\gamma)^{-k/2} j(\gamma, z)^k \varphi(z)$  for all  $\gamma \in \Gamma$ .

### 3.6. Forms of weight $(0, l)$ in characteristic $p$

Let us now return to characteristic  $p$  and give sufficient hypotheses for the sheaf  $\overline{\mathcal{L}}_U^{0,l}$  on the special fibre  $\overline{Y}_U$  to be globally free, even when  $(0, l)$  is not paritious. Suppose that  $\mu_N(\overline{\mathbb{Q}}) \subset \mathcal{O}$ , so the geometric components of  $Y_{J,N}$  are defined over  $\mathcal{O}$ . Recall that the set of geometric components is in bijection with  $Z_{J,N}(\mathcal{O})$ , with  $(v, u)$  acting by  $v \det(u)^{-1}$ , so the stabiliser of each component of  $Y_{J,N}$  is  $\{(v, u) \in O_{F,+}^\times \times U \mid v \equiv \det(u) \pmod{N}\}$ . Letting  $H_{U,N}$  denote the corresponding subgroup of  $G_{U,N}$ , we see that if  $v^l \equiv 1 \pmod{\pi}$  for each  $v \in O_{F,+}^\times \cap \det(U)$ , then the trivialisation of  $\overline{\mathcal{L}}_{J,N}^{0,l}$  on  $\overline{Y}_{J,N}$  is invariant under  $H_{U,N}$ , so descends to the quotient  $\overline{Y}_{J,N}/H_{U,N}$ . Note that this hypothesis also implies that  $\mu^{2l} \equiv 1 \pmod{\pi}$  for all  $\mu \in O_F^\times \cap U$ , so that  $\overline{\mathcal{L}}_{J,N}^{0,l}$  descends to  $\overline{Y}_U$ ; since the projection from  $\coprod \overline{Y}_{J,N}/H_{U,N}$  is an isomorphism on each connected component, it follows that  $\overline{\mathcal{L}}_U^{0,l}$  is (globally) free on  $\overline{Y}_U$ .

We record this as follows (recall that  $Z_U$  is defined in see §2.6):

**Proposition 3.6.1.** *Suppose that  $\mu_N(\overline{\mathbb{Q}}) \subset \mathcal{O}$  for some  $N$  prime to  $p$  such that  $U(N) \subset U$ . If  $v^l \equiv 1 \pmod{\pi}$  for all  $v \in O_{F,+}^\times \cap \det(U)$ , then the sheaf  $\overline{\mathcal{L}}_U^{0,l}$  on  $\overline{Y}_U$  is (non-canonically) isomorphic to  $\mathcal{O}_{\overline{Y}_U}$ , and  $M_{0,l}(U; E)$  to the space of functions  $Z_U(\mathcal{O}) \rightarrow E$ .*

Note that the hypotheses of the proposition are satisfied for all  $l \in \mathbb{Z}^\Sigma$  if  $O_{F,+}^\times \cap \det(U)$  is contained in the kernel of reduction modulo  $p$ . This holds for example if  $U \subset {}^1U_1(\mathfrak{n})$  for some ideal  $\mathfrak{n}$  such that the kernel of  $O_F^\times \rightarrow (O_F/\mathfrak{n})^\times$  is contained in the kernel

of  $O_F^\times \rightarrow (O_F/p)^\times$ . In this case  $U$  is also  $p$ -neat, so the sheaves  $\overline{\mathcal{L}}_U^{k,l}$  are defined for all pairs  $(k, l)$ , and the spaces of mod  $p$  Hilbert modular forms  $H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,l})$  for fixed  $k$  and varying  $l$  are (non-canonically) isomorphic.

#### 4. Hecke operators

In this section, we define Hecke operators geometrically on spaces of mod  $p$  Hilbert modular forms.

##### 4.1. Adelic action on Hilbert modular varieties

Suppose that  $U_1$  and  $U_2$  are open compact subgroups of  $\mathrm{GL}_2(\widehat{O}_F)$ ; we assume as usual that Caveat 2.4.2 holds, so  $U_1$  and  $U_2$  contain  $\mathrm{GL}_2(O_{F,p})$  and are sufficiently small in the sense that the conclusion of Lemma 2.4.1 holds.

Suppose that  $g \in \mathrm{GL}_2(\mathbb{A}_F^\infty) = \mathrm{GL}_2(\widehat{O}_F \otimes \mathbb{Q})$  with  $g_p \in \mathrm{GL}_2(O_{F,p})$  and  $g^{-1}U_1g \subset U_2$ . We now proceed to define a morphism  $\rho_g : Y_{U_1} \rightarrow Y_{U_2}$  which corresponds to the one defined by right multiplication by  $g$  on the associated Hilbert modular varieties; i.e. on complex points it is given by  $\mathrm{GL}_2(F)xU_1U_\infty \mapsto \mathrm{GL}_2(F)xgU_2U_\infty$ .

We first choose

- $\alpha \in O_F$  such that  $\alpha g \in M_2(\widehat{O}_F)$  and  $\alpha \in O_{F,p}^\times$ ;
- $N_2$  prime to  $p$  such that  $U(N_2) \subset U_2$ ;
- $N_1$  prime to  $p$  such that  $U(N_1) \subset U_1$  and  $(\alpha g)^{-1}N_1/N_2 \in M_2(\widehat{O}_F)$ .

We will define a morphism  $\tilde{\rho}_g : \coprod Y_{J,N_1} \rightarrow \coprod Y_{J,N_2}$  whose composite with the projection to  $Y_{U_2}$  factors through  $Y_{U_1}$ , yielding the desired morphism  $\rho_g : Y_{U_1} \rightarrow Y_{U_2}$ , independent of the above choices of  $\alpha, N_1$  and  $N_2$ .

We first note that the conditions above imply that  $N_2 \mid N_1, g^{-1}U(N_1)g \subset U(N_2)$ , and (right) multiplication by  $(\alpha g)^{-1}N_1/N_2$  induces an injective  $O_F$ -linear map

$$j : (O_F/N_2)^2 \rightarrow (O_F/N_1)^2 / (O_F/N_1)^2 \cdot (\alpha g)^{-1}N_1.$$

Let  $(A_1, \iota_1, \lambda_1, \eta_1)$  denote the universal HBAV over  $S = Y_{J_1, N_1}$  where  $J_1 = J_{t_1}$  for some  $t_1 \in T$ , and let  $A'_1 = A_1/\eta_1(C)$  where  $C = (O_F/N_1)^2 \cdot (\alpha g)^{-1}N_1$ . Then  $A'_1$  inherits an  $O_F$ -action  $\iota'_1$  from  $A_1$ , and  $\eta_1$  induces an  $O_F$ -linear closed immersion

$$(O_F/N_1)^2 / (O_F/N_1)^2 \cdot (\alpha g)^{-1}N_1 \rightarrow A'_1$$

whose composite with  $j$  defines an isomorphism  $\eta'_1 : (O_F/N_2)^2 \rightarrow A'_1[N_2]$ .

Now consider the injective  $O_F$ -linear map  $\pi^* : \mathrm{Sym}(A'_1/S) \rightarrow \mathrm{Sym}(A_1/S)$  defined by  $f \mapsto \pi^\vee \circ f \circ \pi$ , where  $\pi$  is the natural projection  $A_1 \rightarrow A'_1$ .

**Lemma 4.1.1.** *The image of  $\pi^*$  is  $(\det(\alpha g))\mathrm{Sym}(A_1/S)$  where  $(\det(\alpha g))$  denotes the ideal  $O_F \cap \det(\alpha g)\widehat{O}_F$  of  $O_F$ .*

*Proof.* Note that since  $\text{Sym}(A_1/S)$  is an invertible  $O_F$ -module, the image of  $\pi^*$  is (locally on  $S$ ) of the form  $I\text{Sym}(A_1/S)$  for some ideal  $I$  of  $O_F$ , non-zero since  $\pi^*$  is injective. Moreover, since  $\ker(\pi) \subset A_1[N_1]$ , there is an isogeny  $\phi : A'_1 \rightarrow A_1$  such that  $\phi \circ \pi$  is multiplication by  $N_1$ ; since  $\pi^* \circ \phi^*$  is multiplication by  $N_1^2$ , it follows that  $N_1^2 \in I$ , so  $I$  can only be divisible by primes dividing  $N_1$ .

We now determine  $I \otimes \mathbb{Z}_\ell$  for each prime  $\ell \mid N_1$ . Note in particular that  $\ell \neq p$ , so  $\ell$  is invertible in  $\mathcal{O}_S$ . Consider the commutative diagram

$$\begin{CD} \text{Sym}(A'_1/S) \otimes \mathbb{Z}_\ell @>>> \text{Sym}(A_1/S) \otimes \mathbb{Z}_\ell \\ @VV\wr V @VV\wr V \\ \text{Hom}_{\mathbb{Z}_\ell}(\wedge^2_{O_{F,\ell}} T_\ell(A'_1), \mathbb{Z}_\ell(1)) @>>> \text{Hom}_{\mathbb{Z}_\ell}(\wedge^2_{O_{F,\ell}} T_\ell(A_1), \mathbb{Z}_\ell(1)) \end{CD}$$

of  $O_F$ -linear maps of  $\ell$ -adic sheaves on  $S$ , where the top map is  $\pi^* \otimes \mathbb{Z}_\ell$ , the vertical isomorphisms are induced by the Weil pairings, and the bottom map is given by the map  $T_\ell(\pi) : T_\ell(A_1) \rightarrow T_\ell(A'_1)$  on  $\ell$ -adic Tate modules induced by  $\pi$ . The cokernel of  $\pi^* \otimes \mathbb{Z}_\ell$  is therefore isomorphic to that of the bottom map, which in turn is isomorphic to  $\text{Hom}_{\mathbb{Z}_\ell}(M_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ , where  $M_\ell$  is the cokernel of  $\wedge^2_{O_{F,\ell}} T_\ell(\pi)$ . Since the  $\ell$ -adic sheaves  $T_\ell(A_1)$  and  $T_\ell(A'_1)$  are locally free of rank 2 over  $O_{F,\ell}$  and the cokernel of  $T_\ell(\pi)$  is isomorphic to

$$\ker(\pi) \otimes \mathbb{Z}_\ell \simeq C \otimes \mathbb{Z}_\ell \simeq O_{F,\ell}^2 \cdot (\alpha g)^{-1} / O_{F,\ell}^2,$$

it follows that  $M_\ell$  is isomorphic to  $O_{F,\ell}/\det(\alpha g)O_{F,\ell}$ .

We have now shown that the cokernel of  $\pi^* \otimes \mathbb{Z}_\ell$  is (étale locally) isomorphic to  $O_{F,\ell}/\det(\alpha g)O_{F,\ell}$  for all  $\ell$ . Since the cokernel of  $\pi^*$  is also étale locally isomorphic  $O_F/I$ , it follows that  $O_F/I$  is isomorphic to  $\widehat{O}_F/\det(\alpha g)\widehat{O}_F$ , and hence  $I = (\det(\alpha g))$ . ■

It follows from the lemma that  $\lambda_1 : J_1 \simeq \text{Sym}(A_1/S)$  restricts to an isomorphism  $IJ_1 \rightarrow \pi^*\text{Sym}(A'_1/S)$ , where  $I = (\det(\alpha g))$ . Moreover, since  $f$  is a section of  $\text{Pol}(A'_1/S)$  if and only if  $\pi^*f$  is a section of  $\text{Pol}(A_1/S)$ , we see that  $\lambda_1$  further restricts to an isomorphism  $(IJ_1)_+ \rightarrow \pi^*\text{Pol}(A'_1/S)$ . Now let  $J_2 = J_{t_2}$  where  $t_2 \in T$  is the fixed representative of  $IJ_1$  in the strict class group of  $F$ , and choose an element  $\beta \in F_+^\times$  such that  $\beta J_2 = IJ_1$ . Thus  $\beta$  is uniquely determined up to  $O_{F,+}^\times$ , and the composite of  $\lambda_1 \circ \beta$  with the inverse of  $\pi^*$  yields an isomorphism

$$\begin{aligned} (J_2, (J_2)_+) &\simeq (IJ_1, (IJ_1)_+) \simeq \pi^*(\text{Sym}(A'_1/S), \text{Pol}(A'_1/S)) \\ &\simeq (\text{Sym}(A'_1/S), \text{Pol}(A'_1/S)), \end{aligned}$$

which we denote by  $\lambda'_1$ .

Finally, we note that since  $A$  satisfies the Deligne–Pappas condition, so does  $A'$ . This follows for example from the commutative diagram



$$\begin{array}{ccc}
 A_1 \otimes_{\mathcal{O}_F} IJ_1 & \longrightarrow & A_1 \otimes_{\mathcal{O}_F} J_1 \simeq A_1^\vee \\
 \pi \otimes 1 \downarrow & & \uparrow \pi^\vee \\
 A'_1 \otimes_{\mathcal{O}_F} IJ_1 & \longrightarrow & (A'_1)^\vee
 \end{array}$$

and the observation that the top left map is an isogeny with kernel  $A_1[I] \otimes_{\mathcal{O}_F} IJ_1$ , hence (constant) degree  $|O_F/I|^2$ , while  $\deg(\pi \otimes 1) = \deg(\pi^\vee) = \deg(\pi) = |O_F/I|$ , so the bottom map must be an isomorphism.

Now  $(A'_1, \iota'_1, \lambda'_1, \eta'_1)$  is a  $J_2$ -polarised HBAV with level  $N_2$  structure over  $Y_{J_1, N_1}$ , so corresponds to a morphism  $Y_{J_1, N_1} \rightarrow Y_{J_2, N_2}$  such that the pull-back of the universal HBAV over  $Y_{J_2, N_2}$  is  $(A'_1, \iota'_1, \lambda'_1, \eta'_1)$ . Taking the union over  $t_1 \in T$  yields the desired morphism  $\tilde{\rho}_g : \coprod Y_{J, N_1} \rightarrow \coprod Y_{J, N_2}$ .

It is straightforward to check that the composite of  $\tilde{\rho}_g$  with the projection to  $Y_{U_2}$  is independent of the choices of  $\alpha, N_2$  and  $\beta$ , and indeed of  $N_1$  in the sense that if  $N_1$  is replaced by a multiple  $N$ , then the resulting morphism is obtained by composing with the natural projection  $\coprod Y_{J, N} \rightarrow \coprod Y_{J, N_1}$ . (The only non-trivial point is that if  $\alpha$  is replaced by a multiple  $\delta\alpha$ , then the resulting  $J_2$ -polarised HBAV with level  $N_2$  structure on  $Y_{J_1, N_1}$  is isomorphic to the original  $(A'_1, \iota'_1, \lambda'_1, \eta'_1)$  via the map induced by  $\iota_1(\delta)$ .) Moreover, the resulting morphism to  $Y_{U_2}$  is invariant under the action of  $G_{U_1, N_1}$  on  $\coprod Y_{J, N_1}$  (indeed, we have  $\tilde{\rho}_g \circ (\mu, u) = (\mu, g^{-1}ug) \circ \tilde{\rho}_g$  for all  $(\mu, u) \in O_{F,+}^\times \times U_1$  on each  $Y_{J_1, N_1}$  for any choice of  $\beta$  as above), hence factors through  $Y_{U_1}$ , yielding the desired morphism  $\rho_g : Y_{U_1} \rightarrow Y_{U_2}$ .

Suppose that  $U_1, U_2$  and  $U_3$  are open compact subgroups of  $GL_2(\hat{O}_F)$  with  $g_1, g_2 \in GL_2(\mathbb{A}_F^\infty)$  as above with  $g_1^{-1}U_1g_1 \subset U_2$  and  $g_2^{-1}U_2g_2 \subset U_3$ , so that  $\rho_{g_1} : Y_{U_1} \rightarrow Y_{U_2}$  and  $\rho_{g_2} : Y_{U_2} \rightarrow Y_{U_3}$  are defined. Note that choosing  $\alpha_2, N_2$  and  $N_3$  to define  $\rho_{g_2}$ , and then  $\alpha_1, N_1$  and (the same)  $N_2$  to define  $\rho_{g_1}$ , we may use  $\alpha_1\alpha_2, N_1$  and  $N_3$  to define  $\rho_{g_1g_2}$ . Let  $(A_i, \iota_i, \lambda_i, \eta_i)$  denote the universal HBAV over  $Y_{N_i, J_i}$  for  $i = 1, 2, 3$ , where  $J_i = J_{t_i}$  for  $t_i \in T$  such that  $t_{i+1}$  represents the class of  $(\det(\alpha_i g_i))J_i$  for  $i = 1, 2$ . The above construction of  $\tilde{\rho}_{g_i}$  then yields a  $J_{i+1}$ -polarised abelian variety  $(A'_i, \iota'_i, \lambda'_i, \eta'_i)$  with level  $N_{i+1}$  structure over  $Y_{J_i, N_i}$ , where  $A'_i = A_i/\eta_i(C_i)$  with  $C_i = (O_F/N_i)^2 \cdot (\alpha_i g_i)^{-1}N_i$ . It is straightforward to check that the pull-back via  $\tilde{\rho}_{g_1}$  of  $(A'_2, \iota'_2, \lambda'_2, \eta'_2)$  is isomorphic to a  $J_3$ -polarised HBAV with level  $N_3$  structure defining  $\tilde{\rho}_{g_1g_2}$ , so that we may take  $\tilde{\rho}_{g_1g_2} = \tilde{\rho}_{g_2} \circ \tilde{\rho}_{g_1}$  and conclude that  $\rho_{g_1g_2} = \rho_{g_2} \circ \rho_{g_1}$ .

#### 4.2. Adelic action on Hilbert modular forms

We revert to the original setting of §4.1, with  $g, U_1$  and  $U_2$  satisfying  $g^{-1}U_1g \subset U_2$ , and use the notation in the definition of  $\rho_g$  (and in particular a choice of  $N_1, N_2, \alpha$  and  $\beta$ ), but writing  $S_i = Y_{J_i, N_i}$  for  $i = 1, 2$  and  $s_i : A_i \rightarrow S_i$  and  $s'_1 : A'_1 \rightarrow S_1$  for the structural morphisms. We let  $\pi_\alpha$  denote the canonical projection  $A_1 \rightarrow A'_1 \simeq \tilde{\rho}_g^* A_2$ ; the dependence on  $\alpha$  is such that if  $\delta \in O_F \cap O_{F,p}^\times$  (and  $N_1$  is such that  $(\delta\alpha g)^{-1}N_1/N_2 \in M_2(\hat{O}_F)$ ),

then  $\pi_{\delta\alpha} = i(\delta)\pi_{\alpha}$ . It follows that the  $O_F \otimes \mathcal{O}_{S_1}$ -linear morphisms

$$\begin{aligned} \tilde{\rho}_g^* s_{2,*} \Omega_{A_2/S_2}^1 &\simeq s'_{1,*} \Omega_{A'_1/S_1}^1 \rightarrow s_{1,*} \Omega_{A_1/S_1}^1, \\ \tilde{\rho}_g^* R^1 s_{2,*} \Omega_{A_2/S_2}^\bullet &\simeq R^1 s'_{1,*} \Omega_{A'_1/S_1}^\bullet \rightarrow R^1 s_{1,*} \Omega_{A_1/S_1}^\bullet \end{aligned} \tag{4.1}$$

induced by  $(\alpha \otimes 1)^{-1} \pi_{\alpha}^*$  are independent of the choice of  $\alpha$  (as well as  $N_2$  and  $\beta$ , and even  $N_1$  in the sense of compatibility with pull-back by the natural projection). Note that these are in fact isomorphisms since the degree of the isogeny  $\pi_{\alpha}$  is invertible in  $\mathcal{O}_S$ . Furthermore, the commutativity of the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_{v,u}} & (v,u)^* A_1 \\ \pi_{\alpha} \downarrow & & \downarrow (v,u)^* \pi_{\alpha} \\ \tilde{\rho}_g^* A_2 & \xrightarrow{\tilde{\rho}_g^*(\alpha_{v,g^{-1}ug})} & \tilde{\rho}_g^*(v,g^{-1}ug)^* A_2 \simeq (v,u)^* \tilde{\rho}_g^* A_2 \end{array}$$

implies that the isomorphisms in (4.1) are compatible with the action of  $G_{U_1, N_1}$  (where  $G_{U_1, N_1}$  acts on the sources via the homomorphism  $(v, u) \mapsto (v, g^{-1}ug)$  to  $G_{U_2, N_2}$  and pull-back by  $\tilde{\rho}_g^*$ ). It follows that the same is true for the  $\mathcal{O}_{S_1}$ -linear isomorphisms  $\tilde{\rho}_g^* \mathcal{L}_{J_2, N_2}^{k,l} \xrightarrow{\sim} \mathcal{L}_{J_1, N_1}^{k,l}$  induced by those in (4.1) for  $k, l \in \mathbb{Z}^{\Sigma}$ , which therefore descend to define isomorphisms

$$\rho_g^* \mathcal{L}_{U_2, R}^{k,l} \xrightarrow{\sim} \mathcal{L}_{U_1, R}^{k,l} \tag{4.2}$$

for any  $\mathcal{O}$ -algebra  $R$  in which the image of  $\mu^{k+2l}$  is trivial for all  $\mu \in O_F^{\times} \cap U_2$  (and hence all  $\mu \in O_F^{\times} \cap U_1$ ). We thus obtain an  $R$ -linear map  $[U_1 g U_2]: M_{k,l}(U_2; R) \rightarrow M_{k,l}(U_1; R)$  defined as the product of  $\|\det(g)\| = \text{Nm}_{F/\mathbb{Q}}(\det(g))^{-1}$  with the composite

$$H^0(Y_{U_2, R}, \mathcal{L}_{U_2, R}^{k,l}) \rightarrow H^0(Y_{U_1, R}, \rho_g^* \mathcal{L}_{U_2, R}^{k,l}) \xrightarrow{\sim} H^0(Y_{U_1, R}, \mathcal{L}_{U_1, R}^{k,l}).$$

Returning now to the setting where  $U_1, U_2$  and  $U_3$  are open compact subgroups of  $\text{GL}_2(\hat{O}_F)$  and  $g_1, g_2 \in \text{GL}_2(\mathbb{A}_F^{\infty})$  are such that  $g_1^{-1}U_1g_1 \subset U_2$  and  $g_2^{-1}U_2g_2 \subset U_3$ , we find that the composite

$$A_1 \xrightarrow{\pi_{\alpha_1}} \tilde{\rho}_{g_1}^* A_2 \xrightarrow{\tilde{\rho}_{g_1}^* \pi_{\alpha_2}} \tilde{\rho}_{g_1}^* \tilde{\rho}_{g_2}^* A_3 \simeq \tilde{\rho}_{g_1 g_2}^* A_3$$

is  $\pi_{\alpha_1 \alpha_2}$ . This in turn implies that the composite

$$\rho_{g_1 g_2}^* \mathcal{L}_{U_3, R}^{k,l} \simeq \rho_{g_1}^* \rho_{g_2}^* \mathcal{L}_{U_3, R}^{k,l} \xrightarrow{\sim} \rho_{g_1}^* \mathcal{L}_{U_2, R}^{k,l} \xrightarrow{\sim} \mathcal{L}_{U_1, R}^{k,l}$$

is the isomorphism in (4.2) used to define  $[U_1 g_1 g_2 U_3]$ , which therefore coincides with  $[U_1 g_1 U_2] \circ [U_2 g_2 U_3]$ .

For  $R = \mathcal{O}$  and  $(k, l)$  parituous, we thus obtain an action of the group  $\{g \in \text{GL}_2(\mathbb{A}_F^{\infty}) \mid g_{\mathfrak{p}} \in \text{GL}_2(\mathcal{O}_{F, \mathfrak{p}})\}$  on

$$M_{k,l}(\mathcal{O}) := \varinjlim M_{k,l}(U; \mathcal{O}),$$

where the direct limit is over all sufficiently small open compact subgroups  $U$  of  $\mathrm{GL}_2(\mathbb{A}_F^\infty)$  containing  $\mathrm{GL}_2(\mathcal{O}_{F,p})$ . Similarly we have an action on  $M_{k,l}(\mathbb{C}) := \varinjlim M_{k,l}(U; \mathbb{C})$ , which is compatible by extension of scalars with the one just defined on  $M_{k,l}(\mathcal{O})$ . One can check that the action is also compatible under the isomorphisms  $M_{k,l}(U; \mathbb{C}) \simeq A_{k,l}(U)$  with the usual action defined by right multiplication on the space of automorphic forms  $A_{k,l} := \varinjlim A_{k,l}(U)$ .

Recall that for  $R = E$  and arbitrary  $(k, l)$ , the space  $M_{k,l}(U; E)$  is defined for sufficiently small  $U$  (for example  $p$ -neat as in Definition 3.2.3), so we may similarly define

$$M_{k,l}(E) := \varinjlim M_{k,l}(U; E).$$

Then  $M_{k,l}(E)$  is a smooth admissible representation of  $\{g \in \mathrm{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \mathrm{GL}_2(\mathcal{O}_{F,p})\}$  over  $E$ , and we recover  $M_{k,l}(U; E) = M_{k,l}(E)^U$  for sufficiently small  $U \supset \mathrm{GL}_2(\mathcal{O}_{F,p})$ . (Note that  $M_{k,l}(E)^U = 0$  if  $\overline{\mu}^{k+2l} \neq 1$  for some  $\mu \in U \cap \mathcal{O}_F^\times$ .)

We may similarly define  $M_{k,l}(R)$  for any  $(k, l)$  and  $R$  in which  $p$  is nilpotent. We again have  $M_{k,l}(U; R) = M_{k,l}(R)^U$  for sufficiently small  $U$  (indeed, for any  $U$  for which we have already defined  $M_{k,l}(U; R)$ ), so we may define  $M_{k,l}(U; R)$  to be  $M_{k,l}(R)^U$  for any open compact subgroup  $U$  of  $\mathrm{GL}_2(\mathbb{A}_F^\infty)$  containing  $\mathrm{GL}_2(\mathcal{O}_{F,p})$ . Note then that  $M_{k,l}(U; R) = 0$  if  $\overline{\mu}^{k+2l} \neq 1$  for some  $\mu \in U \cap \mathcal{O}_F^\times$ , but not necessarily under the weaker assumption (if  $pR \neq 0$ ) that  $\mu^{k+2l}$  has non-trivial image in  $R$  for some  $\mu \in U \cap \mathcal{O}_F^\times$ . We shall restrict our attention however to the case  $R = E$ .

### 4.3. Hecke operators

Suppose that  $U_1$  and  $U_2$  are open compact subgroups of  $\mathrm{GL}_2(\mathbb{A}_F^\infty)$  containing  $\mathrm{GL}_2(\mathcal{O}_{F,p})$  and that  $g$  is an element of  $\mathrm{GL}_2(\mathbb{A}_F^\infty)$  such that  $g_p \in \mathrm{GL}_2(\mathcal{O}_{F,p})$ . We may then define the double coset operator

$$[U_1 g U_2] : M_{k,l}(U_2; E) \rightarrow M_{k,l}(U_1; E)$$

to be the map  $f \mapsto \sum_{i \in I} g_i f$  where  $U_1 g U_2 = \coprod_{i \in I} g_i U_2$ . It is straightforward to check that the map is independent of the choice of representatives  $g_i$ , that the image is indeed in  $M_{k,l}(U_1; E)$ , and that the definition agrees with the one already made when  $U_1$  and  $U_2$  are sufficiently small and  $g^{-1}U_1 g \subset U_2$ . (Recall that a normalising factor of  $\|\det(g)\|$  is incorporated into the definition of the action.)

If  $U_1$  and  $U_2$  are sufficiently small we may reinterpret  $[U_1 g U_2]$  in the usual way using trace morphisms as follows. Letting  $U'_1 = U_1 \cap g U_2 g^{-1}$ , we have  $g^{-1}U'_1 g \subset U_2$ , so that  $[U_1 g U_2] = [U_1 1 U'_1] \circ [U'_1 g U_2]$  and  $[U'_1 g U_2]$  is the composite

$$H^0(\overline{Y}_{U_2}, \overline{\mathcal{L}}_{U'_1}^{k,l}) \rightarrow H^0(\overline{Y}_{U'_1}, \rho_g^* \overline{\mathcal{L}}_{U_2}^{k,l}) \rightarrow H^0(\overline{Y}_{U'_1}, \overline{\mathcal{L}}_{U'_1}^{k,l})$$

where the second map is  $\|\det(g)\|$  times the one induced from (4.2). On the other hand,  $[U_1 1 U'_1]$  is precisely the composite

$$H^0(\overline{Y}_{U'_1}, \overline{\mathcal{L}}_{U'_1}^{k,l}) \rightarrow H^0(\overline{Y}_{U'_1}, \rho_1^* \overline{\mathcal{L}}_{U_1}^{k,l}) \rightarrow H^0(\overline{Y}_{U_1}, \overline{\mathcal{L}}_{U_1}^{k,l}),$$

where the first map is given by the inverse of the isomorphism  $\rho_1^* \mathcal{L}_{U_1}^{k,l} \rightarrow \mathcal{L}_{U_1}^{k,l}$  (from (4.2)), and the last map is the trace times the index of  $U'_1 \cap O_F^\times$  in  $U_1 \cap O_F^\times$ .

For primes  $v$  of  $F$  such that  $v \nmid p$  and  $\text{GL}_2(O_{F,v}) \subset U$ , we define the Hecke operators

$$T_v := \left[ U \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} U \right] \quad \text{and} \quad S_v := \left[ U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U \right] \tag{4.3}$$

on  $M_{k,l}(U; E)$ , where  $\varpi_v$  is a uniformiser of  $O_{F,v}$ . These operators are independent of the choice of  $\varpi_v$ , and commute with each other (for varying  $v$ ). Note that under the above interpretation via the trace map (for sufficiently small  $U$ ), we have  $U'_1 = U \cap U_0(v)$  and  $U'_1 \cap O_F^\times = U \cap O_F^\times$ , so that  $T_v$  can be written as  $\text{Nm}_{F/\mathbb{Q}}(v)^{-1}$  times the composite

$$H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,l}) \rightarrow H^0(\overline{Y}_{U'}, \rho_g^* \overline{\mathcal{L}}_U^{k,l}) \rightarrow H^0(\overline{Y}_{U'}, \rho_1^* \overline{\mathcal{L}}_U^{k,l}) \rightarrow H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,l}),$$

where  $U' = U \cap U_0(v)$ , the first map is the natural pull-back, the second map is induced by the maps  $\rho_g^* \overline{\mathcal{L}}_U^{k,l} \rightarrow \overline{\mathcal{L}}_{U'}^{k,l} \simeq \rho_1^* \overline{\mathcal{L}}_U^{k,l}$  of (4.2), and the last map is the trace. We remark also that if  $(k, l)$  is paritious, then the above definitions with  $E$  replaced by  $\mathcal{O}$  gives Hecke operators compatible with the usual ones denoted  $T_v$  and  $S_v$  on the corresponding spaces of automorphic forms.

#### 4.4. Adelic action on components

We will describe below the action of the group  $\{g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \text{GL}_2(O_{F,p})\}$  on the spaces  $M_{0,l}(E)$ , but first we consider the right action via  $\rho_g$  on geometric components. More precisely, suppose as usual that  $g^{-1}U_1g \subset U_2$  and  $N_1, N_2$  and  $\alpha$  are as in the definition of  $\rho_g$ ; assume moreover that  $\mu_{N_1}(\mathbb{Q}) \subset \mathcal{O}$  and consider the map  $Z_{U_1}(\mathcal{O}) \rightarrow Z_{U_2}(\mathcal{O})$  induced by  $\rho_g$  (where  $Z_{U_i}$  was defined in §2.6). Maintaining the notation in the construction of  $\rho_g$ , one finds that the commutativity of the diagram in the proof of Lemma 4.1.1 implies that of

$$\begin{array}{ccccc} J_1 \otimes_{O_F} \widehat{O}_F & \xrightarrow{\sim} & \text{Sym}(A_1/S_1) \otimes_{O_F} \widehat{O}_F & \twoheadrightarrow & \text{Hom}(\wedge_{O_F}^2 A_1[N_1], \mu_{N_1}) & \xrightarrow{\sim} & \text{Hom}(O_F/N_1, \mu_{N_1}) \\ \uparrow \wr & & \uparrow \wr & & \downarrow & & \downarrow \\ J_2 \otimes_{O_F} \widehat{O}_F & \xrightarrow{\sim} & \text{Sym}(A'_1/S_1) \otimes_{O_F} \widehat{O}_F & \twoheadrightarrow & \text{Hom}(\wedge_{O_F}^2 A'_1[N_2], \mu_{N_2}) & \xrightarrow{\sim} & \text{Hom}(O_F/N_2, \mu_{N_2}) \end{array}$$

where the horizontal arrows of the top (resp. bottom) row are induced (from left to right) by  $\lambda_1$  (resp.  $\lambda'_1$ ), the Weil pairing on  $A_1$  (resp.  $A'_1$ ), and  $\eta_1$  (resp.  $\eta'_1$ ), the first vertical arrow by  $\beta \det(\alpha g)^{-1}$ , the second by  $\det(\alpha g)^{-1} \pi^*$ , the third by the surjections  $\wedge_{O_F}^2 A_1[N_1] \rightarrow \wedge_{O_F}^2 A'_1[N_2]$  (arising from the isomorphisms  $\det(\alpha g_\ell)^{-1} \wedge_{O_{F,\ell}}^2 T_\ell(\pi)$  for  $\ell \mid N_2$ ) and  $\cdot^{N_1/N_2} : \mu_{N_1} \rightarrow \mu_{N_2}$ , and the last by the natural projection and  $\cdot^{N_1/N_2}$ . It follows that if  $\zeta \in Z_{J_1, N_1}(\mathcal{O})$  (i.e.,  $\zeta : J_1/N_1 \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mu_{N_1}(\mathcal{O})$ ) then  $\tilde{\rho}_g(\zeta)$  is the isomorphism  $J_2/N_2 \simeq \mathfrak{d}^{-1} \otimes \mu_{N_2}(\mathcal{O})$  induced by  $x \mapsto \zeta(\beta \alpha^{-2} \det(g)^{-1} x)^{N_1/N_2}$ . It follows in turn that the map  $Z_{U_1}(\mathcal{O}) \rightarrow Z_{U_2}(\mathcal{O})$  induced by  $\rho_g$  corresponds to multiplication by  $\det(g)^{-1}$  under the bijections of (2.2), with  $\zeta_{N_2}$  chosen to be  $\zeta_{N_1}^{N_1/N_2}$ ,

4.5. *Adelic action on forms of weight  $(0, l)$*

Recall that the map  $[U_1 g U_2]$  arises by descent (and reduction mod  $\pi$ ) from maps

$$H^0(S_2, \mathcal{L}_{J_2, N_2}^{0, l}) \rightarrow H^0(S_1, \tilde{\rho}_g^* \mathcal{L}_{J_2, N_2}^{0, l}) \rightarrow H^0(S_1, \mathcal{L}_{J_1, N_1}^{0, l}) \tag{4.4}$$

where  $S_i = Y_{J_i, N_i}$ . Moreover, we have isomorphisms  $\mathcal{L}_{J_i, N_i}^{0, l} \simeq \mathcal{O}_{S_i}$  obtained by tensoring powers of the components of the composite

$$\bigoplus_{\tau \in \Sigma} \delta_\tau = \wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_{S_i}} \mathcal{H}_{\text{DR}}^1(A_i/S_i) \simeq J_i \mathfrak{d}^{-1} \otimes \mathcal{O}_{S_i} \simeq \bigoplus_{\tau \in \Sigma} \mathcal{O}_{S_i},$$

where the first isomorphism is the canonical one following (3.2), and the second arises from the isomorphisms  $J_i \mathfrak{d}^{-1} \otimes \mathcal{O} \simeq \mathcal{O}_F \otimes \mathcal{O} \simeq \bigoplus_{\tau \in \Sigma} \mathcal{O}$  induced by the inclusions  $J_i \mathfrak{d}^{-1} \subset F$  (a choice permitted by our assumption that  $J_1, J_2$  and  $\mathfrak{d}$  are prime to  $p$ ). Since the global sections of  $\mathcal{O}_{S_i}$  are constant on components, we may realise (4.4) as a map

$$\{Z_{J_2, N_2}(\mathcal{O}) \rightarrow \mathcal{O}\} \rightarrow \{Z_{J_1, N_1}(\mathcal{O}) \rightarrow \mathcal{O}\}.$$

Under the canonical isomorphisms  $\wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_{S_i}} \mathcal{H}_{\text{DR}}^1(A_i/S_i) \simeq J_i \mathfrak{d}^{-1} \otimes \mathcal{O}_{S_i}$ , we find that the map

$$\tilde{\rho}_g^*(\wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_{S_2}} \mathcal{H}_{\text{DR}}^1(A_2/S_2)) \rightarrow \wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_{S_1}} \mathcal{H}_{\text{DR}}^1(A_1/S_1)$$

in the definition of  $[U_1 g U_2]$  corresponds to the map  $J_2 \mathfrak{d}^{-1} \otimes \mathcal{O}_{S_1} \rightarrow J_1 \mathfrak{d}^{-1} \otimes \mathcal{O}_{S_1}$  induced by multiplication by  $\beta \alpha^{-2} \in (\mathcal{O}_F \otimes \mathcal{O})^\times$ . We therefore realise (4.1) as the map sending  $s : Z_{J_2, N_2}(\mathcal{O}) \rightarrow \mathcal{O}$  to the map  $Z_{J_1, N_1}(\mathcal{O}) \rightarrow \mathcal{O}$  sending  $\zeta$  to  $\|\det(g)\|(\beta \alpha^{-2})^l s(\tilde{\rho}_g(\zeta))$ . Note in particular that if  $\det(g) = 1$  and  $U_1 \subset U_2$ , then we may choose  $\beta = \alpha^2$  and conclude that  $[U_1 g U_2]$  coincides with the natural inclusion  $M_{0, l}(U_2; E) \rightarrow M_{0, l}(U_1; E)$  defined by  $[U_1 l U_2]$ . It follows that the action of  $\{g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \text{GL}_2(\mathcal{O}_{F, p})\}$  on  $M_{0, l}(E)$  factors via  $\det$  through that of  $\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in \mathcal{O}_{F, p}^\times\}$ , giving an action of  $\{g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \text{GL}_2(\mathcal{O}_{F, p})\}$  on  $M_{0, l}(U; E)$  factoring through

$$\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in \mathcal{O}_{F, p}^\times\} / \det(U).$$

We now determine the corresponding representation of the latter group on  $M_{0, l}(U; E)$ . Note that we have an exact sequence

$$1 \rightarrow \mathcal{O}_{F, +}^\times \cap \det(U) \rightarrow F_+^\times \cap \mathcal{O}_{F, p}^\times \rightarrow \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in \mathcal{O}_{F, p}^\times\} / \det(U) \rightarrow (\mathbb{A}_F^\infty)^\times / F_+^\times \det(U) \rightarrow 1, \tag{4.5}$$

where the maps are all induced by the canonical inclusions. Note that the last quotient is finite. If  $\bar{v}^l = 1$  for all  $v \in \det(U) \cap \mathcal{O}_{F, +}^\times$ , then  $\mu \mapsto \bar{\mu}^l$  defines an  $E^\times$ -valued character of  $(F_+^\times \cap \mathcal{O}_{F, p}^\times) / (\mathcal{O}_{F, +}^\times \cap \det(U))$ , hence of a finite index subgroup of

$$\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in \mathcal{O}_{F, p}^\times\} / \det(U).$$

**Lemma 4.5.1.** *If  $\bar{v}^l = 1$  for all  $v \in \det(U) \cap O_{F,+}^\times$ , then  $M_{0,l}(U; E)$  is isomorphic, as a representation of  $\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\}/\det(U)$ , to the induction of the character*

$$\psi_l : (F_+^\times \cap O_{F,p}^\times)/(O_{F,+}^\times \cap \det(U)) \rightarrow E^\times$$

defined by  $\psi_l(\mu) = \text{Nm}_{F/\mathbb{Q}}(\mu)^{-1}\bar{\mu}^l$ ; otherwise  $M_{0,l}(U; E) = 0$ .

*Proof.* Note that the conclusion of the lemma is equivalent to the assertion that  $M_{0,l}(U; E)$  is isomorphic to

$$I_U = \{f : G \rightarrow E \mid f(\mu x w) = \psi_l(\mu) f(x) \text{ for all } \mu \in G \cap F_+^\times, x \in G, w \in \det(U)\}$$

as a representation of  $G = \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\}$ . We may therefore replace  $L$  by a finite extension and  $U$  by an open subgroup  $U_2$  for which the hypotheses of Proposition 3.6.1 are satisfied.

Next observe that if  $\det(g) = \mu \in F_+^\times \cap O_{F,p}^\times$  and  $g^{-1}U_1g \subset U_2$ , then we may take  $\beta = \mu\alpha^2$  in the definition of  $[U_1gU_2]$ , so that  $\tilde{\rho}_g$  induces the natural projection  $Z_{J,N_1}(\mathcal{O}) \rightarrow Z_{J,N_2}(\mathcal{O})$  for each  $J$ , and the map in (4.4) is the composite of the natural inclusion with multiplication by  $\text{Nm}_{F/\mathbb{Q}}(\mu)^{-1}\bar{\mu}^l$ . Therefore  $F_+^\times \cap O_{F,p}^\times$  acts on  $M_{0,l}(U_2; E)$  via the character  $\psi_l$ .

Let  $e$  be a non-zero element of  $M_{0,l}(U_2; E)$  supported on a single component of  $Z_{U_2}(\mathcal{O})$ . Since  $F_+^\times \cap O_{F,p}^\times$  acts via  $\psi_l$  on  $e$ , there is a  $G$ -equivariant homomorphism  $I_{U_2} \rightarrow M_{0,l}(U_2; E)$  whose image contains  $e$ . Since  $G$  acts transitively on  $Z_{U_2}(\mathcal{O})$ , the  $G$ -orbit of  $e$  spans  $M_{0,l}(U_2; E)$ , so the homomorphism is surjective. Since  $I_{U_2}$  and  $M_{0,l}(U_2; E)$  both have dimension equal to the cardinality of  $(\mathbb{A}_F^\infty)^\times/F_+^\times \det(U_2)$ , it follows that the map is in fact an isomorphism. ■

#### 4.6. Twisting by characters

It follows from Lemma 4.5.1 that for any character

$$\xi : \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\}/\det(U) \rightarrow E^\times$$

such that  $\xi(\alpha) = \bar{\alpha}^l$  for all  $\alpha \in F_+^\times \cap O_{F,p}^\times$ , the eigenspace consisting of those  $e \in M_{0,l}(U; E)$  satisfying

$$ge = \|\det(g)\|\xi(\det(g))e \quad \text{for all } g \in \text{GL}_2(\mathbb{A}_F^\infty) \text{ such that } g_p \in \text{GL}_2(O_{F,p})$$

is one-dimensional. We let  $e_\xi$  be a basis element.

**Lemma 4.6.1.** *If  $U, l$  and  $\xi$  are as above, then for any  $k, m \in \mathbb{Z}^\Sigma$ , the map  $f \mapsto e_\xi \otimes f$  defines an isomorphism  $M_{k,m}(U; E) \rightarrow M_{k,l+m}(U; E)$  such that*

$$[UgU](e_\xi \otimes f) = \xi(\det(g))e_\xi \otimes [UgU]f$$

for all  $f \in M_{k,m}(U; E)$ ,  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  such that  $g_p \in \text{GL}_2(O_{F,p})$ ; in particular, we have  $T_v(e_\xi \otimes f) = \xi(\varpi_v)e_\xi \otimes T_v f$  and  $S_v(e_\xi \otimes f) = \xi(\varpi_v)^2 e_\xi \otimes S_v f$  for all  $v$  such that  $v \nmid p$  and  $\text{GL}_2(O_{F,v}) \subset U$ .

*Proof.* We first prove that the map is an isomorphism. The existence of  $\xi$  implies that  $\bar{v}^l = 1$  for all  $v \in \det(U) \cap O_{F,+}^\times$ , so replacing  $L$  by a finite extension, we may assume that the hypotheses of Proposition 3.6.1 are satisfied and hence view  $e_\xi$  as a function  $Z_U(\mathcal{O}) \rightarrow E$ . Since  $e_\xi$  is non-zero and the action of the group  $\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\}$  on  $Z_U(\mathcal{O})$  is transitive, it follows that  $e_\xi$  is everywhere non-zero. We therefore have a section  $e_\xi^{-1} \in M_{0,-1}(U; E)$  such that  $f \mapsto f \otimes e_\xi^{-1}$  defines the inverse of our map.

We now establish the compatibility with the Hecke action. The definition of  $[UgU]$  gives

$$[UgU](e_\xi \otimes f) = \sum_i g_i(e_\xi \otimes f) = \sum_i \|\det(g)\|^{-1} g_i e_\xi \otimes g_i f,$$

where  $UgU = \coprod_i g_i U$ . Noting that  $g_i e_\xi = g e_\xi = \|\det(g)\| \xi(\det(g)) e_\xi$  since  $\det(g_i) \in \det(g) \det(U)$ , it follows that

$$[UgU](e_\xi \otimes f) = \xi(\det(g)) e_\xi \otimes \sum_i g_i f = \xi(\det(g)) e_\xi \otimes [UgU]f$$

as required. ■

### 5. Partial Hasse invariants

We next adapt the definition of partial Hasse invariants from [1] to our setting.

#### 5.1. Definition of partial Hasse invariants

We write  $\text{Ver}_A$  for the Verschiebung isogeny of an abelian scheme  $A$  over a base  $S$  of characteristic  $p$ , i.e., the morphism  $A^{(p)} \rightarrow A$  defined as the dual of the relative Frobenius morphism  $A^\vee \rightarrow (A^\vee)^{(p)} = (A^{(p)})^\vee$ , where  $A^{(p)}$  denotes the pull-back  $A \times_S S$  with respect to the absolute Frobenius morphism  $\text{Fr}_S : S \rightarrow S$ . Taking  $A$  to be the universal HBAV over  $S = \bar{Y}_{J,N}$ , the pull-back  $\text{Ver}_A^*$  defines an  $O_F \otimes \mathcal{O}_S$ -linear morphism

$$s_* \Omega_{A/S}^1 \rightarrow s_* \Omega_{A^{(p)}/S}^1 = \text{Fr}_S^* s_* \Omega_{A/S}^1,$$

where  $s : A \rightarrow S$  denotes the structure morphism. Writing  $s_* \Omega_{A/S}^1 = \bigoplus_\tau \bar{\omega}_\tau$ , we see that the  $\tau$ -component of  $\text{Fr}_S^* s_* \Omega_{A/S}^1$  is canonically isomorphic to  $\bar{\omega}_{\text{Fr}^{-1} \circ \tau}^{\otimes p}$ , where  $\text{Fr}$  denotes the absolute Frobenius on  $\bar{\mathbb{F}}_p$ . The  $\tau$ -component of  $\text{Ver}_A^*$  is therefore a section of  $\bar{\mathcal{L}}_{J,N}^{k,0} = \bar{\omega}_{\text{Fr}^{-1} \circ \tau}^{\otimes p} \bar{\omega}_\tau^{\otimes (-1)}$ , where

- if  $\text{Fr} \circ \tau = \tau$ , then  $k_\tau = p - 1$  and  $k_{\tau'} = 0$  if  $\tau' \neq \tau$ ;
- if  $\text{Fr} \circ \tau \neq \tau$ , then  $k_\tau = -1$ ,  $k_{\text{Fr}^{-1} \circ \tau} = p$ , and  $k_{\tau'} = 0$  if  $\tau' \notin \{\text{Fr}^{-1} \circ \tau, \tau\}$ .

For each  $\tau$ , we denote this weight by  $k_{\text{Ha}_\tau}$ , and let  $\text{Ha}_{J,N,\tau} \in H^0(\bar{Y}_{J,N}, \bar{\mathcal{L}}_{J,N}^{k_{\text{Ha}_\tau}, 0})$  be the element just constructed. Then  $\text{Ha}_{J,N,\tau}$  has non-zero restriction to each component of  $\bar{Y}_{J,N}$ ; moreover, if we let  $Z_\tau$  denote the associated divisor of zeros, then

$Z_\tau$  is non-trivial on each component and  $\sum_\tau Z_\tau$  is reduced. (This follows from the corresponding result proved in [1, §8] for the partial Hasse invariants on the variety they denote  $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ : Choosing  $\zeta_N \in \mu_N(E)$  for sufficiently large  $E$  yields an étale cover  $\bar{Y}_{J,N} \rightarrow \mathfrak{M}(E, \mu_N)$  which identifies  $\mathfrak{M}(E, \mu_N)$  with the quotient of  $\bar{Y}_{J,N}$  by the image of  $U_1(N)$  in  $G_{U_1(N),N}$  and our  $\text{Ha}_\tau$  with the pull-back of their partial Hasse invariant  $h_{\mathfrak{B},i}$  for the pair  $(\mathfrak{B}, i)$  corresponding to  $\tau$ .)

Note that  $\mu^{k_{\text{Ha}_\tau}} \equiv 1 \pmod{\pi}$  for all  $\mu \in O_F^\times$ , so the line bundle  $\bar{\mathcal{L}}_U^{k_{\text{Ha}_\tau},0}$  is defined for all  $U$  under consideration. By the compatibility of the Verschiebung with base-change and isomorphisms, we see that the sections  $\text{Ha}_{J,N,\tau}$  on  $\coprod \bar{Y}_{J,N}$  descend to define a mod  $p$  Hilbert modular form of weight  $(k_{\text{Ha}_\tau}, 0)$  and level  $U$ , which we denote by  $\text{Ha}_{U,\tau}$ . Moreover, from the compatibility of Verschiebung with isogenies, in particular with  $\pi_\alpha$  as defined in §4.2, we see that  $[U_1 g U_2] \text{Ha}_{U_2,\tau} = \|\det(g)\| \text{Ha}_{U_1,\tau}$  for any  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  such that  $g_p \in \text{GL}_2(O_{F,p})$  and  $g^{-1}U_1 g \subset U_2$ . In particular, the element

$$\text{Ha}_{U,\tau} \in M_{k_{\text{Ha}_\tau},0}(E) := \lim_{\rightarrow} M_{k_{\text{Ha}_\tau},0}(U; E)$$

is independent of the choice of  $U$ , so we henceforth omit the subscript  $U$  and write simply  $\text{Ha}_\tau$  for this mod  $p$  Hilbert modular form, which we call the *partial Hasse invariant (associated to  $\tau$ )*.

We record the following immediate consequence of the assertions above:

**Proposition 5.1.1.** *The partial Hasse invariant  $\text{Ha}_\tau$  satisfies  $g \text{Ha}_\tau = \|\det(g)\| \text{Ha}_\tau$  for all  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  such that  $g_p \in \text{GL}_2(O_{F,p})$ . For any weight  $(k, l)$ , multiplication by  $\text{Ha}_\tau$  defines an injective map*

$$M_{k,l}(E) \rightarrow M_{k+k_{\text{Ha}_\tau},l}(E)$$

*commuting with the action of  $g$  for all such  $g$ . In particular, for any open compact subgroup  $U$  of  $\text{GL}_2(\mathbb{A}_F^\infty)$  containing  $\text{GL}_2(O_{F,p})$ , multiplication by  $\text{Ha}_\tau$  defines an injective map*

$$M_{k,l}(U; E) \rightarrow M_{k+k_{\text{Ha}_\tau},l}(U; E)$$

*commuting with the operators  $T_v$  and  $S_v$  for all  $v \nmid p$  such that  $\text{GL}_2(O_{F,v}) \subset U$ .*

### 5.2. Minimal weights

We now recall the definition of the minimal weight of a mod  $p$  Hilbert modular form, again adapting notions from [1] to our setting (see also [19]). This is an analogue of the weight filtration for mod  $p$  modular forms in the classical setting  $F = \mathbb{Q}$ . For  $F = \mathbb{Q}$ , the vanishing of the spaces of mod  $p$  modular forms of negative weight forces the weight filtration to be non-negative, but in the Hilbert case, the partial negativity of the weights of partial Hasse invariants already shows the situation is more subtle. We let

$$\Xi_{\text{AG}} = \left\{ \sum_{\tau \in \Sigma} n_\tau k_{\text{Ha}_\tau} \mid n_\tau \in \mathbb{Z}_{\geq 0} \text{ for all } \tau \in \Sigma \right\}$$



be the set of non-negative integer linear combinations of the weights of the partial Hasse invariants. Note that the weights  $k_{\text{Ha}_\tau}$  are linearly independent, so each  $k \in \Xi_{\text{AG}}$  is of the form  $\sum_{\tau \in \Sigma} n_\tau k_{\text{Ha}_\tau}$  for a unique  $n \in \mathbb{Z}_{\geq 0}^\Sigma$ . We define a partial ordering  $\leq_{\text{Ha}}$  on  $\mathbb{Z}^\Sigma$  by stipulating that  $k' \leq_{\text{Ha}} k$  if and only if  $k - k' \in \Xi_{\text{AG}}$ .

For any non-zero  $f \in M_{k,l}(U; E)$ , consider the set  $W(f)$  defined as

$$\left\{ k' = k - \sum_{\tau} n_\tau k_{\text{Ha}_\tau} \mid n \in \mathbb{Z}_{\geq 0}^\Sigma, f = f' \prod_{\tau} \text{Ha}_\tau^{n_\tau} \text{ for some } f' \in M_{k',l}(U; E) \right\}.$$

Since the divisor  $\sum_{\tau} Z_\tau$  is reduced, the set  $W(f)$  contains a unique minimal element under the partial ordering  $\leq_{\text{Ha}}$  (cf. [1, 8.19, 8.20]), which we call the *minimal weight* of  $f$ , and denote  $\nu(f)$ . Note that replacing  $U$  by an open compact subgroup  $U' \subset U$  does not alter  $\nu(f)$ , since any  $f' \in M_{k',l}(U'; E)$  satisfying  $f = f' \prod_{\tau} \text{Ha}_\tau^{n_\tau}$  will be invariant under  $U$ , hence in  $M_{k',l}(U; E)$ . We may therefore define  $\nu(f)$  for  $f \in M_{k,l}(E)$  without reference to  $U$ . Note also that  $\nu(f)$  is not affected by replacing  $E$  by an extension  $E'$ .

We note also that the minimal weight of a form is independent of  $l$  in the following sense: Recall from Lemma 4.6.1 that we have isomorphisms  $M_{k,l}(E) \rightarrow M_{k,l'}(E)$  defined by multiplication by eigenvectors  $e_\xi \in M_{0,l'-l}(E)$  associated to suitable characters  $\xi$  of  $(\mathbb{A}_F^\times)^\times$ . Since these isomorphisms commute with multiplication by the partial Hasse invariants, it follows that  $\nu(e_\xi \otimes f) = \nu(f)$  for all  $f \in M_{k,l}(E)$ .

Finally we define the *minimal cone* in  $\mathbb{Z}^\Sigma$  to be

$$\Xi_{\min} = \{ k \in \mathbb{Z}^\Sigma \mid pk_\tau \geq k_{\text{Fr}^{-1} \circ \tau} \text{ for all } \tau \in \Sigma \}.$$

(Note that  $\Xi_{\min} \subset \mathbb{Z}_{\geq 0}^\Sigma$ .) A recent result of the first author and Kassaei [19] shows that in fact  $\nu(f) \in \Xi_{\min}$  for all non-zero mod  $p$  Hilbert modular forms  $f$ .

### 6. Associated Galois representations

The aim of this section is to prove the existence of Galois representations associated to Hecke eigenforms of arbitrary weight. We first state the theorem and review some ingredients needed for the proof.

#### 6.1. Statement of the theorem

**Theorem 6.1.1.** *Suppose that  $U$  is an open compact subgroup of  $\text{GL}_2(\widehat{O}_F)$  containing  $\text{GL}_2(O_{F,p})$ , and  $Q$  is a finite set of primes containing all  $v \mid p$  and all  $v$  such that  $\text{GL}_2(O_{F,v}) \not\subset U$ . Suppose that  $k, l \in \mathbb{Z}^\Sigma$  and  $f \in M_{k,l}(U; E)$  is an eigenform for  $T_v$  and  $S_v$  (defined in (4.3)) for all  $v \notin Q$ . Then there is a Galois representation*

$$\rho_f : G_F \rightarrow \text{GL}_2(E)$$

such that if  $v \notin Q$ , then  $\rho_f$  is unramified at  $v$  and the characteristic polynomial of  $\rho_f(\text{Frob}_v)$  is

$$X^2 - a_v X + d_v \text{Nm}_{F/\mathbb{Q}}(v),$$

where  $T_v f = a_v f$  and  $S_v f = d_v f$ .

This has been proved for paritious weights  $(k, l)$ , independently by Emerton–Reduzzi–Xiao [26] and Goldring–Koskivirta [36]; in fact, their methods yield the result under a weaker parity condition. The contribution here is to remove the parity hypothesis altogether, and the new ingredient is to use congruences to forms of level divisible by  $p$ . For this we will need to work with the integral models for Hilbert modular varieties with level structure  $U_1(p)$  at  $p$  studied by Pappas [51].

6.2. Hilbert modular varieties of level  $U' = U \cap U_1(p)$

Suppose that  $J$  is a fractional ideal of  $F$  and  $N \geq 3$  is an integer, with  $J$  and  $N$  both prime to  $p$ . We let  $\mathcal{M}_{J,N}^0$  denote the functor which associates to an  $\mathcal{O}$ -scheme  $S$  the set of isomorphism classes of pairs  $(\underline{A}, H)$ , where

- $\underline{A} = (A, i, \lambda, \eta)$  is a  $J$ -polarised HBAV with level  $N$  structure over  $S$ , and
- $H$  is a free rank 1  $(\mathcal{O}_F/p)$ -submodule scheme of  $A[p]$  over  $S$  such that the quotient isogeny  $A \rightarrow A' = A/H$  induces an isomorphism  $\text{Sym}(A'/S) \rightarrow p\text{Sym}(A/S)$ .

Then  $\mathcal{M}_{J,N}^0$  is represented by an  $\mathcal{O}$ -scheme which we denote  $Y_{J,N}^0$ , the forgetful morphism  $Y_{J,N}^0 \rightarrow Y_{J,N}$  is projective and  $Y_{J,N}^0$  is a flat local complete intersection over  $\mathcal{O}$  of relative dimension  $[F : \mathbb{Q}]$  [51, Thm. 2.2.2]. We let  $\mathcal{M}_{J,N}^1$  denote the functor which associates to an  $\mathcal{O}$ -scheme  $S$  the set of isomorphism classes of triples  $(\underline{A}, H, P)$  where  $\underline{A}$  and  $H$  are as above and

- $P \in H(S)$  is an  $(\mathcal{O}_F/p)$ -generator of  $H$  in the sense of Drinfeld–Katz–Mazur [42, 1.10].

Then  $\mathcal{M}_{J,N}^1$  is represented by an  $\mathcal{O}$ -scheme which we denote  $Y_{J,N}^1$ , and the forgetful morphism  $Y_{J,N}^1 \rightarrow Y_{J,N}^0$  is finite flat; therefore  $Y_{J,N}^1$  is flat and Cohen–Macaulay over  $\mathcal{O}$  [51, Thm. 2.3.3].

Suppose  $U$  is an open compact subgroup of  $\text{GL}_2(\widehat{\mathcal{O}}_F)$  containing  $\text{GL}_2(\mathcal{O}_{F,p})$ , and let  $U' = U \cap U_1(p)$ . We suppose that  $U$  is sufficiently small, and in particular that  $U$  is  $p$ -neat (see Definition 3.2.3). The action of the group  $G_{U,N}$  on  $Y_{J,N}$  then lifts to one on  $Y_{J,N}^1$ , corresponding to the action on  $\mathcal{M}_{J,N}^1$  defined by  $(v, u) \cdot (\underline{A}, H, P) = ((v, u) \cdot \underline{A}, H, P)$ . It follows from the corresponding assertions for  $Y_{J,N}$  that  $G_{U,N}$  acts freely on  $\coprod_{t \in T} Y_{J_t,N}^1$ , the quotient is representable by a scheme  $Y_{U'}$ , and the quotient map is étale and Galois with group  $G_{U,N}$ . Since the  $Y_{J,N}^1$  are flat and Cohen–Macaulay over  $\mathcal{O}$ , so is  $Y_{U'}$ , and let  $\pi_U : Y_{U'} \rightarrow Y_U$  denote the natural projection (writing just  $\pi$  when  $U$  is clear from the context). We let  $\mathcal{K}_{U'}$  denote the dualising sheaf on  $Y_{U'}$  over  $\mathcal{O}$  (see [15, §3.5]), and similarly let  $\mathcal{K}_U$  denote the dualising sheaf on  $Y_U$  over  $\mathcal{O}$ . Since  $Y_U$  is smooth over  $\mathcal{O}$ , its dualising sheaf  $\mathcal{K}_U$  is canonically identified with  $\Omega_{Y_U/\mathcal{O}}^{[F:\mathbb{Q}]} = \wedge_{\mathcal{O}_{Y_U}}^{[F:\mathbb{Q}]} \Omega_{Y_U/\mathcal{O}}^1$ .

Suppose now that  $g, U_1$  and  $U_2$  are as in §4, so in particular  $g^{-1}U_1g \subset U_2$ , and assume further that  $g_p \in U_1(p)$ . We then obtain exactly as before finite étale  $\rho'_g : Y_{U'_1} \rightarrow Y_{U'_2}$ , by descent from morphisms  $\tilde{\rho}'_g : \coprod Y_{J_t,N_1}^1 \rightarrow \coprod Y_{J_t,N_2}^1$ , and compatible with  $\rho_g : Y_{U_1} \rightarrow Y_{U_2}$  via the projections  $\pi_{U_i} : Y_{U'_i} \rightarrow Y_{U_i}$ . Since  $\rho'_g$  is étale, we have a canonical isomorphism  $(\rho'_g)^* \mathcal{K}_{U'_2} \xrightarrow{\sim} \mathcal{K}_{U'_1}$ .

6.3. Hilbert modular forms of level  $U' = U \cap U_1(p)$

For  $(m, n) \in \mathbb{Z}^2$  (viewed also as an element of  $(\mathbb{Z}^\Sigma)^2$ ) and  $p$ -neat  $U$ , we let  $\mathcal{L}_{U'}^{m,n} = \pi_U^* \mathcal{L}_U^{m,n}$ , and we similarly define  $\mathcal{L}_{U',R}^{m,n}$  for  $\mathcal{O}$ -algebras  $R$ , writing also  $\overline{\mathcal{L}}_{U'}^{m,n}$  if  $R = E$ . For  $k, l \in \mathbb{Z}$ , we define the space of Hilbert modular forms over  $R$  of weight  $k$  and level  $U'$  to be

$$M_{k,l}(U'; R) := H^0(Y_{U',R}, \mathcal{K}_{U',R} \otimes_{\mathcal{O}_{Y_{U',R}}} \mathcal{L}_{U',R}^{k-2,l+1}).$$

Note that we could have made this definition for more general weights  $(k, l)$ , but we will in fact only need the case of parallel weight. Recall also from [15] that formation of the dualising sheaf is compatible with base change, so  $\mathcal{K}_{U',R}$  can be identified with the dualising sheaf of  $Y_{U',R}$  over  $R$ .

For  $g, U_1, U_2$  as above, we define an  $R$ -linear map

$$[U'_1 g U'_2] : M_{k,l}(U'_2; R) \rightarrow M_{k,l}(U'_1; R)$$

as  $\|\det(g)\|$  times the composition of the pull-back from  $Y_{U'_2}$  to  $Y_{U'_1}$  with the map on sections induced by the tensor product of the canonical isomorphism  $(\rho'_g)^* \mathcal{K}_{U'_2} \xrightarrow{\sim} \mathcal{K}_{U'_1}$  with the map

$$(\rho'_g)^* \mathcal{L}_{U'_2}^{k-2,l+1} = \pi_{U'_1}^* \rho_g^* \mathcal{L}_{U_2}^{k-2,l+1} \rightarrow \pi_{U'_1}^* \mathcal{L}_{U_1}^{k-2,l+1} = \mathcal{L}_{U'_1}^{k-2,l+1}$$

given by  $\pi_{U'_1}^*$  of (4.2). We again have the compatibility  $[U'_1 g_1 U'_2] \circ [U'_2 g_2 U'_3] = [U'_1 g_1 g_2 U'_3]$ , giving rise to an  $R$ -linear action of the group  $\{g \in \mathrm{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in U_1(p)\}$  on  $M'_{k,l}(R) := \varinjlim M_{k,l}(U'; R)$ . As before we may identify  $M_{k,l}(U'; R)$  with  $(M'_{k,l}(R))^{U'}$ , and define commuting  $R$ -linear Hecke operators  $T_v$  and  $S_v$  on  $M_{k,l}(U'; R)$  for all  $v$  such that  $\mathrm{GL}_2(\mathcal{O}_{F,v})$  is contained in  $U'$ .

Let  $S = Y_{J,N}$ , and  $A = A_{J,N}$  the universal HBAV over  $S$ . Since  $A$  is smooth over  $S$  and  $S$  is smooth over  $\mathcal{O}$ , we have an exact sequence

$$0 \rightarrow s^* \Omega_{S/\mathcal{O}}^1 \rightarrow \Omega_{A/\mathcal{O}}^1 \rightarrow \Omega_{A/S}^1 \rightarrow 0$$

of locally free sheaves on  $A$ . Applying  $R^i s_*$ , we obtain the connecting homomorphism

$$s_* \Omega_{A/S}^1 \rightarrow R^1 s_* s^* \Omega_{S/\mathcal{O}}^1 \simeq \Omega_{S/\mathcal{O}}^1 \otimes_{\mathcal{O}_S} R^1 s_* \mathcal{O}_A. \tag{6.1}$$

Combined with the canonical isomorphisms

$$\mathcal{H}om_{\mathcal{O}_F \otimes \mathcal{O}_S}(\wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_S} \mathcal{H}_{\mathrm{DR}}^1(A/S), s_* \Omega_{A/S}^1) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_S}(R^1 s_* \mathcal{O}_A, \mathcal{O}_S)$$

induced by the inclusion  $\mathcal{O}_F \subset \mathfrak{d}^{-1} \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_F, \mathbb{Z})$  and the isomorphism (3.1), we obtain an  $\mathcal{O}_S$ -linear homomorphism

$$\mathcal{H}om_{\mathcal{O}_F \otimes \mathcal{O}_S}(\wedge^2_{\mathcal{O}_F \otimes \mathcal{O}_S} \mathcal{H}_{\mathrm{DR}}^1(A/S), \otimes_{\mathcal{O}_F \otimes \mathcal{O}_S}^2 (s_* \Omega_{A/S}^1)) \rightarrow \Omega_{S/\mathcal{O}}^1, \tag{6.2}$$

which is in fact an isomorphism (see [41, 1.0.21]), called the *Kodaira–Spencer isomorphism*. Taking  $\wedge_{\mathcal{O}_S}^{[F:\mathbb{Q}]}$ , we obtain an isomorphism

$$\xi_{J,N} : \mathcal{L}_{J,N}^{2,-1} = \bigotimes_{\tau} (\omega_{\tau}^2 \otimes_{\mathcal{O}_S} \delta_{\tau}^{-1}) \rightarrow \wedge_{\mathcal{O}_S}^{[F:\mathbb{Q}]} \Omega_{S/\mathcal{O}}^1 = \Omega_{S/\mathcal{O}}^d.$$

The functoriality of the morphisms in the construction ensures that the isomorphism is compatible with the action of  $G_{U,N}$ , and therefore descends to an isomorphism

$$\xi_U : \mathcal{L}_U^{2,-1} \simeq \mathcal{K}_U.$$

Moreover, for  $g, U_1$  and  $U_2$  such that  $g^{-1}U_1g \subset U_2, g_p \in \text{GL}_2(\mathcal{O}_{F,p})$ , one finds similarly that the canonical isomorphism  $\rho_g^* \mathcal{K}_{U_2} \rightarrow \mathcal{K}_{U_1}$  is compatible with the morphism of (4.2). It follows that the isomorphisms

$$M_{k,l}(U; R) \simeq H^0(Y_{U,R}, \mathcal{K}_{U,R} \otimes_{\mathcal{O}_{Y_{U,R}}} \mathcal{L}_{U,R}^{k-2,l+1})$$

induced by  $\xi_U$  are compatible with the operators  $[U_1gU_2]$ . Moreover, the generic fibre of  $Y_{J,N}^1$  is smooth over  $L$ , so that if  $p$  is invertible in  $R$ , the same constructions apply to give isomorphisms

$$H^0(Y_{U',R}, \mathcal{L}_{U',R}^{k,l}) \simeq M_{k,l}(U'; R)$$

such that the operators  $[U'_1gU'_2]$  are compatible by extension of scalars with those on the spaces  $A_{k,l}(U')$  of automorphic forms of weight  $(k, l)$  and level  $U'$ .

#### 6.4. Minimal compactifications

We will also make use of minimal compactifications of Hilbert modular varieties, whose properties we now recall. The minimal compactification  $X_{J,N}$  of  $Y_{J,N}$  is constructed by Chai [11] (see also [21, 22]), and we define  $X_U$  to be the quotient of  $\coprod X_{J,N}$  under the natural action of  $G_{U,N}$ . Then  $X_U$  is a flat, projective scheme over  $\mathcal{O}$  with  $j : Y_U \rightarrow X_U$  as an open subscheme whose complement is finite over  $\mathcal{O}$ , and the line bundle  $\mathcal{L}_U^{1,0}$  extends to an ample line bundle on  $X_U$  which we denote by  $\mathcal{L}_U$ . The Koecher Principle in this setting means that the natural map  $\mathcal{O}_{X_U} \rightarrow j_* \mathcal{O}_{Y_U}$  is an isomorphism.

**Definition 6.4.1.** Assuming as usual that  $\mathcal{O}$  is sufficiently large (i.e., containing the  $N$ th roots of unity), then each (reduced) connected component  $C$  of  $X_U - Y_U$  is isomorphic to  $\text{Spec } \mathcal{O}$ . We call  $C$  a *cuspidal* of  $X_U$ .

If  $U$  is of the form  $U(\mathfrak{n}) := \ker(\text{GL}_2(\hat{\mathcal{O}}_F) \rightarrow \text{GL}_2(\mathcal{O}_F/\mathfrak{n}))$  for a sufficiently small, prime-to- $p$  ideal  $\mathfrak{n}$  of  $\mathcal{O}_F$ , then the completion of  $X_U$  along  $C$  is canonically isomorphic to  $\text{Spf } \hat{S}_C$ , where

$$\hat{S}_C := \mathcal{O}[[q^\alpha]]_{\alpha \in I_+ \cup \{0\}}^{U_{\mathfrak{n},+}^\times} \tag{6.3}$$

for a fractional ideal  $I$  depending on  $C$ , and  $\mu \in U_{\mathfrak{n},+}^\times = \ker(\mathcal{O}_{F,+}^\times \rightarrow (\mathcal{O}_F/\mathfrak{n})^\times)$  acts via  $q^\alpha \mapsto q^{\mu\alpha}$ . (The  $\mathcal{O}$ -algebra  $\hat{S}_C$  is obtained from the corresponding one in [11] by working over  $\mathcal{O}$  instead of  $\mathbb{Z}[\mu_N, 1/N]$  and taking invariants under the stabiliser in  $G_{U,N}$  of a cusp

$\tilde{C}$  of  $X_{J,N}$  mapping to  $C$ . In particular, the class of the ideal  $I$  in (6.3) is given by  $\alpha \mathfrak{b} \mathfrak{n}^{-1}$  where  $\alpha$  and  $\mathfrak{b}$  are as in [11]; a more detailed discussion in the case of arbitrary  $U$  is provided below in §9, where Proposition 9.1.2 gives (6.3) as a special case.)

The minimal compactification of  $Y_{U'}$  is then obtained as follows. First one constructs a toroidal compactification  $X_{U'}^{\text{tor}}$  of  $Y_{U'}$  as the quotient of a toroidal compactification of  $\coprod Y_{J,N}^1$  defined exactly as for  $\coprod Y_{J,N}$ , but using the functors  $\mathcal{M}_{J,N}^1$  and  $\Gamma(Np)$ -admissible polyhedral cone decompositions (in the terminology of [11]). Then  $\pi : Y_{U'} \rightarrow Y_U$  extends to a projective morphism  $X_{U'}^{\text{tor}} \rightarrow X_U$  such that the connected components of the pre-image of a cusp  $C$  correspond to pairs  $(\mathfrak{f}, P)$  where  $p\mathcal{O}_F \subset \mathfrak{f} \subset \mathcal{O}_F$  and  $P$  is an  $(\mathcal{O}_F/p)$ -generator of  $\mathcal{O}_F/\mathfrak{f}$  (or more canonically,  $\mathfrak{b}/\mathfrak{b}\mathfrak{f}$ ). Moreover, a similar calculation to the case of level  $U$  shows that if  $U = U(\mathfrak{n})$ , then the ring of global sections of the completion of  $X_{U'}^{\text{tor}}$  along the component over  $C$  corresponding to  $(\mathfrak{f}, P)$  is isomorphic to the  $\widehat{S}_C$ -algebra

$$\mathcal{O}'_{\mathfrak{f}}[[q^\alpha]]_{\alpha \in (\mathfrak{f}^{-1}I)_+ \cup \{0\}}^{U_{\mathfrak{n},+}}, \tag{6.4}$$

where  $\text{Spec } \mathcal{O}'_{\mathfrak{f}}$  is the finite flat  $\mathcal{O}$ -scheme representing  $(\mathcal{O}_F/p)$ -generators of  $\mu_p \otimes \mathfrak{f}/p\mathcal{O}_F$  (or more canonically,  $\mu_p \otimes \mathfrak{f} \alpha^{-1} \mathfrak{b}^{-1}/p \alpha^{-1} \mathfrak{b}^{-1}$ ).

Now let  $X_U^{\text{ord}}$  denote the ordinary locus of  $X_U$ , so  $X_{U'}^{\text{ord}}$  is an open subscheme of  $X_U$  containing the cusps, and let  $Y_U^{\text{ord}} = Y_U \cap X_U^{\text{ord}}$ . Let  $X_{U'}^{\text{ord}}$  (resp.  $Y_{U'}^{\text{ord}}$ ) denote the pre-image of  $X_U^{\text{ord}}$  (resp.  $Y_U^{\text{ord}}$ ) in  $X_{U'}^{\text{tor}}$  (resp.  $Y_{U'}$ ), and define

$$X_{U'}^{\text{ord}} = \mathbf{Spec} f_* \mathcal{O}_{X_{U'}^{\text{tor}}},$$

where  $f : X_{U'}^{\text{tor}} \rightarrow X_{U'}^{\text{ord}}$  is the restriction of  $X_{U'}^{\text{tor}} \rightarrow X_U$ . Since  $f$  is proper,  $X_{U'}^{\text{ord}}$  is finite over  $X_U^{\text{ord}}$ , and since  $Y_U^{\text{ord}} \rightarrow Y_U^{\text{ord}}$  is finite, we can identify  $Y_{U'}^{\text{ord}}$  with an open subscheme of  $X_{U'}^{\text{ord}}$ . We then define the minimal compactification  $j' : Y_{U'} \rightarrow X_{U'}$  by gluing  $Y_{U'}$  and  $X_{U'}^{\text{ord}}$  along  $Y_{U'}^{\text{ord}}$ .

Thus  $X_{U'}$  is flat over  $\mathcal{O}$ , and the morphism  $\pi$  extends to a projective morphism  $\tilde{\pi} : X_{U'} \rightarrow X_U$ , so in particular  $X_{U'}$  is projective over  $\mathcal{O}$ . Furthermore, the restriction  $\tilde{\pi}^{\text{ord}} : X_{U'}^{\text{ord}} \rightarrow X_U^{\text{ord}}$  is finite, and  $\pi^{\text{ord}} : Y_{U'}^{\text{ord}} \rightarrow Y_U^{\text{ord}}$  is finite flat. The cusps  $C'$  of  $X_{U'}$  (i.e., the reduced connected components of  $X_{U'} - Y_{U'}$ ) lying over a cusp  $C$  of  $X_U$  correspond to pairs  $(\mathfrak{f}, P)$  as above, and in the case  $U = U(\mathfrak{n})$ , the completion of  $X_{U'}$  along  $C'$  is isomorphic to  $\text{Spf } \widehat{S}_{C'}$  where  $\widehat{S}_{C'}$  is the  $\widehat{S}_C$ -algebra defined by (6.4) above. Note in particular that if  $\mathfrak{f} = \mathcal{O}_F$ , then  $\widehat{S}_{C'} = \mathcal{O}'_{\mathcal{O}_F} \otimes_{\mathcal{O}} \widehat{S}_C$  is flat over  $\widehat{S}_C$ . The Koecher Principle carries over to show that  $j'_* \mathcal{O}_{Y_{U'}} = \mathcal{O}_{X_{U'}}$ , and we let  $\mathcal{L}_{U'}$  denote the pull-back  $\tilde{\pi}^* \mathcal{L}_U$  of the ample line bundle  $\mathcal{L}_U$ .

6.5. Proof of Theorem 6.1.1

We begin the proof with some preliminary reductions.

First we claim we can replace the field  $E$  by a finite extension  $E'$ . Indeed, if  $\rho : G_F \rightarrow \text{GL}_2(E')$  satisfies the conclusion of the theorem with  $E$  replaced by  $E'$ , then in fact  $\rho$  is defined over  $E$ . For  $p > 2$ , this follows by an elementary argument using an element

of  $g \in G_F$  (a complex conjugation, for example) such that  $\rho(g)$  has distinct eigenvalues in  $E$ . For  $p = 2$ , one can twist by the character  $\xi : G_F \rightarrow E^\times$  such that  $\xi^2 = \det \rho$  so as to assume  $\det \rho = 1$ , and then use the classification of subgroups of  $SL_2(E') \simeq PGL_2(E')$  to arrive at the desired conclusion.

Next we claim that we can assume  $U = U(\mathfrak{n})$  for a sufficiently small ideal  $\mathfrak{n}$  prime to  $p$ . Indeed, by the proof of Chevalley’s Theorem on congruence subgroups of  $O_F^\times$ , we can choose ideals  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$  relatively prime to each other and to  $p$  so that the kernels of reduction mod  $\mathfrak{n}_i$  for  $i = 1, 2$  are contained in that of reduction mod  $p$ . We may then apply the theorem with  $U$  replaced by  $U(\mathfrak{m}\mathfrak{n}_i)$ , where  $U(\mathfrak{m}) \subset U$  and  $\mathfrak{m}$  is divisible only by primes such that  $GL_2(O_{F,v}) \not\subset U$ . This produces representations  $\rho_i$  satisfying the conclusions with  $Q$  augmented by the set of primes dividing  $\mathfrak{n}_i$ . Moreover, we can replace the  $\rho_i$  by their semisimplifications, which are isomorphic to each other by the Brauer–Nesbitt and Chebotarev Density Theorems. We therefore obtain the desired conclusion for all  $v \notin Q$ .

Next we show that we can assume  $l = -1$ ,<sup>3</sup> i.e.,  $l_\tau = -1$  for all  $\tau \in \Sigma$ . Given any  $l$ , define  $l' \in \mathbb{Z}^\Sigma$  by  $l'_\tau = l_\tau + 1$ . Recall from the discussion before Lemma 4.5.1 that our hypothesis on  $U$  ensures that  $\mu \mapsto \bar{\mu}^{l'}$  is a well-defined  $E^\times$ -valued character on the finite index subgroup  $(F_+^\times \cap O_{F,p}^\times)/(O_{F,+}^\times \cap \det(U))$  of  $\{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\}/\det(U)$ , for which we may choose an extension  $\xi$  as in Lemma 4.6.1 (enlarging  $E$  if necessary). The case  $l = -1$  of the theorem then furnishes a Galois representation  $\rho_{f \otimes e_\xi^{-1}}$  unramified at all  $v \notin Q$  with  $\text{Frob}_v$  having characteristic polynomial

$$X^2 - \xi(\varpi_v)^{-1} a_v X + \xi(\varpi_v)^{-2} d_v \text{Nm}_{F/\mathbb{Q}}(v).$$

Let  $V = \{b \in \det(U) \mid b_p \equiv 1 \pmod{p}\}$ , and define

$$\xi' : \mathbb{A}_F^\times / F^\times F_{\infty,+}^\times V \rightarrow E^\times$$

by  $\xi'(\alpha z a) = \xi(a) \bar{a}_p^{-l'}$  for  $\alpha \in F^\times$ ,  $z \in F_{\infty,+}^\times$  and  $a \in (\mathbb{A}_F^\infty)^\times$  with  $a_p \in O_{F,p}^\times$ . Letting  $\rho_{\xi'} : G_F \rightarrow E^\times$  be the character corresponding to  $\xi'$  by class field theory, we have  $\rho_{\xi'}(\text{Frob}_v) = \xi(\varpi_v)$  for all  $v \notin Q$ , so the representation  $\rho_{\xi'} \otimes \rho_{f \otimes e_\xi^{-1}}$  satisfies the conclusion of the theorem.

Now we reduce to the case where  $f$  is of arbitrarily large, “nearly parallel” weight. More precisely, we claim that, given any  $M \in \mathbb{Z}$ , we can assume that  $k = (k_\tau)_{\tau \in \Sigma}$  has the form  $k = m + 2 - \kappa = (m + 2 - \kappa_\tau)_\tau$ , where

- $m \in \mathbb{Z}, m \geq M$ ;
- $0 \leq \kappa_\tau \leq p - 1$  for all  $\tau \in \Sigma$ ;
- for each  $v \mid p$ ,  $\kappa_\tau < p - 1$  for some  $\tau \in \Sigma_v$ .

Here we have identified  $\Sigma$  with the set of embeddings  $O_F \rightarrow \mathcal{O}$  and written

$$\Sigma = \coprod_{v \mid p} \Sigma_v, \quad \text{where } \Sigma_v = \{\tau \in \Sigma \mid v = \tau^{-1}(\pi \mathcal{O})\}. \tag{6.5}$$

<sup>3</sup>In fact, any parallel  $l$  will do; the choice of  $l = -1$  is made for later convenience.

To prove the claim, suppose  $f \in M_{k,-1}(U; E)$ , and choose any  $m \in \mathbb{Z}$  such that  $m \geq M$  and  $m \geq k_\tau + p - 3$  for all  $\tau \in \Sigma$ . For each  $v \mid p$ , choose some  $\tau_{v,0} \in \Sigma_v$  and let  $\tau_{v,i} = \text{Fr}^i \circ \tau_{v,0}$ , so  $\Sigma_v = \{\tau_{v,i} \mid i = 0, \dots, f_v - 1\}$  where  $f_v = [O_F/v : \mathbb{F}_p]$ . Now let  $r \in \mathbb{Z}$  be such that  $0 \leq r < p^{f_v} - 1$  and

$$r \equiv \sum_{i=0}^{f_v-1} (m + 2 - k_{\tau_{v,i}}) p^i \pmod{(p^{f_v} - 1)}.$$

We then define  $\kappa_\tau$  for  $\tau \in \Sigma_v$  by requiring that  $0 \leq \kappa_{\tau_{v,i}} \leq p - 1$  for  $i = 0, \dots, f_v - 1$  and  $r = \sum \kappa_{\tau_{v,i}} p^i$ . Note that the resulting  $\kappa_\tau$  is independent of the choice of  $\tau_{v,0}$  and that  $\kappa_\tau < p - 1$  for some  $\tau \in \Sigma_v$ . Now define  $k' = (k'_\tau)_\tau \in \mathbb{Z}^\Sigma$  by setting  $k'_\tau = m + 2 - \kappa_\tau$ . We then have  $k' - k = \sum n_\tau k_{\text{Ha}\tau}$  where

$$n_\tau = (p^{f_v} - 1)^{-1} \sum_{i=0}^{f_v-1} (k'_{\text{Fr}^i \circ \tau} - k_{\text{Fr}^i \circ \tau}) p^i$$

for  $\tau \in \Sigma_v$ . Note that  $n_\tau \in \mathbb{Z}_{\geq 0}$  for all  $\tau \in \Sigma$ , so  $k' - k \in \Xi_{\text{AG}}$ . By Proposition 5.1.1 there is a Hecke-equivariant injection  $M_{k,-1}(U; E) \rightarrow M_{k',-1}(U; E)$ , so the theorem for forms of weight  $(k, -1)$  follows from the case of weight  $(k', -1)$ .

The heart of the proof is to construct, for  $k = m + 2 - \kappa$  as above, a Hecke-equivariant injective homomorphism

$$M_{k,-1}(U; E) \rightarrow M_{m+2,-1}(U'; E).$$

Letting  $\underline{A}$  denote the universal HBAV over  $S = \overline{Y}_{J,N}$ ,  $\text{Frob}_A : A \rightarrow A^{(p)}$  the relative Frobenius morphism and  $H^\mu = \ker \text{Frob}_A$ , the pair  $(\underline{A}, H^\mu)$  defines a section  $\overline{Y}_{J,N} \rightarrow \overline{Y}_{J,N}^0$ , where as usual we use  $\overline{Y}$  to denote the special fibre of an  $\mathcal{O}$ -scheme  $Y$ . Moreover, the section identifies  $\overline{Y}_{J,N}$  with a union of irreducible components of  $\overline{Y}_{J,N}^0$ , whose pre-image in  $\overline{Y}_{J,N}^1$  we denote by  $Y_{J,N}^\mu$ . The action of  $G_{U,N}$  on  $\overline{Y}_{J,N}^1$  restricts to one on  $Y_{J,N}^\mu$ , and we let  $Y_{U'}^\mu$  denote the corresponding quotient of  $\coprod Y_{J_t,N}^\mu$ . Thus  $i : Y_{U'}^\mu \rightarrow \overline{Y}_{U'}$  is a closed immersion identifying  $Y_{U'}^\mu$  with a union of irreducible components of  $\overline{Y}_{U'}$ , and  $\overline{\pi} \circ i : Y_{U'}^\mu \rightarrow \overline{Y}_{U'}$  is finite flat. In particular,  $Y_{U'}^\mu$  is Cohen–Macaulay (over  $E$ ), and we let  $\mathcal{K}_U^\mu$  denote its dualising sheaf. By Grothendieck–Serre duality ([15, Thm. 3.4.4], and the compatibility [15, (3.3.14)]) applied to the finite morphisms  $i$  and  $\overline{\pi} \circ i$ , we have canonical isomorphisms

$$i_* \mathcal{K}_U^\mu \simeq \mathcal{H}om_{\mathcal{O}_{\overline{Y}_{U'}}} (i_* \mathcal{O}_{Y_{U'}^\mu}, \overline{\mathcal{K}}_{U'}) , \quad \overline{\pi}_* i_* \mathcal{K}_U^\mu \simeq \mathcal{H}om_{\mathcal{O}_{\overline{Y}_{U'}}} (\overline{\pi}_* i_* \mathcal{O}_{Y_{U'}^\mu}, \overline{\mathcal{K}}_U). \tag{6.6}$$

Since  $i$  is a closed immersion, the first of these isomorphisms identifies  $i_* \mathcal{K}_U^\mu$  with a subsheaf of  $\overline{\mathcal{K}}_{U'}$ . To exploit the second isomorphism, we recall that [51, Prop. 5.1.5] identifies  $Y_{J,N}^1$  with a closed subscheme of the universal submodule scheme  $H$  over  $Y_{J,N}^0$ . In particular, if  $\underline{A}$  is the universal HBAV on  $S = \overline{Y}_{J,N}$ , then

$$H^\mu = \mathbf{Spec}(\text{Sym}_{\mathcal{O}_S}(\bigoplus_{\tau \in \Sigma} \mathcal{L}_\tau) / \langle \mathcal{L}_\tau^{\otimes p} \text{ for } \tau \in \Sigma \rangle)$$

as a Raynaud  $(O_F/p)$ -module scheme (i.e., the morphisms  $\Delta_\tau : \mathcal{L}_\tau^{\otimes p} \rightarrow \mathcal{L}_{\text{For}\tau}$  of [51, 4.4.1] are zero), so that

$$Y_{J,N}^\mu = \mathbf{Spec}(\text{Sym}_{\mathcal{O}_S}(\bigoplus_{\tau \in \Sigma} \mathcal{L}_\tau) / (\mathcal{L}_\tau^{\otimes p} \text{ for } \tau \in \Sigma, \bigotimes_{\tau \in \Sigma_v} \mathcal{L}_\tau^{\otimes(p-1)} \text{ for } v \mid p)),$$

where the  $\mathcal{L}_\tau$  are line bundles on  $S$ . Moreover, the inclusion  $H^\mu \rightarrow A$  induces a canonical  $\mathcal{O}_S \otimes O_F$ -linear isomorphism

$$s_*\Omega_{A/S}^1 \simeq e^*\Omega_{A/S}^1 \simeq e^*\Omega_{H^\mu/S}^1 \simeq \bigoplus_{\tau \in \Sigma} \mathcal{L}_\tau,$$

and hence isomorphisms  $\bar{\omega}_\tau \simeq \mathcal{L}_\tau$  of line bundles on  $S$  for  $\tau \in \Sigma$ . These isomorphisms are compatible with the action of  $G_{U,N}$ , and so give rise to an isomorphism

$$Y_U^\mu \simeq \mathbf{Spec}(\text{Sym}_{\mathcal{O}_{\bar{Y}_U}}(\bigoplus_{\tau \in \Sigma} \bar{\omega}_\tau) / (\bar{\omega}_\tau^{\otimes p} \text{ for } \tau \in \Sigma, \bigotimes_{\tau \in \Sigma_v} \bar{\omega}_\tau^{\otimes(p-1)} \text{ for } v \mid p)),$$

which in turn gives an isomorphism  $\bar{\pi}_*i_*\mathcal{O}_{Y_U^\mu} \simeq \bigoplus_\kappa \bar{\mathcal{L}}_U^{\kappa,0}$  where the direct sum is over  $\kappa = (\kappa_\tau)_{\tau \in \Sigma}$  such that  $0 \leq \kappa_\tau \leq p - 1$  for each  $\tau$ , and  $\kappa_\tau < p - 1$  for some  $\tau$  in each  $\Sigma_v$ . Combined with the Kodaira–Spencer isomorphism on  $\bar{Y}_U$ , we deduce from (6.6) that

$$\bar{\pi}_*i_*\mathcal{K}_U^\mu \simeq \mathcal{H}om_{\mathcal{O}_{\bar{Y}_U}}(\bigoplus_\kappa \bar{\mathcal{L}}_U^{\kappa,0}, \bar{\mathcal{K}}_U) \simeq \bigoplus_\kappa \bar{\mathcal{L}}_U^{2-\kappa,-1}.$$

Tensoring with  $\bar{\mathcal{L}}_U^{m,0}$ , we get injective morphisms  $\bar{\mathcal{L}}_U^{k,-1} \rightarrow \bar{\pi}_*i_*(\mathcal{K}_U^\mu \otimes_{\mathcal{O}_Y} i^*\bar{\mathcal{L}}_{U'}^{m,0})$  for  $k = m + 2 - \kappa$  as above. Composing the homomorphism on sections with the one induced by the inclusion  $i_*\mathcal{K}_U^\mu \rightarrow \bar{\mathcal{K}}_{U'}$  obtained from (6.6), we obtain the desired injective homomorphism

$$H^0(\bar{Y}_U, \bar{\mathcal{L}}_U^{k,-1}) \rightarrow H^0(Y_U^\mu, \mathcal{K}_U^\mu \otimes_{\mathcal{O}_Y} i^*\bar{\mathcal{L}}_{U'}^{m,0}) \rightarrow H^0(\bar{Y}_{U'}, \bar{\mathcal{K}}_{U'} \otimes_{\mathcal{O}_{\bar{Y}_{U'}}} \bar{\mathcal{L}}_{U'}^{m,0}).$$

Moreover, one finds that for  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  with  $g_p \in U_1(p)$ , the isomorphisms  $\bar{\pi}_*i_*\mathcal{O}_{Y_U^\mu} \simeq \bigoplus_\kappa \bar{\mathcal{L}}_U^{\kappa,0}$  are compatible with (4.2) under the restriction of  $\rho'_g$  to the subschemes  $Y_U^\mu$ , and deduces that the maps  $M_{k,-1}(U; E) \rightarrow M_{m+2,-1}(U'; E)$  are compatible with the Hecke action; in particular, they commute with the operators  $T_v$  and  $S_v$  for  $v \notin Q$ .

Next we show that for sufficiently large  $m$ , the image of  $M_{k,-1}(U; E)$  in  $M_{m+2,-1}(U'; E)$  is contained in that of the reduction map from  $M_{m+2,-1}(U'; \mathcal{O})$  to  $M_{m+2,-1}(U'; E)$ . For this we will make use of the minimal compactifications  $j : Y_U \rightarrow X_U$  and  $j' : Y_{U'} \rightarrow X_{U'}$  and their properties recalled above.

We first compute the completion of  $j'_*\mathcal{K}_{U'}$  along the cusps of  $X_{U'}$ . We let  $j^{\text{ord}} : Y_U^{\text{ord}} \rightarrow X_U^{\text{ord}}$  denote the restriction of  $j$ . Recall also the notation  $\tilde{\pi} : X_{U'} \rightarrow X_U$ ,  $\tilde{\pi}^{\text{ord}} : X_{U'}^{\text{ord}} \rightarrow X_U^{\text{ord}}$  and  $\pi^{\text{ord}} : Y_{U'}^{\text{ord}} \rightarrow Y_U^{\text{ord}}$  for the morphisms extending and restricting  $\pi$ . Since  $\pi^{\text{ord}}$  is finite flat, we have



$$\begin{aligned}
 \tilde{\pi}_*^{\text{ord}}(j'_* \mathcal{K}_{U'})|_{X_U^{\text{ord}}} &= j_*^{\text{ord}} \pi_*^{\text{ord}}(\mathcal{K}_{U'}|_{Y_U^{\text{ord}}}) \\
 &\simeq j_*^{\text{ord}}(\mathcal{H}om_{\mathcal{O}_{Y_U^{\text{ord}}}}(\pi_*^{\text{ord}} \mathcal{O}_{Y_U^{\text{ord}}}, \mathcal{O}_{Y_U^{\text{ord}}}) \otimes_{\mathcal{O}_{Y_U^{\text{ord}}}} (\mathcal{L}_U^{2,-1}|_{Y_U^{\text{ord}}})) \\
 &\simeq \mathcal{H}om_{j_*^{\text{ord}} \mathcal{O}_{Y_U^{\text{ord}}}}(j_*^{\text{ord}} \pi_*^{\text{ord}} \mathcal{O}_{Y_U^{\text{ord}}}, j_*^{\text{ord}} \mathcal{O}_{Y_U^{\text{ord}}}) \otimes_{\mathcal{O}_{X_U^{\text{ord}}}} (\mathcal{L}_U^2|_{X_U^{\text{ord}}}) \\
 &= \mathcal{H}om_{\mathcal{O}_{X_U^{\text{ord}}}}(\tilde{\pi}_*^{\text{ord}} \mathcal{O}_{X_U^{\text{ord}}}, \mathcal{O}_{X_U^{\text{ord}}}) \otimes_{\mathcal{O}_{X_U^{\text{ord}}}} (\mathcal{L}_U^2|_{X_U^{\text{ord}}}),
 \end{aligned}$$

where we make use of the canonical trivialisation of  $\mathcal{L}_U^{0,1}$  and the Koecher Principle (for the last equality). Moreover, the isomorphism is of  $\tilde{\pi}_*^{\text{ord}} \mathcal{O}_{X_U^{\text{ord}}}$ -modules.

Since  $\tilde{\pi}^{\text{ord}}$  is finite, it follows that the completion of  $\tilde{\pi}_* j'_* \mathcal{K}_{U'} \otimes_{\mathcal{O}_{X_U}} \mathcal{L}_U^{-2}$  along a cusp  $C \subset X_U$  is canonically isomorphic to the coherent sheaf on  $\text{Spf } \hat{S}_C$  associated to the  $\bigoplus \hat{S}_{C'}$ -module

$$\text{Hom}_{\hat{S}_C}(\bigoplus \hat{S}_{C'}, \hat{S}_C),$$

where the direct sums are over the cusps  $C'$  of  $X_{U'}$  in the pre-image of  $C$  and the rings  $\hat{S}_C$  and  $\hat{S}_{C'}$  are defined by (6.3) and (6.4) above. The completion of  $j'_* \mathcal{K}_{U'} \otimes_{\mathcal{O}_{X_U}} \mathcal{L}_U^{-2}$  along a cusp  $C'$  of  $X_{U'}$  is therefore canonically isomorphic to  $\text{Hom}_{\hat{S}_C}(\hat{S}_{C'}, \hat{S}_C)$  as an  $\hat{S}_{C'}$ -module if  $C' \subset \tilde{\pi}^{-1}(C)$ .

Now consider the natural inclusion  $\overline{j'_* \mathcal{K}_{U'}} \rightarrow \overline{j'_* \mathcal{K}_{U'}}$  of coherent sheaves on  $\overline{X_{U'}}$ , where as usual we write  $\overline{\phantom{x}}$  for the special fibres of (quasi-coherent sheaves on and morphisms of) schemes over  $\mathcal{O}$ . This inclusion is an isomorphism on  $\overline{Y_{U'}}$ , so its cokernel is supported on the cusps of  $\overline{X_{U'}}$ . The same computation as above shows that the completion of  $\overline{j'_* \mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{X_{U'}}}} \overline{\mathcal{L}_{U'}^{-2}}$  along  $\overline{C'} \subset \overline{X_{U'}}$  is canonically isomorphic to the sheaf associated to  $\text{Hom}_{\hat{S}_{\overline{C'}}}(\hat{S}_{\overline{C'}}, \hat{S}_{\overline{C}})$ , where  $\hat{S}_{\overline{C}} = \hat{S}_C \otimes_{\mathcal{O}} E$  and  $\hat{S}_{\overline{C'}} = \hat{S}_{C'} \otimes_{\mathcal{O}} E$ . Let  $X_U^\mu$  denote the closure of  $Y_U^\mu$  in  $\overline{X_{U'}}$ , so  $X_U^\mu$  is a union of irreducible components of  $\overline{X_{U'}}$ . If  $C' \subset \tilde{\pi}^{-1}(C)$  is a cusp of  $X_{U'}$  such that  $\overline{C'} \subset X_U^\mu$ , then  $\mathfrak{f} = \mathcal{O}_F$ , so  $\hat{S}_{C'}$  is flat over  $\hat{S}_C$  and the natural inclusion

$$\text{Hom}_{\hat{S}_C}(\hat{S}_{C'}, \hat{S}_C) \otimes_{\mathcal{O}} E \rightarrow \text{Hom}_{\hat{S}_{\overline{C'}}}(\hat{S}_{\overline{C'}}, \hat{S}_{\overline{C}})$$

is an isomorphism. It follows that  $\overline{j'_* \mathcal{K}_{U'}} \rightarrow \overline{j'_* \mathcal{K}_{U'}}$  is an isomorphism after completing along  $C'$ , and so an isomorphism on stalks at the (closed points of) cusps of  $X_U^\mu$ . Therefore the cokernel of  $\overline{j'_* \mathcal{K}_{U'}} \rightarrow \overline{j'_* \mathcal{K}_{U'}}$  is supported on the complement of  $X_U^\mu$ . It follows that  $\overline{j'_*}$  of the inclusion  $i_* \mathcal{K}_{U'}^\mu \rightarrow \overline{\mathcal{K}_{U'}}$  factors through  $\overline{j'_* \mathcal{K}_{U'}}$ , and hence the image of  $M_{k,-1}(U; E)$  is contained in the subspace

$$\begin{aligned}
 H^0(\overline{X_{U'}}, \overline{j'_* \mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{X_{U'}}}} \overline{\mathcal{L}_{U'}^m}) &\subset H^0(\overline{X_{U'}}, \overline{j'_* \mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{X_{U'}}}} \overline{\mathcal{L}_{U'}^m}) \\
 &= H^0(\overline{X_{U'}}, \overline{j'_*}(\overline{\mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{Y_{U'}}}} \overline{\mathcal{L}_{U'}^m})) \\
 &= H^0(\overline{Y_{U'}}, \overline{\mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{Y_{U'}}}} \overline{\mathcal{L}_{U'}^m}) = M_{m+2,-1}(U'; E).
 \end{aligned}$$

A key ingredient we need at this point is [20, Th. E], which states that  $R^i \pi_* \mathcal{K}_{U'} = 0$  for  $i > 0$ , so in particular  $R^1 \pi_* \mathcal{K}_{U'} = 0$ . Since  $\tilde{\pi}^{\text{ord}}$  is finite, it follows that  $R^1 \tilde{\pi}_*(j'_* \mathcal{K}_{U'})$

vanishes, and hence the morphism  $\tilde{\pi}_* j'_* \mathcal{K}_{U'} \rightarrow \tilde{\pi}_*(\overline{j'_* \mathcal{K}_{U'}})$  is surjective. Since  $\mathcal{L}_U$  is ample, we have  $H^1(X_U, \tilde{\pi}_* j'_* \mathcal{K}_{U'} \otimes_{\mathcal{O}_{X_U}} \mathcal{L}_U^m) = 0$  for sufficiently large  $m$ , and it follows that the homomorphism

$$\begin{array}{ccc} H^0(X_U, \tilde{\pi}_* j'_* \mathcal{K}_{U'} \otimes_{\mathcal{O}_{X_U}} \mathcal{L}_U^m) & \longrightarrow & H^0(X_U, \tilde{\pi}_*(\overline{j'_* \mathcal{K}_{U'}}) \otimes_{\mathcal{O}_{X_U}} \mathcal{L}_U^m) \\ \parallel & & \parallel \\ M_{m+2,-1}(U'; \mathcal{O}) & \longrightarrow & H^0(\overline{X}_{U'}, \overline{j'_* \mathcal{K}_{U'}} \otimes_{\mathcal{O}_{\overline{X}_{U'}}} \overline{\mathcal{L}_{U'}^m}) \end{array}$$

is surjective. This completes the proof of the claim that the image of  $M_{k,-1}(U; E)$  in  $M_{m+2,-1}(U'; E)$  is contained in that of  $M_{m+2,-1}(U'; \mathcal{O})$ .

The theorem now follows from a standard argument. Let  $\mathbb{T}$  denote the ring of endomorphisms of  $M_{m+2,-1}(U'; \mathcal{O})$  generated over  $\mathcal{O}$  by the operators  $T_v$  and  $S_v$  for  $v \notin Q$ . Then  $\mathbb{T}$  is a finite flat  $\mathcal{O}$ -algebra, and  $M_{m+2,-1}(U'; \mathcal{O})$  is a faithful  $\mathbb{T}$ -module with  $M_{k,-1}(U; E)$  as a subquotient. The formula  $Tf = \theta_f(T)f$  defines an  $E$ -algebra homomorphism  $\mathbb{T} \rightarrow E$  whose kernel is a maximal ideal  $\mathfrak{m}$  generated by the operators  $T_v - a_v$  and  $S_v - d_v$  for  $v \notin Q$ . By the Going Down Theorem, there is a prime ideal  $\mathfrak{p} \subset \mathfrak{m}$  such that  $\mathfrak{p} \cap \mathcal{O} = 0$ , and hence (enlarging  $L$ ,  $\mathcal{O}$  and  $E$  if necessary), an  $\mathcal{O}$ -algebra homomorphism  $\tilde{\theta} : \mathbb{T} \rightarrow \mathcal{O}$  whose kernel is  $\mathfrak{p} \subset \mathfrak{m}$ . Since  $\mathfrak{p}$  is in the support of  $M_{m+2,-1}(U'; L) = M_{m+2,-1}(U'; \mathcal{O}) \otimes_{\mathcal{O}} L$ , there is an eigenform  $\tilde{f} \in M_{m+2,-1}(U'; L)$  such that  $T\tilde{f} = \tilde{\theta}(T)\tilde{f}$  for all  $T \in \mathbb{T}$ . By the existence of Galois representations associated to characteristic zero eigenforms [9,58] (together with the usual association of reducible representations to Eisenstein series), we have a representation

$$\rho_{\tilde{f}} : G_F \rightarrow \text{GL}_2(L)$$

such that if  $v \notin Q$ , then  $\rho_{\tilde{f}}$  is unramified at  $v$  and the characteristic polynomial of  $\rho_{\tilde{f}}(\text{Frob}_v)$  is

$$X^2 - a_v X + d_v \text{Nm}_{F/\mathbb{Q}}(v)$$

where  $a_v = \tilde{\theta}(T_v)$  and  $d_v = \tilde{\theta}(S_v)$ . Choosing a stable lattice and reducing modulo  $\mathfrak{p}$  gives the desired representation  $\rho_f$ . This concludes the proof of Theorem 6.1.1.

**Remark 6.5.1.** Note that by construction, if  $\alpha \in F^\times \cap \mathcal{O}_{F,p}^\times$ , then  $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$  acts on  $M_{k,l}(E)$  as  $\bar{\alpha}^{k+2l-2}$ . Therefore if  $f \in M_{k,l}(U; E)$  is an eigenform for  $S_v$  with eigenvalue  $d_v$  for all  $v \notin Q$ , then there is a character

$$\psi : (\mathbb{A}_F^\infty)^\times / (U \cap (\mathbb{A}_F^\infty)^\times) \rightarrow E^\times$$

such that  $\psi(\alpha) = \bar{\alpha}^{k+2l-2}$  for all  $\alpha \in F^\times \cap \mathcal{O}_{F,p}^\times$  and  $\psi(\varpi_v) = d_v$  for all  $v \notin Q$ . It follows from the description of  $\rho_f$  in Theorem 6.1.1 that  $\det(\rho_f)\chi_{\text{cyc}}$  (where  $\chi_{\text{cyc}}$  is the cyclotomic character) corresponds via class field theory to the character

$$\psi' : \mathbb{A}_F^\times / F^\times F_\infty^\times V \rightarrow E^\times$$

defined by  $\psi'(\alpha z a) = \psi(a)\bar{a}_p^{2-k-2l}$  for  $\alpha \in F^\times, z \in F_\infty^\times$  and  $a \in (\mathbb{A}_F^\infty)^\times$  with  $a_p \in \mathcal{O}_{F,p}^\times$ , where  $V = \{a \in \mathbb{A}_F^\infty \mid a \in U, a_p \equiv 1 \pmod{p}\}$ .

### 7. Geometric weight conjectures

In this section we formulate our geometric Serre weight conjectures and discuss the relation with [5].

#### 7.1. Geometric modularity

Let

$$\rho : G_F = \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$$

be an irreducible, continuous, totally odd representation of the absolute Galois group of  $F$ .

**Definition 7.1.1.** We say that  $\rho$  is *geometrically modular of weight  $(k, l)$*  if  $\rho$  is equivalent to the extension of scalars of  $\rho_f$  for some open compact subgroup  $U \subset \text{GL}_2(\widehat{O}_F)$  and eigenform  $f \in M_{k,l}(U; E)$  as in the statement of Theorem 6.1.1.

Note that the level  $U$  is unspecified, but required to contain  $\text{GL}_2(O_{F,p})$ . Also unspecified are the field  $E$  (and thus implicitly the field  $L \subset \overline{\mathbb{Q}}_p$ , by which we view  $E = \mathcal{O}/\pi \subset \overline{\mathbb{F}}_p$ ) and the finite set of primes of  $\mathcal{O}$ . Thus  $\rho$  is geometrically modular of weight  $(k, l)$  if there is a non-zero element  $f \in M_{k,l}(U; E)$  for some  $U \supset \text{GL}_2(O_{F,p})$  and  $E \subset \overline{\mathbb{F}}_p$  such that

$$T_v f = \text{tr}(\rho(\text{Frob}_v))f \quad \text{and} \quad \text{Nm}_{F/\mathbb{Q}}(v)S_v f = \det(\rho(\text{Frob}_v))f$$

for all but finitely many primes  $v$ . (Note that both sides of both equations are defined whenever  $v \nmid p$ ,  $\text{GL}_2(O_{F,v}) \subset U$  and  $\rho$  is unramified at  $v$ .)

**Remark 7.1.2.** Folklore conjectures predict that every  $\rho$  as above is indeed geometrically modular of *some* weight  $(k, l)$ . The focus of this paper is to give a conjectural recipe for *all* such weights  $(k, l)$  in terms of the local behaviour of  $\rho$  at primes over  $p$ .

#### 7.2. Crystalline lifts

In order to formulate our conjectures, we recall the notion of labelled Hodge–Tate weights. Let  $K$  be a finite extension of  $\mathbb{Q}_p$ , and let

$$\sigma : G_K \rightarrow \text{GL}_d(L) = \text{Aut}_L(V)$$

be a continuous representation on a  $d$ -dimensional  $L$ -vector space  $V$ . Recall that  $V$  is *crystalline* if  $D_{\text{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K}$  is free of rank  $d$  over

$$(L \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K} = L \otimes_{\mathbb{Q}_p} K_0$$

where  $B_{\text{crys}}$  is Fontaine’s ring of crystalline periods [27] and  $K_0$  is the maximal unramified subfield of  $K$ . One similarly defines the notion of a *de Rham* (resp. *Hodge–Tate*) representation and an associated filtered (resp. graded) free module  $D_{\text{dR}}(V)$  (resp.  $D_{\text{HT}}(V)$ ) of rank  $d$  over  $L \otimes_{\mathbb{Q}_p} K$  in terms of the rings  $B_{\text{dR}}$  (resp.  $B_{\text{HT}}$ ). Moreover, if  $V$  is crystalline,

then it is de Rham, and if  $V$  is de Rham then it is also Hodge–Tate. Thus if  $V$  is crystalline, then  $D_{\text{HT}}(V)$  is a graded free module of rank  $d$  over  $L \otimes_{\mathbb{Q}_p} K$ . If  $L$  is sufficiently large to contain the image of each embedding of  $K$  into  $\overline{\mathbb{Q}_p}$ , then  $L \otimes_{\mathbb{Q}_p} K \simeq \prod_{\tau \in \Sigma_K} L$  where  $\Sigma_K = \{\tau : K \rightarrow L\}$ , and for each  $\tau \in \Sigma_K$ , the corresponding component of  $D_{\text{HT}}(V)$  is a graded  $d$ -dimensional vector space over  $L$ .

**Definition 7.2.1.** If  $V$  is crystalline, then the  $\tau$ -labelled weights of  $V$  are defined as the  $d$ -tuple of integers  $(w_1, \dots, w_d) \in \mathbb{Z}^d$  such that  $w_1 \geq \dots \geq w_d$  and the  $\tau$ -component of  $D_{\text{HT}}(V)$  is isomorphic to  $\bigoplus_{i=1}^d L[w_i]$ , where  $L[w_i]$  has degree  $w_i$ . We define the Hodge–Tate type of  $V$  to be the element of  $(\mathbb{Z}^d)^{\Sigma_K}$  whose  $\tau$ -component is given by the  $\tau$ -labelled weights of  $V$ ; thus to give the Hodge–Tate type of  $V$  is equivalent to giving the isomorphism class of  $D_{\text{HT}}(V)$  as a graded  $L \otimes_{\mathbb{Q}_p} K$ -module.

We now specialise to the case  $d = 2$  and  $K = F_v$  where  $v$  is a prime of  $F$  dividing  $p$ , so  $\Sigma_K$  is identified with the subset  $\Sigma_v \subset \Sigma = \{\tau : F \rightarrow L\}$  defined by (6.5), and consider the representation

$$\sigma : G_K \rightarrow \text{GL}_2(\overline{\mathbb{F}_p}).$$

**Definition 7.2.2.** For a pair  $(k, l) \in \mathbb{Z}_{\geq 1}^{\Sigma_v} \times \mathbb{Z}^{\Sigma_v}$ , we say that  $\sigma$  has a crystalline lift of weight  $(k, l)$  if for some sufficiently large extension  $L \subset \overline{\mathbb{Q}_p}$  of  $\mathbb{Q}_p$  with ring  $\mathcal{O}$  of integers and residue field  $E \subset \overline{\mathbb{F}_p}$ , there exists a continuous representation

$$\tilde{\sigma} : G_K \rightarrow \text{GL}_2(\mathcal{O})$$

such that  $\tilde{\sigma} \otimes_{\mathcal{O}} \overline{\mathbb{F}_p}$  is isomorphic to  $\sigma$ , and  $\tilde{\sigma} \otimes_{\mathcal{O}} L$  is crystalline with Hodge–Tate type  $(k + l - 1, l)$ .

### 7.3. Statement of the conjectures

First recall from §5.2 the definition of the minimal cone:

$$\mathfrak{E}_{\min} = \{k \in \mathbb{Z}^{\Sigma} \mid pk_{\tau} \geq k_{\text{Fr}^{-1} \circ \tau} \text{ for all } \tau \in \Sigma\},$$

and let  $\mathfrak{E}_{\min}^+ = \mathfrak{E}_{\min} \cap \mathbb{Z}_{\geq 1}^{\Sigma}$ .

**Conjecture 7.3.1.** Let  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}_p})$  be an irreducible, continuous, totally odd representation, and let  $l \in \mathbb{Z}^{\Sigma}$ . There exists  $k_{\min} = k_{\min}(\rho, l) \in \mathfrak{E}_{\min}^+$  such that

- (1)  $\rho$  is geometrically modular of weight  $(k, l)$  if and only if  $k \geq_{\text{Ha}} k_{\min}$ ;
- (2) if  $k \in \mathfrak{E}_{\min}^+$ , then  $k \geq_{\text{Ha}} k_{\min}$  if and only if  $\rho|_{G_{F_v}}$  has a crystalline lift of weight  $(k_{\tau}, l_{\tau})_{\tau \in \Sigma_v}$  for all  $v \mid p$ .

Note that the conjecture, in particular the existence of  $k_{\min}$  as in (1), incorporates the “folklore conjecture” (see Remark 7.1.2) that  $\rho$  is geometrically modular of some weight  $(k, l)$ . Moreover, for any  $l \in \mathbb{Z}^{\Sigma}$  there should be weights  $(k, l)$  for which  $\rho$  is geometrically modular. In fact, one can show using partial  $\Theta$ -operators (defined below in §8) that for any given  $l$  and  $l'$ , if  $\rho$  is irreducible and geometrically modular of some

weight  $(k, l)$ , then  $\rho$  is geometrically modular of some weight  $(k', l')$ . We explain this, and the dependence of  $k_{\min}$  on  $l$  (for fixed  $\rho$ ), in §10.4. A simpler observation is that the conjecture is compatible with twists by arbitrary characters  $\xi : G_F \rightarrow \overline{\mathbb{F}}_p^\times$ . More precisely, by Lemma 4.6.1 and the well-known computation of reductions of crystalline characters (see for example [14, Prop. B4]), we see that the conjecture holds for the pair  $(\rho, l)$  if and only if it holds for the pair  $(\rho \otimes \xi, l - m)$  for any  $m \in \mathbb{Z}^\Sigma$  such that  $\xi|_{I_{F_v}} = \prod_{\tau \in \Sigma_v} \epsilon_\tau^{m_\tau}$  for all  $v \mid p$ , where  $I_{F_v}$  is the inertia subgroup of  $G_{F_v}$  and

$$\epsilon_\tau : I_{F_v} \rightarrow O_{F,v}^\times \rightarrow \overline{\mathbb{F}}_p^\times$$

is the fundamental character defined as the composite of the maps induced by  $\tau$  and local class field theory. Thus Conjecture 7.3.1 for all pairs  $(\rho, l)$  reduces to the case  $l = 0$ , with the resulting minimal weights related by  $k_{\min}(\rho, l) = k_{\min}(\rho \otimes \xi, 0)$  for any character  $\xi$  chosen so that  $\xi|_{I_{F_v}} = \prod_{\tau \in \Sigma_v} \epsilon_\tau^{l_\tau}$  for all  $v \mid p$ . We remark also that  $\rho|_{G_{F_v}}$  always has a crystalline lift of some weight  $(k_\tau, l_\tau)_{\tau \in \Sigma_v}$  with  $2 \leq k_\tau \leq p + 1$  for all  $\tau \in \Sigma_v$ , from which it follows that  $\rho$  has a twist for which  $k_{\min}$  as in (2) would satisfy  $k_\tau \leq p + 1$  for all  $\tau \in \Sigma$ .

Assuming that  $\rho$  is geometrically modular of some weight, the existence of a weight  $k_{\min}$  satisfying (1) in Conjecture 7.3.1 is strongly suggested by Corollary 1.2 of [19], which implies that the minimal weight  $\nu(f)$  of the eigenform  $f$  satisfies  $\nu(f) \in \Xi_{\min}$ , but it is not an immediate consequence. Indeed, there are two issues: firstly, we would need  $\nu(f) \in \Xi_{\min}^+$  (which we expect to hold if  $\rho_f$  is irreducible), and secondly, the eigenform  $f$  giving rise to  $\rho$  is not unique. However, if we grant the existence of  $k_{\min}$  as in (1), then Conjecture 7.3.1 reduces to the following:

**Conjecture 7.3.2.** *Suppose that  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible and geometrically modular of some weight, and that  $k \in \Xi_{\min}^+$ . Then  $\rho$  is geometrically modular of weight  $(k, l)$  if and only if  $\rho|_{G_{F_v}}$  has a crystalline lift of weight of  $(k_\tau, l_\tau)_{\tau \in \Sigma_v}$  for all  $v \mid p$ .*

**Remark 7.3.3.** The existence of  $k_{\min}$  satisfying part (2) of Conjecture 7.3.1 is a purely  $p$ -adic Hodge-theoretic statement, and it is strongly suggested by the Breuil–Mézard Conjecture (of [4] as generalised by [33]) and the modular representation theory of  $\text{GL}_2(O_F/p)$ , but again not an immediate consequence. We remark also that the condition  $k \in \Xi_{\min}^+$  is needed; indeed, R. Bartlett has constructed local Galois representations with crystalline lifts of weight<sup>4</sup>  $(k, l)$ , but none of weight  $(k', l)$ , where  $k' = k + k_{\text{Ha}_\tau}$  is in  $\mathbb{Z}_{\geq 1}^\Sigma$  but not in  $\Xi_{\min}^+$ . Granting the existence of a weight  $k_{\min}$  as in (2), then Conjecture 7.3.1 follows from Conjecture 7.3.2 under the assumptions that  $\rho$  is geometrically modular of some weight and that  $\nu(f) \in \Xi_{\min}^+$  if  $\rho_f \sim \rho$  is irreducible.

#### 7.4. The case $k = 1$

We now consider a special case of Conjecture 7.3.2. Since a representation  $G_K \rightarrow \text{GL}_d(L)$  is crystalline of Hodge–Tate type 0  $\in (\mathbb{Z}^d)^{\Sigma_K}$  if and only if it is unramified, it follows that

<sup>4</sup>For notational consistency, assume here that  $p$  is inert in  $F$ .

$\sigma : G_K \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  has a crystalline lift of weight  $(1, 0)$  if and only if it is unramified. Thus Conjecture 7.3.2 incorporates the prediction that  $\rho$ , assumed to be geometrically modular, is of weight  $(1, 0)$  if and only if it is unramified at all primes  $v \mid p$ . One direction of this, that if  $\rho$  is geometrically modular of weight  $(1, 0)$  then it is unramified at all  $v \mid p$ , is a theorem of Dimitrov and Wiese [23] (also proved independently by Emerton, Reduzzi and Xiao [25] under additional hypotheses), and the other direction is proved under technical hypotheses by Gee and Kassaei [32]. By twisting, these results extend to the case of weight  $(1, l)$  for arbitrary  $l$ .

7.5. Relation to algebraic modularity

We now explain how our conjecture is consistent with results on the weight part of Serre’s Conjecture as formulated by Buzzard, Jarvis and one of the authors in [5]. These results provide information about *algebraic weights*, meaning weights  $(k, l)$  such that  $k_\tau \geq 2$  for all  $\tau$ , but with a different notion of modularity, which we call *algebraic modularity*. We will next explain this notion and its relation to the conjectures above. The remainder of the paper will then focus on developing methods applicable to the case of partial weight 1, which lies outside both settings just mentioned, namely weights that are algebraic or of the form  $(1, l)$ .

Recall that in [5], a *Serre weight* is an irreducible representation of  $\mathrm{GL}_2(O_F/p)$  over  $\overline{\mathbb{F}}_p$ . For an algebraic weight  $(k, l) \in \mathbb{Z}_{\geq 2}^\Sigma \times \mathbb{Z}^\Sigma$ , we let  $V_{k,l}$  denote the representation

$$\bigotimes_{\tau \in \Sigma} (\det^{l_\tau} \otimes \mathrm{Sym}^{k_\tau - 2} \overline{\mathbb{F}}_p^2),$$

where  $\mathrm{GL}_2(O_F/p)$  acts on the factor indexed by  $\tau$  via the homomorphism to  $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$  induced by  $\tau$ . The irreducible representations of  $\mathrm{GL}_2(O_F/p)$  (i.e., Serre weights) are precisely the  $V_{k,l}$  such that  $2 \leq k_\tau \leq p + 1$  for all  $\tau \in \Sigma$ ; moreover, for such  $(k, l)$ ,  $V_{k,l}$  is isomorphic to  $V_{k',l'}$  if and only if  $k = k'$  and  $l - l' \in \bigoplus_{\tau \in \Sigma} \mathbb{Z} \cdot \mathrm{Ha}_\tau$ . (More concretely, the latter condition means that  $\sum_{i=0}^{f_\tau - 1} l_{\mathrm{Fr}^i \circ \tau} p^i \equiv \sum_{i=0}^{f_\tau - 1} l'_{\mathrm{Fr}^i \circ \tau} p^i \pmod{(p^{f_\tau} - 1)}$  for all  $\tau \in \Sigma$ , where  $f_\tau = [\tau(O_F) : \mathbb{F}_p]$ .)

For an irreducible representation  $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  and an arbitrary finite-dimensional representation  $V$  of  $\mathrm{GL}_2(O_F/p)$  over  $\overline{\mathbb{F}}_p$ , we say  $\rho$  is *modular of weight  $V$*  if it arises in the étale cohomology of a suitable quaternionic Shimura curve over  $F$  with coefficients in a lisse sheaf associated to  $V$ ; we refer the reader to [5, Section 2] for the precise definition.<sup>5</sup> It is also proved in *loc. cit.* that  $\rho$  is modular of weight  $V$  if and only if it is modular of weight  $W$  for some Jordan–Hölder constituent  $W$  of  $V$ , so the determination of the weights  $V$  for which  $\rho$  is modular reduces to the consideration of Serre weights.

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<sup>5</sup>Alternatively, but not a priori equivalently, one can define the notion of modularity of weight  $V$  in terms of the presence of the corresponding system of Hecke eigenvalues on spaces of mod  $p$  automorphic forms on totally definite quaternion algebras over  $F$ .

**Definition 7.5.1.** For an algebraic weight  $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ , we will say that  $\rho$  is *algebraically modular* of weight  $(k, l)$  if it is modular of weight  $V_{k,1-k-l}$  in the sense of [5] (the presence of the twist being to reconcile the conventions of this paper with the ones of [5]).

A conjecture is formulated in [5] for the set of Serre weights for which  $\rho$  is modular. Under the assumption that  $\rho$  is algebraically modular of some weight and mild technical hypotheses, the conjecture is proved in a series of papers by Gee and coauthors, culminating in [33, 34], with an independent alternative to the latter (deducing the conjecture from its analogue in the context of certain unitary groups) provided by Newton [50]. They also prove variants of the conjecture (under the same hypotheses), including that if  $2 \leq k_{\tau} \leq p + 1$  for all  $\tau$ , then  $\rho$  is algebraically modular of weight  $(k, l)$  if and only if  $\rho|_{G_{F_v}}$  has a crystalline lift of weight of  $(k_{\tau}, l_{\tau})_{\tau \in \Sigma_v}$  for all  $v \mid p$ . The generalised Breuil–Mézard Conjecture (as in [33]) would imply that this result extends to arbitrary algebraic weights. We are therefore led to conjecture:

**Conjecture 7.5.2.** *Let  $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  be an irreducible, continuous, totally odd representation, and let  $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$ . If  $\rho$  is algebraically modular of weight  $(k, l)$ , then  $\rho$  is geometrically modular of weight  $(k, l)$ . Moreover, if in addition  $k \in \mathbb{E}_{\min}^+$ , then the converse holds.*

**Remark 7.5.3.** The assumption  $k \in \mathbb{E}_{\min}^+$  appears for reasons related to its presence in part (2) of Conjecture 7.3.1; here however one can see its necessity more readily from modular representation-theoretic considerations. Indeed, if  $k, k' \in \mathbb{Z}_{\geq 2}^{\Sigma}$  with  $k' = k + k_{\mathrm{Ha}_\tau} \notin \mathbb{E}_{\min}^+$ , then  $V_{k,l}$  may have Jordan–Hölder constituents not present in  $V_{k',l}$ , so there are representations which are algebraically modular of weight  $(k, l)$ , but not of weight  $(k', l)$ . Note though that if  $2 \leq k_{\tau} \leq p + 1$  for all  $\tau$ , then  $k \in \mathbb{E}_{\min}^+$ , so we conjecture that algebraic and geometric modularity are equivalent for weights associated to Serre weights.

From our construction of the Galois representation associated to an eigenform  $f$ , we see that  $\rho_f$  is the reduction of some representation associated to a characteristic zero eigenform, from which it follows (e.g. from [5, Prop. 2.10]) that  $\rho_f$  is modular of some weight  $V$ . Thus if  $\rho$  is geometrically modular of some weight, then it is algebraically modular of some weight.

Conversely, suppose that  $\rho$  is algebraically modular of some *paritious* weight  $(k, l) \in \mathbb{Z}_{\geq 2}^{\Sigma} \times \mathbb{Z}^{\Sigma}$  (see Definition 3.2.1). Then [5, Prop. 2.5] implies that  $\rho$  is the reduction of some representation associated to a characteristic zero eigenform of weight  $(k, l)$  and level prime to  $p$ , and hence that  $\rho$  is geometrically modular of weight  $(k, l)$ . More generally, if  $(k, l)$  is any algebraic weight such  $k_{\tau} \equiv k_{\tau'} \pmod{2}$  for all  $\tau, \tau' \in \Sigma$ , then we can choose  $l'$  so that  $(k, l')$  is paritious and a character  $\xi$  so that  $\xi|_{I_{F_v}} = \prod_{\tau \in \Sigma_v} \epsilon_{\tau}^{l_{\tau} - l'_{\tau}}$ . If  $\rho$  is algebraically modular of weight  $(k, l)$ , then  $\rho \otimes \xi$  is algebraically modular of weight  $(k, l')$  (by [5, Prop. 2.11]), so the above argument shows that  $\rho \otimes \xi$  is geometrically modular of weight  $(k, l')$  and hence that  $\rho$  is geometrically modular of weight  $(k, l)$ . We have thus proved the following:

**Proposition 7.5.4.** *If  $\rho$  is geometrically modular of some weight, then it is algebraically modular of some (algebraic) weight. Conversely, if  $\rho$  is algebraically modular of some algebraic weight  $(k, l)$  such that  $k_\tau \equiv k_{\tau'} \pmod 2$  for all  $\tau, \tau' \in \Sigma$ , then  $\rho$  is geometrically modular of the same weight  $(k, l)$ .*

### 8. $\Theta$ -operators

In this section we recall the definition due to Andreatta and Goren of partial  $\Theta$ -operators (see [1, §12]), with some simplifications and adaptations to our setting. See also [18], where the approach taken here is applied in the context of Pappas–Rapoport models defined in [52], leading to a substantial refinement of the results in [1] when  $p$  is ramified in  $F$ .

#### 8.1. Igusa level structure

We assume that  $U$  is  $p$ -neat, as in Definition 3.2.3. For  $\tau \in \Sigma$ , we will write  $\overline{\omega}_\tau$  (resp.  $\overline{\delta}_\tau$ ) for the line bundle  $\overline{\mathcal{L}}_U^{k,0}$  (resp.  $\overline{\mathcal{L}}_U^{0,k}$ ) on  $\overline{Y}_U$ , where  $k$  is such that  $k_\tau = 1$  and  $k_{\tau'} = 0$  for  $\tau' \neq \tau$ . We view the partial Hasse invariant

$$\text{Ha}_\tau \in H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k_{\text{Ha}_\tau}, 0}) = H^0(\overline{Y}_U, \overline{\omega}_\tau^{-1} \otimes \overline{\omega}_{\text{Fr}^{-1}\circ\tau}^p)$$

as a morphism  $\overline{\omega}_{\text{Fr}^{-1}\circ\tau}^{-p} \rightarrow \overline{\omega}_\tau^{-1}$ . For each  $v \mid p$ , we let  $\text{Ha}_v = \prod_{\tau \in \Sigma_v} \text{Ha}_\tau$ , which we view as a morphism  $(\bigotimes_{\tau \in \Sigma_v} \overline{\omega}_{\text{Fr}^{-1}\circ\tau}^{-p}) \otimes (\bigotimes_{\tau \in \Sigma_v} \overline{\omega}_\tau) \rightarrow \mathcal{O}_{\overline{Y}_U}$ , i.e.,  $\bigotimes_{\tau \in \Sigma_v} \overline{\omega}_\tau^{1-p} \rightarrow \mathcal{O}_{\overline{Y}_U}$  (where  $\Sigma_v$  is defined in (6.5)).

We define the scheme

$$Y_U^{\text{lg}} = \text{Spec}(\text{Sym}_{\mathcal{O}_{\overline{Y}_U}}(\bigoplus_{\tau \in \Sigma} \overline{\omega}_\tau^{-1})/\mathcal{I}),$$

where  $\mathcal{I}$  is the sheaf of ideals of  $\text{Sym}_{\mathcal{O}_{\overline{Y}_U}}(\bigoplus_{\tau \in \Sigma} \overline{\omega}_\tau^{-1})$  generated by the sheaves of  $\mathcal{O}_{\overline{Y}_U}$ -submodules

$$(\text{Ha}_\tau - 1)\omega_{\text{Fr}^{-1}\circ\tau}^{-p} \text{ for } \tau \in \Sigma, \quad (\text{Ha}_v - 1)\left(\bigotimes_{\tau \in \Sigma_v} \omega_\tau^{1-p}\right) \text{ for } v \mid p.$$

We define an action of  $(\mathcal{O}_F/p\mathcal{O}_F)^\times$  on  $Y_U^{\text{lg}}$  over  $\overline{Y}_U$  by having  $\alpha \in (\mathcal{O}_F/p\mathcal{O}_F)^\times$  act on the structure sheaf as the  $\mathcal{O}_{\overline{Y}_U}$ -algebra automorphism defined by multiplication by  $\tau(\alpha)^{-1}$  on the summand  $\overline{\omega}_\tau^{-1}$ . (Note that the action is well-defined since the  $\text{Ha}_\tau$  are invariant under this action and hence  $\mathcal{I}$  is preserved.)

**Proposition 8.1.1.** *Let  $\pi_U : Y_U^{\text{lg}} \rightarrow \overline{Y}_U$  denote the natural projection. Then*

- (1) *the morphism  $\pi_U$  is finite and flat, and identifies  $\overline{Y}_U$  with the quotient of  $Y_U^{\text{lg}}$  by the action of  $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ ;*
- (2) *the restriction of  $\pi_U$  to the preimage of  $Y_U^{\text{ord}}$  is étale;*
- (3) *the scheme  $Y_U^{\text{lg}}$  is normal.*



*Proof.* Each assertion can be checked over affine open subschemes  $V \subset \overline{Y}_U$  on which the line bundles  $\overline{\omega}_\tau^{-1}$  are trivial. For each  $\tau \in \Sigma$ , let  $x_\tau$  be a generator of  $M_\tau = \Gamma(V, \overline{\omega}_\tau^{-1})$  over  $R = \Gamma(V, \mathcal{O}_{\overline{Y}_U})$ . Then  $\text{Ha}_\tau(x_{\text{Fr}^{-1} \circ \tau}^p) = r_\tau x_\tau$  for some  $r_\tau \in R$ , and  $\pi_U^{-1}(R) = \text{Spec } T$  where

$$T = R[x_\tau]_{\tau \in \Sigma} / \left\langle x_{\text{Fr}^{-1} \circ \tau}^p - r_\tau x_\tau \text{ for } \tau \in \Sigma, \prod_{\tau \in \Sigma_v} x_\tau^{p-1} - \prod_{\tau \in \Sigma_v} r_\tau \text{ for } v \mid p \right\rangle.$$

Thus  $T$  is free over  $R$  with basis  $\{\prod_{\tau \in \Sigma} t_\tau^{\kappa_\tau}\}$ , where  $t_\tau$  denotes the image of  $x_\tau$  in  $T$  and the tuples  $\kappa = (\kappa_\tau)_{\tau \in \Sigma}$  are those satisfying

- $0 \leq \kappa_\tau \leq p - 1$  for each  $\tau \in \Sigma$ ;
- $\kappa_\tau < p - 1$  for some  $\tau$  in each  $\Sigma_v$ .

Note that  $(\mathcal{O}_F/p\mathcal{O}_F)^\times$  acts on  $\prod_{\tau \in \Sigma} t_\tau^{\kappa_\tau}$  by the character  $\prod_{\tau \in \Sigma} \tau^{-\kappa_\tau}$ , and these are precisely the distinct characters of the  $(\mathcal{O}_F/p\mathcal{O}_F)^\times$ . Therefore  $T^{(\mathcal{O}_F/p\mathcal{O}_F)^\times} = R$ , and (1) follows.

To prove (2), recall that  $Y_U^{\text{ord}}$  is the complement of  $\bigcup_{\tau \in \Sigma} Z_{U,\tau}$  where  $Z_{U,\tau}$  is vanishing locus of  $\text{Ha}_\tau$  on  $\overline{Y}_U$ . We must therefore show that if all  $r_\tau$  are invertible in  $R$ , then  $T$  is étale over  $R$ . From the above description of  $T$ , we see that

$$r_\tau dt_\tau = d(x_{\text{Fr}^{-1} \circ \tau}^p - r_\tau t_\tau) = 0$$

in  $\Omega_{T/R}^1$ . It follows that  $dt_\tau = 0$  for all  $\tau$ , and hence  $\Omega_{T/R}^1 = 0$ , so  $T$  is étale over  $R$ .

To prove (3), we use Serre’s Criterion. Since  $R$  is regular and  $T$  is finite and flat over  $R$ ,  $T$  is Cohen–Macaulay, so it suffices to prove that  $T$  is regular in codimension 1. Thus it suffices to prove that the semilocal ring  $T_{\mathfrak{p}}$  is regular for every height 1 prime  $\mathfrak{p}$  of  $R$ . If  $\mathfrak{p} \in Y_U^{\text{ord}}$ , then  $T_{\mathfrak{p}}$  is étale over  $R_{\mathfrak{p}}$ , so  $T_{\mathfrak{p}}$  is regular. Otherwise  $\mathfrak{p}$  defines an irreducible component of  $V \cap Z_{U,\tau}$  for some  $\tau = \tau_0 \in \Sigma$ . Since  $V \cap Z_{U,\tau}$  is defined by  $r_\tau$  and  $\sum Z_{U,\tau}$  is reduced, the DVR  $R_{\mathfrak{p}}$  has uniformiser  $r_{\tau_0}$ , and  $r_\tau \in R_{\mathfrak{p}}^\times$  for  $\tau \neq \tau_0$ . Letting  $\tau_0 \in \Sigma_{v_0}$ ,  $f = \#\Sigma_{v_0}$  and

$$S = R[x_\tau]_{\tau \in \Sigma_{v_0}} / \left\langle x_{\text{Fr}^{-1} \circ \tau}^p - r_\tau x_\tau \text{ for } \tau \in \Sigma_{v_0}, \prod_{\tau \in \Sigma_{v_0}} x_\tau^{p-1} - \prod_{\tau \in \Sigma_{v_0}} r_\tau \right\rangle,$$

the formulas  $t_{\text{Fr}^i \circ \tau} = r_{\text{Fr}^i \circ \tau}^{-1} t_{\text{Fr}^{i-1} \circ \tau}^p$  for  $i = 1, \dots, f - 1$  show that

$$S_{\mathfrak{p}} = R_{\mathfrak{p}}[x_{\tau_0}] / \left\langle x_{\tau_0}^{p^f - 1} - r_{\tau_0} \prod_{i=1}^{f-1} r_{\text{Fr}^{f-i} \circ \tau_0} \right\rangle,$$

which is a DVR with uniformiser  $t_{\tau_0}$ . Since  $r_\tau$  is invertible in  $S_{\mathfrak{p}}$  for  $\tau \notin \Sigma_{v_0}$ , we see as above that  $T_{\mathfrak{p}}$  is étale over  $S_{\mathfrak{p}}$ , and is therefore also regular. ■

**Remark 8.1.2.** We could similarly have defined schemes  $Y_{J,N}^{\text{lg}}$  as above by replacing  $\overline{Y}_U$  with  $\overline{Y}_{J,N}$ . Then  $Y_{J,N}^{\text{lg}}$  is isomorphic to the closed subscheme (in fact, a union of irreducible components) of  $\overline{Y}_{J,N}^1$  for which the subgroup scheme  $H \subset A[p]$  is generically

étale. However, under this isomorphism, the natural projection  $Y_{J,N}^{\text{lg}} \rightarrow \bar{Y}_{J,N}$  corresponds to the restriction of the morphism  $\bar{Y}_{J,N}^1 \rightarrow \bar{Y}_{J,N}$  defined by  $(\underline{A}, H, P) \mapsto \underline{A}/H$ . Furthermore, we can realise  $Y_U^{\text{lg}}$  as the quotient of  $\coprod Y_{J,N}^{\text{lg}}$  by the action of  $G_{U,N}$  obtained from the one on  $\bar{Y}_{J,N}^1$  defined by  $(v, u) \cdot (\underline{A}, H, P) = ((v, u) \cdot \underline{A}, H, vP)$ ; as this differs from the one already defined, it does not yield an identification of  $Y_U^{\text{lg}}$  with a union of irreducible components of  $\bar{Y}_{U'}$ .

**Remark 8.1.3.** We note also that the ordinary locus of  $Y_{J,N}^{\text{lg}}$  can instead be viewed as parametrising pairs  $(\underline{A}, \iota)$  where  $\iota : \mu_p \otimes \mathcal{O}_F \xrightarrow{\sim} \ker \text{Frob}_A$ . Since  $Y_{J,N}^{\text{lg}}$  is normal, it is essentially the scheme defined as  $\overline{\mathfrak{M}}(E, \mu_{pN})^{\text{Kum}}$  in [1, §9]; the differences are that we are working with full level  $N$  structure and not including the cusps. We will not however make any direct use of the fact that  $Y_U^{\text{lg}}$  or  $Y_{J,N}^{\text{lg}}$  is normal; in particular, we will not compute divisors on them as in [1, §12], appealing instead in the proof of Theorem 8.2.2 below to general properties of logarithmic differentiation in order to descend the problem to  $\bar{Y}_U$ .

### 8.2. Construction of $\Theta$ -operators

For each  $\tau \in \Sigma$ , we consider the inclusion  $\bar{\omega}_\tau^{-1} \subset \text{Sym}_{\mathcal{O}_{\bar{Y}_U}}(\bigoplus_{\tau \in \Sigma} \bar{\omega}_\tau^{-1})$ , which induces an injective morphism

$$\bar{\omega}_\tau^{-1} \rightarrow \pi_{U,*} \mathcal{O}_{Y_U^{\text{lg}}} = \text{Sym}_{\mathcal{O}_{\bar{Y}_U}}(\bigoplus_{\tau \in \Sigma} \bar{\omega}_\tau^{-1})/I,$$

hence an injective morphism  $\pi_U^* \bar{\omega}_\tau^{-1} \rightarrow \mathcal{O}_{Y_U^{\text{lg}}}$ , which we view as a section of  $\pi_U^* \bar{\omega}_\tau$ . We denote this section by  $h_\tau$ , and call it a *fundamental Hasse invariant*. The definition of  $Y_U^{\text{lg}}$  implies that these satisfy the relation

$$h_{\text{Fr}^{-1} \circ \tau}^p = h_\tau \pi_U^*(\text{Ha}_\tau).$$

Recall now the Kodaira–Spencer isomorphism (6.2). Taking  $A$  to be the universal HBAV over  $S = \bar{Y}_{J,N}$  and decomposing over embeddings  $\tau$  yields a  $G_{U,N}$ -equivariant isomorphism

$$\bigoplus_{\tau \in \Sigma} (\bar{\omega}_\tau^2 \otimes_{\mathcal{O}_{\bar{Y}_{J,N}}} \bar{\delta}_\tau^{-1}) \simeq \Omega_{\bar{Y}_{J,N}/E}^1$$

whose union over  $J$  descends to an isomorphism

$$\bigoplus_{\tau \in \Sigma} (\bar{\omega}_\tau^2 \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\delta}_\tau^{-1}) \simeq \Omega_{\bar{Y}_U/E}^1 \tag{8.1}$$

of vector bundles on  $\bar{Y}_U$ . We let  $\text{KS}_\tau : \Omega_{\bar{Y}_U/E}^1 \rightarrow \bar{\omega}_\tau^2 \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\delta}_\tau^{-1}$  denote the composite of its inverse with the projection to the  $\tau$ -component.

Let  $\mathcal{F}_U$  denote the sheaf of total fractions on  $\bar{Y}_U$ , and  $F_U = H^0(\bar{Y}_U, \mathcal{F}_U)$  the ring of meromorphic functions on  $\bar{Y}_U$ , so  $F_U$  is the product of the function fields of the components of  $\bar{Y}_U$ . Similarly let  $\mathcal{F}_U^{\text{lg}}$  be the sheaf of total fractions on  $Y_U^{\text{lg}}$  (so  $\mathcal{F}_U^{\text{lg}} = \pi_U^* \mathcal{F}_U$ )

and let  $F_U^{\text{lg}}$  be the ring of meromorphic functions on  $Y_U^{\text{lg}}$ , so  $F_U^{\text{lg}}$  is Galois over  $F_U$  with Galois group  $(O_F/pO_F)^\times$ . Since the natural map  $\pi_U^* \Omega_{Y_U/E}^1 \rightarrow \Omega_{Y_U^{\text{lg}}/E}^1$  is generically an isomorphism, i.e.,

$$\pi_U^*(\Omega_{Y_U/E}^1 \otimes_{\mathcal{O}_{\bar{Y}_U}} \mathcal{F}_U) \simeq \pi_U^* \Omega_{Y_U/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}} \simeq \Omega_{Y_U^{\text{lg}}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}},$$

the pull-back of  $\text{KS}_\tau$  induces a morphism

$$\Omega_{Y_U^{\text{lg}}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}} \rightarrow \pi_U^*(\bar{\omega}_\tau^2 \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\delta}_\tau^{-1}) \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}},$$

which we will denote by  $\text{KS}_\tau^{\text{lg}}$ .

Suppose now that  $f \in M_{k,l}(U; E)$ . Let  $h^k = \prod_\tau h_\tau^{k_\tau}$  and  $g^l = \prod_\tau g_\tau^{l_\tau}$ , where  $h_\tau$  is the fundamental Hasse invariant and  $g_\tau$  is any trivialisation of  $\bar{\delta}_\tau$ . Then  $h^{-k} \pi_U^*(g^{-l} f) \in F_U^{\text{lg}}$ , so we may apply  $\text{KS}_\tau^{\text{lg}}$  to

$$d(h^{-k} \pi_U^*(g^{-l} f)) \in \Omega_{F_U^{\text{lg}}/E}^1 = H^0(Y_U^{\text{lg}}, \Omega_{Y_U^{\text{lg}}/E}^1 \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}}).$$

**Definition 8.2.1.** We define

$$\Theta_\tau^{\text{lg}}(f) = h^k \pi_U^*(g^l \text{Ha}_\tau) \text{KS}_\tau^{\text{lg}}(d(h^{-k} \pi_U^*(g^{-l} f))) \in H^0(Y_U^{\text{lg}}, \pi_U^* \bar{\omega}^{k',l'} \otimes_{\mathcal{O}_{Y_U^{\text{lg}}}} \mathcal{F}_U^{\text{lg}}),$$

where

- if  $\text{Fr} \circ \tau = \tau$ , then  $k'_\tau = k_\tau + p + 1$  and  $k'_{\tau'} = k_{\tau'}$  if  $\tau' \neq \tau$ ;
- if  $\text{Fr} \circ \tau \neq \tau$ , then  $k'_\tau = k_\tau + 1$ ,  $k'_{\text{Fr}^{-1} \circ \tau} = k_{\text{Fr}^{-1} \circ \tau} + p$ , and  $k_{\tau'} = k_{\tau'}$  if  $\tau' \notin \{\text{Fr}^{-1} \circ \tau, \tau\}$ ;
- $l'_\tau = l_\tau - 1$ , and  $l'_{\tau'} = l_{\tau'}$  if  $\tau' \neq \tau$ .

Since the ratio of any two trivialisations of  $\bar{\delta}_\tau$  is locally constant, we see that  $\Theta_\tau^{\text{lg}}(f)$  is independent of the choice of  $g_\tau$ . Moreover, it is straightforward to check that  $\Theta_\tau^{\text{lg}}(f)$  is invariant under the action of  $(O_F/p)^\times$ , hence descends to a section of  $\bar{\omega}^{k',l'} \otimes_{\mathcal{O}_{\bar{Y}_U}} \mathcal{F}_U$ , which we denote by  $\Theta_\tau(f)$ . What is more difficult is that  $\Theta_\tau(f)$  is in fact a section of  $\bar{\omega}^{k',l'}$ . In fact, we have the following result, essentially due to Andreatta and Goren [1]:

**Theorem 8.2.2.** *If  $f \in M_{k,l}(U; E)$  and  $\tau \in \Sigma$ , then  $\Theta_\tau(f) \in M_{k',l'}(U; E)$ . Moreover,  $\Theta_\tau(f)$  is divisible by  $\text{Ha}_\tau$  if and only if either  $f$  is divisible by  $\text{Ha}_\tau$ , or  $k_\tau$  is divisible by  $p$ .*

*Proof.* First note that the formula

$$\prod_{\tau' \in \Sigma} h_{\tau'}^{p-1} = \prod_{\tau \in \Sigma} \pi_U^*(\text{Ha}_{\tau'})$$

implies that  $h^k$  is non-vanishing on  $\pi_U^{-1}(Y_U^{\text{ord}})$ , so  $h^{-k} \pi_U^*(g^{-l} f)$  and hence  $d(h^{-k} \pi_U^*(g^{-l} f))$  are regular on  $\pi_U^{-1}(Y_U^{\text{ord}})$ . Since  $\pi_U$  is étale on  $\pi_U^{-1}(Y_U^{\text{ord}})$ , it follows that  $d(h^{-k} \pi_U^*(g^{-l} f))$  restricts to a section of

$$\pi_U^* \Omega_{Y_U^{\text{ord}}/E}^1 = \Omega_{\pi_U^{-1}(Y_U^{\text{ord}})/E}^1.$$

Therefore  $\Theta_\tau^{\text{lg}}(f)$  is regular on  $\pi_U^{-1}(Y_U^{\text{ord}})$ , and  $\Theta_\tau(f)$  is regular on  $Y_U^{\text{ord}}$ .

To complete the proof of the first assertion, we must show that if  $z$  is the generic point of an irreducible component  $Z \subset Z_{U, \tau_0}$ , then (the germ at  $z$ ) of  $\Theta_\tau(f)$  lies in  $\overline{\omega}_z^{k', l'}$ , i.e.,  $\text{ord}_z(\Theta_\tau(f)) \geq 0$ . For the second assertion, it suffices to show further that if  $\tau_0 = \tau$ , then  $\text{ord}_z(\Theta_\tau(f)) > 0$  if and only if either  $p \mid k_\tau$  or  $\text{ord}_z(f) > 0$ .

Let us revert to the notation of the proof of Proposition 8.1.1, so  $\mathcal{O}_{\overline{Y}_{U, z}} = R_{\mathfrak{p}}$  and  $x_{\tau'}$  is a basis for the stalk of  $\overline{\omega}_{\tau'}^{-1}$  at  $z$ . Let  $y_{\tau'}$  be the dual basis for  $\overline{\omega}_{\tau'}$ , so that  $y^k g^l$  is a basis for  $\overline{\omega}_z^{k, l}$  over  $R_{\mathfrak{p}}$  (where as usual  $y^k$  denotes  $\prod_{\tau'} y_{\tau'}^{k_{\tau'}}$ ), and we have  $f = \phi_f y^k g^l$  for some  $\phi_f \in R_{\mathfrak{p}}$ . In terms of the basis  $\pi_{\overline{U}}^*(y_{\tau'})$  for  $\pi_{\overline{U}}^* \overline{\omega}_{\tau', z}$ , the fundamental Hasse invariant  $h_{\tau'}$  is given by  $t_{\tau'} \pi_{\overline{U}}^*(y_{\tau'})$ , so that  $h^{-k} \pi_{\overline{U}}^*(g^{-l} f) = t^{-k} \phi_f$  in the total fraction ring of  $T_{\mathfrak{p}, \tau}$  over which we deduce that

$$\Theta_\tau^{\text{lg}}(f) = \text{KS}_\tau^{\text{lg}}(t^k d(t^{-k} \phi_f)) \pi_{\overline{U}}^*(\text{Ha}_\tau y^k g^l).$$

The formulas  $t_{\text{Fr}^{-1} \circ \tau}^p = r_\tau t_\tau$  in  $T_{\mathfrak{p}}$  imply that  $r_{\tau'} dt_{\tau'} = -t_{\tau'} dr_{\tau'}$  in  $\Omega_{T_{\mathfrak{p}}/E}^1$ , and it follows that

$$\begin{aligned} \Theta_\tau(f) &= \text{KS}_\tau(d\phi_f - \phi_f r^k d(r^{-k})) \text{Ha}_\tau y^k g^l \\ &= \text{KS}_\tau\left(d\phi_f + \phi_f \sum_{\tau' \in \Sigma} k_{\tau'} \frac{dr_{\tau'}}{r_{\tau'}}\right) \text{Ha}_\tau y^k g^l \end{aligned} \tag{8.2}$$

(locally at  $z$ ). Since  $\text{Ha}_\tau = r_\tau y^{k_{\text{Ha}_\tau}}$ , we conclude that

$$\text{ord}_z(\Theta_\tau(f)) = \text{ord}_z\left(r_\tau \text{KS}_\tau(d\phi_f) + k_\tau \phi_f \text{KS}_\tau(dr_\tau) + \sum_{\tau' \neq \tau} k_{\tau'} r_\tau \phi_f \frac{\text{KS}_\tau(dr_{\tau'})}{r_{\tau'}}\right).$$

In particular, if  $\tau = \tau_0$ , then we see immediately that  $\text{ord}_z(\Theta_\tau(f)) \geq 0$ , with equality if and only if  $\text{ord}_z(k_\tau \phi_f \text{KS}_\tau(dr_\tau)) = 0$ , so in this case we are reduced to proving that  $\text{ord}_z(\text{KS}_\tau(dr_\tau)) = 0$ . On the other hand, if  $\tau \neq \tau_0$ , then we are reduced to proving that  $\text{ord}_z(\text{KS}_\tau(dr_{\tau_0})) > 0$ . Both cases are treated by the lemma below. ■

The following is essentially the unramified case of [1, Prop. 12.34], which we prove using a computation of Koblitz [48] (as presented in [39]) instead of the theory of displays.

**Lemma 8.2.3.** *Let  $z$  be a generic point of  $Z_{U, \tau_0}$  and let  $r$  be a generator for the maximal ideal of  $\mathcal{O}_{\overline{Y}_{U, z}}$ . Then  $\text{ord}_z(\text{KS}_\tau(dr)) = 0$  if and only if  $\tau = \tau_0$ .*

*Proof.* First note that since the projection  $\coprod \overline{Y}_{J, N} \rightarrow \overline{Y}_U$  is étale, we may replace  $\overline{Y}_U$  by  $\overline{Y}_{J, N}$  and  $Z_{U, \tau_0}$  by  $Z_{\tau_0}$  in the statement of the lemma. Note also that the conclusion of the lemma is independent of the choice of uniformising parameter  $r$ , since if  $u \in \mathcal{O}_{\overline{Y}_{J, N, z}}^\times$ , then

$$\text{KS}_\tau(d(ur)) = u \text{KS}_\tau(dr) + r \text{KS}_\tau(du).$$

We will prove that for every closed point  $x$  of  $Z_{\tau_0}$ , there is a choice of parameter  $r$ , regular at  $x$ , such that the fibre of  $\text{KS}_\tau(dr)$  at  $x$  vanishes if and only if  $\tau \neq \tau_0$ . By the formula above, the equivalence then holds for all  $r$  regular at  $x$ , hence for all  $x$  at which any given  $r$  is regular, and this implies the lemma.

Let  $R = \widehat{\mathcal{O}}_{\overline{Y}_{J,N},x}$ ,  $A$  the pull-back to  $\text{Spec } R$  of the universal HBAV over  $Y_{J,N}$  and  $M = H^1_{\text{DR}}(A/R)$ . Letting

$$\nabla : M \rightarrow \Omega^1_{R/E} \otimes_R M$$

denote the Gauss–Manin connection and  $\phi : M \rightarrow M$  the morphism induced by the absolute Frobenius morphism on  $A$ , we are in the situation of [39, A2.1].<sup>6</sup> We then have  $L = H^0(A, \Omega^1_{A/R})$  and  $N = H^1(A, \mathcal{O}_A)$ , and the  $O_F$ -action on  $A$  yields decompositions

$$L = \bigoplus L_\tau, \quad M = \bigoplus M_\tau \quad \text{and} \quad N = \bigoplus N_\tau$$

into free  $R$ -modules indexed by  $\tau \in \Sigma$ . The Hasse–Witt endomorphism of [39, (A2.1.1)] decomposes as the sum of Frobenius semi-linear morphisms  $N_{\text{Fr}^{-1} \circ \tau} \rightarrow N_\tau$  such that, after choosing a generator for  $J/pJ$  and applying the isomorphisms  $\text{Lie}(A) \otimes_{O_F} J \simeq N$ , the induced morphism  $N^{\otimes p}_{\text{Fr}^{-1} \circ \tau} \rightarrow N_\tau$  corresponds to the completion at  $x$  of the partial Hasse invariant  $\text{Ha}_{N,J,\tau}$ . The general properties of the construction of the Gauss–Manin connection ensure its compatibility with the  $O_F$ -action, so that it decomposes as a direct sum of connections  $\nabla_\tau$  on  $M_\tau$ . Furthermore, the morphism [39, (A2.1.2)] induced by  $\nabla$  is the completion at  $x$  of the reduction mod  $\pi$  of (6.1), and hence the induced morphism

$$\bigoplus_\tau \text{Hom}_R(\wedge^2_R M_\tau, L_\tau^{\otimes R^2}) = \bigoplus_\tau \text{Hom}_R(N_\tau, L_\tau) \rightarrow \Omega^1_{R/E}$$

is the completion at  $x$  of the Kodaira–Spencer isomorphism on  $\overline{Y}_{J,N}$ .

Following [39], we let  $R_i = R/\mathfrak{m}_R^{i+1}$  (the cases of interest being  $i = 0, 1$ ), and similarly use subscript  $i$  for reductions mod  $\mathfrak{m}^{i+1}$  of  $R$ -modules, morphisms and matrices. We now choose a basis for  $M_1$  as in [39, (A2.1.6)] as follows: First choose a basis for  $M_0$  consisting of vectors  $e_{\tau,0} \in L_{\tau,0}$ ,  $f_{\tau,0} \in M_{\tau,0}$  for  $\tau \in \Sigma$ . Then

$$\phi_0(e_{\text{Fr}^{-1} \circ \tau,0}) = 0 \quad \text{and} \quad \phi_0(f_{\text{Fr}^{-1} \circ \tau,0}) = c_\tau e_{\tau,0} + d_\tau f_{\tau,0}$$

for some  $c_\tau, d_\tau \in R_0$  not both zero. Replacing  $f_{\tau,0}$  by  $e_{\tau,0} + f_{\tau,0}$  whenever  $c_\tau = 0$ , we may assume  $c_\tau \neq 0$ , and then replacing  $e_{\tau,0}$  by  $c_\tau^{-1}e_{\tau,0}$ , we may assume  $c_\tau = 1$  for all  $\tau$ . Now lift each pair  $(e_{\tau,0}, f_{\tau,0})$  to a basis  $(e_\tau, f'_\tau)$  of  $M_{\tau,1}$  with  $e_\tau \in L_{\tau,1}$  and let  $f_\tau = P(f'_\tau)$  where  $P$  is defined in [39, (A2.1.3)]. Since  $\nabla$  respects the decomposition  $M = \bigoplus M_\tau$ , so does  $P$ , and hence  $f_\tau \in M_{\tau,1}$ . Moreover,  $f_\tau \equiv f'_\tau \pmod{\mathfrak{m}_R}$ , so in the matrix  $\begin{pmatrix} 0 & B_1 \\ 0 & H_1 \end{pmatrix}$  of [39, (A2.1.7)] representing  $\phi$  on  $M_1$  with respect to the basis  $((e_\tau), (f_\tau))$ ,

- the reduction  $B_0$  of  $B_1$  is defined by  $b_{\tau,\tau'} = \delta_{\tau,\text{Fr} \circ \tau'}$ ;
- the matrix  $H_1$  represents the Hasse–Witt endomorphism  $N_1 \rightarrow N_1$  with respect to the basis induced by  $(f_\tau)$ , so  $h_{\tau,\tau'} = 0$  if  $\tau' \neq \text{Fr}^{-1} \circ \tau$  and  $h_{\tau,\text{Fr}^{-1} \circ \tau} = r_\tau$ , where  $r_\tau$  represents the pull-back of  $\text{Ha}_{J,N,\tau}$  to  $R_1$  with respect to the basis induced by the map sending  $f_{\text{Fr}^{-1} \circ \tau}$  to  $f_\tau$ .

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<sup>6</sup>The notations  $A$  and  $F$  in [39], being in other use here, have been replaced by  $R$  and  $\phi$ .

In particular,  $B_0$  is invertible, and [39, Proposition A2.1.8] implies that the matrix

$$K_0 = (H_0 - H_1)B_0^{-1}$$

with entries in  $\mathfrak{m}_R/\mathfrak{m}_R^2 \simeq \Omega_{R/E}^1 \otimes_R R_0$  is diagonal with  $(\tau, \tau)$ -entry  $-dr_\tau$ . Note that the map  $L_0 \rightarrow (\mathfrak{m}_R/\mathfrak{m}_R^2) \otimes_{R_0} N_0$  is the fibre of (6.1) at  $x$ , and is represented by  $K_0$  with respect to the bases  $(e_{\tau,0})$  of  $L_0$  and  $(f_{\tau,0} \bmod L_0)$  of  $N_0$ . It follows that the fibre at  $x$  of the Kodaira–Spencer isomorphism is the map

$$\bigoplus_{\tau} \text{Hom}_{R_0}(N_{\tau,0}, L_{\tau,0}) \simeq \text{Hom}_{R_0 \otimes_{O_F}}(N_0, L_0) \rightarrow \Omega_{R/E}^1 \otimes_R R_0$$

under which the basis vector in the  $\tau$ -component induced by  $f_{\tau,0} \mapsto e_{\tau,0}$  corresponds to  $-dr_\tau$ . Note that  $r_{\tau_0}$  is the image in  $\mathfrak{m}_R/\mathfrak{m}_R^2$  of a uniformising parameter for  $Z_{\tau_0}$  in a neighbourhood of  $x$ , and the fibre at  $x$  of  $\text{KS}_\tau$  sends  $dr_{\tau_0}$  to 0 if and only if  $\tau \neq \tau_0$ , so this completes the proof of the lemma. ■

It is straightforward to check that the maps  $\Theta_\tau$  are compatible with the maps  $[U_1gU_2]$  for sufficiently small  $U_1, U_2$  and  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  such that  $g^{-1}U_1g \subset U_2$  and  $g_p \in \text{GL}_2(O_{F,p})$ . Taking limits over open compact subgroups  $U$  therefore gives

**Corollary 8.2.4.** *For any weight  $(k, l)$ ,  $\Theta_\tau$  defines a map*

$$M_{k,l}(E) \rightarrow M_{k',l'}(E)$$

*commuting with the action of all  $g \in \text{GL}_2(\mathbb{A}_F^\infty)$  such that  $g_p \in \text{GL}_2(O_{F,p})$ . In particular, for any open compact subgroup  $U$  of  $\text{GL}_2(\mathbb{A}_F^\infty)$  containing  $\text{GL}_2(O_{F,p})$ ,  $\Theta_\tau$  defines a map*

$$M_{k,l}(U; E) \rightarrow M_{k',l'}(U; E)$$

*commuting with the operators  $T_v$  and  $S_v$  for all  $v \nmid p$  such that  $\text{GL}_2(O_{F,v}) \subset U$ .*

### 9. $q$ -expansions

We review the definition and properties of  $q$ -expansions, including the effect on them of Hecke and partial  $\Theta$ -operators, and we generalise a result of Katz on the kernel of  $\Theta$ . See [18] for a further generalisation to the case where  $p$  is ramified in  $F$ .

#### 9.1. Definition and explicit descriptions

Suppose as usual that  $U$  is a sufficiently small open compact subgroup of  $\text{GL}_2(\widehat{O}_F)$  containing  $\text{GL}_2(\widehat{O}_{F,p})$ , with  $k, l \in \mathbb{Z}^\Sigma$  and  $R$  a Noetherian  $\mathcal{O}$ -algebra such that  $v^{k+2l} = 1$  in  $R$  for all  $v \in O_F^\times \cap U$ . Recall from §6.4 that  $X_U$  is the minimal compactification of  $Y_U$ , and a cusp of  $X_U$  is a connected component of  $X_U - Y_U$ . When it is clear and convenient, we will suppress the subscript  $R$  from base-changes of morphisms to which schemes have been extended, writing just  $j$  for example for the inclusion  $Y_{U,R} \hookrightarrow X_{U,R}$ .

**Definition 9.1.1.** For each cusp  $C$  of  $X_U$ , we let  $Q_{\tilde{C},R}^{k,l}$  denote the completion of  $j_*\mathcal{L}_{U,R}^{k,l}$  at  $C_R$ , and for  $f \in M_{k,l}(U; R)$ , we define the  $q$ -expansion of  $f$  at  $C$  to be its image in  $Q_{\tilde{C},R}^{k,l}$ .

We now proceed to describe  $Q_{\tilde{C},R}^{k,l}$  more explicitly. (See [18, Sections 7.1–2] for a more general and detailed analysis.) We first recall (e.g. from [11]) the description in the context of  $X_{J,N}$ , supposing that  $N \geq 3$  and  $\mu_N(\mathbb{Q}) \subset \mathcal{O}$ . The cusps  $\tilde{C}$  of  $X_{J,N}$  are in bijection with equivalence classes of data:

- fractional ideals  $\mathfrak{a}, \mathfrak{b}$  of  $O_F$ ;
- an exact sequence of  $O_F$ -modules  $0 \rightarrow (\mathfrak{a}\mathfrak{b})^{-1} \rightarrow H \rightarrow \mathfrak{b} \rightarrow 0$ ;
- an isomorphism  $J \xrightarrow{\sim} \mathfrak{a}\mathfrak{b}^{-1}$ ;
- an isomorphism  $(O_F/NO_F)^2 \xrightarrow{\sim} H/NH$ .

By the Koecher Principle (as in the discussion preceding [53, Prop. 6.11]; see [49, Thm. 2.5] for a statement in the required generality), we have  $\pi_*\mathcal{L}_{J,N,R}^{k,l,\text{tor}} = \tilde{j}_*\mathcal{L}_{J,N,R}^{k,l}$ , where  $\pi : X_{J,N}^{\text{tor}} \rightarrow X_{J,N}$  is the projection from a toroidal compactification of  $Y_{J,N}$ , and  $\mathcal{L}_{J,N}^{k,l,\text{tor}}$  is the canonical extension of  $\mathcal{L}_{J,N}^{k,l}$  to a line bundle over  $X_{J,N}^{\text{tor}}$  associated to the semiabelian scheme  $A^{\text{tor}}$  extending the universal abelian scheme  $A$  over  $Y_{J,N}$ . Since  $\pi$  is projective, the Formal Functions Theorem identifies the completion  $Q_{\tilde{C},R}^{k,l}$  of the sheaf  $\tilde{j}_*\mathcal{L}_{J,N,R}^{k,l}$  at  $\tilde{C}_R$  with the set of global sections of the completion of the coherent sheaf  $\mathcal{L}_{J,N,R}^{k,l,\text{tor}}$  at the fibre of  $\pi$  over  $\tilde{C}_R$ .

The construction of the toroidal compactification identifies the completion of  $X_{J,N}^{\text{tor}}$  at the fibre over  $\tilde{C}$  with the quotient  $\hat{S}/U_N$ , where  $\hat{S}$  is the formal scheme denoted  $S_N(\{\sigma_\alpha^{\tilde{C}}\})^\wedge$  in [11, 3.4.2] (for a choice of polyhedral cone decomposition  $\{\sigma_\alpha^{\tilde{C}}\}$ ), but we define the action of  $\alpha \in U_N = \ker(O_F^\times \rightarrow (O_F/N)^\times)$  as being induced by multiplication by  $\alpha^2$ . We therefore obtain an identification of the  $R$ -algebra  $Q_{\tilde{C},R}^{0,0}$  with

$$\hat{S}_{\tilde{C},R} = (R[[q^m]]_{m \in (N^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}})^{U_N}$$

where  $\alpha \in U_N$  acts via  $q^m \mapsto q^{\alpha^2 m}$  on power series, under which  $\tilde{C}$  corresponds to the closed subscheme of  $\text{Spec } \hat{S}_{\tilde{C}}$  defined by  $\sum r_m q^m \mapsto r_0$ . Furthermore, the pull-back of  $A^{\text{tor}}$  to  $\hat{S}$  is the Tate semi-HBAV  $T_{\mathfrak{a},\mathfrak{b}}$  associated to the quotient  $(\mathbb{G}_m \otimes (\mathfrak{a}\mathfrak{b})^{-1})/q^{\mathfrak{b}}$  (by Mumford’s construction). The canonical trivialisations

$$\begin{aligned} \text{Lie}(T_{\mathfrak{a},\mathfrak{b}}/\hat{S}) &\simeq (\mathfrak{a}\mathfrak{b})^{-1} \otimes \mathcal{O}_{\hat{S}} \simeq \text{Hom}(\mathfrak{a}, \mathcal{O}_{\hat{S}}), \\ \text{Lie}(T_{\mathfrak{a},\mathfrak{b}}^\vee/\hat{S}) &\simeq (\mathfrak{b}\mathfrak{b})^{-1} \otimes \mathcal{O}_{\hat{S}} \simeq \text{Hom}(\mathfrak{b}, \mathcal{O}_{\hat{S}}) \end{aligned}$$

then give a trivialisation of the pull-back of  $\mathcal{L}_{J,N,R}^{k,l,\text{tor}}$  to  $\hat{S}$ , and hence an identification

$$Q_{\tilde{C},R}^{k,l} = (D^{k,l} \otimes_{\mathcal{O}} R[[q^m]]_{m \in (N^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}})^{U_N},$$

where

$$D^{k,l} = \bigotimes_{\tau} ((\mathfrak{a} \otimes \mathcal{O})_{\tau}^{\otimes k_{\tau}} \otimes_{\mathcal{O}} (J\mathfrak{b}^{-1} \otimes \mathcal{O})_{\tau}^{\otimes l_{\tau}})$$

and  $\alpha \in U_N$  acts as  $\alpha^k$  on  $D^{k,l}$ . Note that  $D^{k,l}$  is free of rank 1 over  $\mathcal{O}$ , and letting  $b$  be a basis, we have

$$Q_{\tilde{C},R}^{k,l} = \left\{ \sum_{m \in (N^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}} (b \otimes r_m)q^m \mid r_{\alpha^2 m} = \alpha^k r_m \text{ for all } \alpha \in U_N \right\}.$$

We will also write  $\overline{D}^{k,l}$  for  $D^{k,l} \otimes_{\mathcal{O}} E$ .

If we fix the data of  $\mathfrak{a}$ ,  $\mathfrak{b}$  and  $\mathfrak{a}\mathfrak{b}^{-1} \simeq J$ , then the corresponding cusps of  $X_{J,N}$  are in bijection with  $P_N \backslash \text{GL}_2(\mathcal{O}_F/N)$ , where

$$P_N = \left\{ \begin{pmatrix} \alpha^{-1} & * \\ 0 & \alpha \end{pmatrix} \bmod N \mid \alpha \in \mathcal{O}_F^\times \right\}.$$

Here we have chosen isomorphisms

$$s : \mathcal{O}_F/\mathcal{N}\mathcal{O}_F \simeq N^{-1}\mathfrak{b}/\mathfrak{b} \quad \text{and} \quad t : \mathcal{O}_F/\mathcal{N}\mathcal{O}_F \simeq \mu_N \otimes (\mathfrak{a}\mathfrak{b})^{-1}$$

to define a level  $N$  structure  $\eta$  on  $T_{\mathfrak{a},\mathfrak{b}}$  by  $\eta(x, y) = t(y)q^{s(x)}$ , and then associated the coset  $P_N g$  to the cusp of  $X_{J,N}$  defined by  $T_{\mathfrak{a},\mathfrak{b}}$  with level  $N$  structure  $\eta \circ r_{g^{-1}}$ . Under this bijection, the (right) action of  $G_{U,N}$  is defined by

$$P_N g \cdot (v, u) = P_N \begin{pmatrix} v^{-1} & 0 \\ 0 & 1 \end{pmatrix} g u.$$

The stabiliser in  $G_{U,N}$  of (the cusp corresponding to)  $P_N g$  is therefore the image of the group

$$\left\{ (v, u) \in \mathcal{O}_{F,+}^\times \times U \mid g u g^{-1} \equiv \begin{pmatrix} v\alpha^{-1} & * \\ 0 & \alpha \end{pmatrix} \bmod N \text{ for some } \alpha \in \mathcal{O}_F^\times \right\}.$$

We find that the (left) action on  $Q_{\tilde{C},R}^{k,l}$  of such an element  $(v, u)$ , with

$$g u g^{-1} = \begin{pmatrix} v\alpha^{-1} & -v\alpha^{-1}x \\ 0 & \alpha \end{pmatrix} \bmod N,$$

is defined by  $\alpha^k v^l$  on  $D^{k,l}$  and

$$\sum r_m q^m \mapsto \sum \zeta(xm) r_m q^{\alpha^2 v^{-1} m}$$

on  $R[[q^m]]_{m \in (N^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}}$ , where  $\zeta : N^{-1}\mathfrak{a}\mathfrak{b}/\mathfrak{a}\mathfrak{b} \rightarrow \mu_N$  is the composite of the  $\mathcal{O}_F$ -linear isomorphism  $N^{-1}\mathfrak{a}\mathfrak{b}/\mathfrak{a}\mathfrak{b} \rightarrow \mathfrak{b}^{-1} \otimes \mu_N$  induced by  $t \circ s^{-1}$  with  $\text{tr}_{F/\mathbb{Q}} \otimes 1$ . The module  $Q_{\tilde{C},R}^{k,l}$  (over  $\widehat{S}_{\tilde{C},R} = Q_{\tilde{C},R}^{0,0}$ ) is then given by the invariants in  $Q_{\tilde{C},R}^{k,l}$  under the action of the above stabiliser. In particular, we note the following two special cases:

**Proposition 9.1.2.** *Suppose that  $\mu^{k+2l} = 1$  in  $R$  for all  $\mu \in \mathcal{O}_F^\times$  such that  $\mu \equiv 1 \pmod{\mathfrak{n}}$ , and let  $b$  be a generator of  $D^{k,l}$ .*



- If  $U = U(\mathfrak{n})$ , then

$$\mathcal{Q}_{C,R}^{k,l} \simeq \left\{ \sum_{m \in (\mathfrak{n}^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}} (b \otimes r_m)q^m \mid r_{vm} = v^{-l}r_m \text{ for all } v \in U_{\mathfrak{n},+} \right\}.$$

- If  $U = U_1(\mathfrak{n})$  and  $g = 1$ , then

$$\mathcal{Q}_{C,R}^{k,l} = \left\{ \sum_{m \in (\mathfrak{a}\mathfrak{b})_+ \cup \{0\}} (b \otimes r_m)q^m \mid r_{vm} = v^{-l}r_m \text{ for all } v \in O_{F,+}^\times \right\}.$$

Note that the isomorphism (in the case of  $U = U(\mathfrak{n})$ ) depends on the choice of representative  $g$ . Note also that the description of  $\mathcal{Q}_{C,R}^{k,l}$  is compatible in the obvious senses with the morphisms induced by base-changes  $R \rightarrow R'$ , and inclusions  $U' \subset U$  (for cusps  $C'$  of  $X_{U'}$  mapping to  $C$ ).

### 9.2. The $q$ -expansion principle

The  $q$ -expansion at  $C$  of a form  $f \in M_{k,l}(U; R)$  vanishes if and only if (the extension to  $X_{U,R}$  of)  $f$  vanishes on a neighbourhood of  $C_R$ , which is equivalent to the vanishing of  $f$  on all connected components of  $X_{U,R}$  intersecting  $C_R$ . (Note that if  $\text{Spec } R$  is connected, then there is a unique component containing  $C_R$ .) Recall from §2.6 that  $Z_U$  is the scheme representing the set of components of  $Y_U$  and hence  $X_U$ , so we have the following:

**Lemma 9.2.1.** *If  $\mathcal{S}$  is any set of cusps of  $X_U$  such that  $\coprod_{C \in \mathcal{S}} C \rightarrow Z_U$  is surjective, then the  $q$ -expansion map*

$$M_{k,l}(U; R) \rightarrow \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{C,R}^{k,l}$$

*is injective.*

If  $U = U_1(\mathfrak{n})$ , then  $\det(U) = \widehat{O}_F^\times$ , so  $Z_U$  is in bijection with the strict class group of  $F$ . For each representative  $J$ , we choose  $\mathfrak{b} = J^{-1}$ ,  $\mathfrak{a} = O_F$ , and  $\mathcal{S}$  consisting of a single  $C$  at infinity (i.e. as in the second part of Proposition 9.1.2) on each component associated to a fixed  $t : O_F/NO_F \simeq \mu_N \otimes \mathfrak{d}^{-1}$  (independent of  $J$ ) and  $s : O_F/NO_F \simeq (NJ)^{-1}/J^{-1}$  (of which  $C$  is independent). Using the isomorphism  $D^{k,l} \simeq \mathcal{O}$  obtained from the inclusion  $J\mathfrak{d}^{-1} \subset F$  for each  $J$ , we obtain an injective  $q$ -expansion map (defined for arbitrary  $\mathfrak{n}$ )

$$M_{k,l}(U_1(\mathfrak{n}); R) \rightarrow \bigoplus_J \left\{ \sum_{m \in J_{\mp}^{-1} \cup \{0\}} r_m q^m \mid r_{vm} = v^{-l}r_m \text{ for all } v \in O_{F,+}^\times \right\}. \tag{9.1}$$

### 9.3. $q$ -expansions of partial Hasse invariants

Let us now return to the case of arbitrary (sufficiently small)  $U$ , take  $R = E$  and consider the  $q$ -expansions of the partial Hasse invariants  $\text{Ha}_\tau$ . Since the pull-back  $\text{Ver}_{T_{\alpha,\mathfrak{b}}}^*$  of the

relative Verschiebung on the Tate semi-HBAV  $T_{\alpha,b}$  (see §9.1) is induced by the canonical isomorphism

$$\bigoplus_{\tau} (\alpha \otimes E)_{\tau} \rightarrow \bigoplus_{\tau} (\alpha \otimes E)_{\text{Fr}^{-1} \circ \tau}^{\otimes p},$$

we see that the  $q$ -expansion of  $\text{Ha}_{\tau}$  at any cusp is the constant 1, or more precisely  $\iota_{\tau} \otimes 1$  where  $\iota_{\tau} \in \overline{D}^{k_{\text{Ha}_{\tau}},0}$  is defined by the canonical isomorphism  $(\alpha \otimes E)_{\tau} \rightarrow (\alpha \otimes E)_{\text{Fr}^{-1} \circ \tau}^{\otimes p}$ .

Furthermore, the same is true for the  $q$ -expansions of  $G_{\tau} \in M_{0,k_{\text{Ha}_{\tau}}}(U; E)$  (with  $\overline{D}^{k_{\text{Ha}_{\tau}},0}$  replaced by  $\overline{D}^{0,k_{\text{Ha}_{\tau}}}$  and  $\alpha$  by  $J\delta^{-1}$ ), where  $G_{\tau}$  is obtained by descent from the canonical trivialisations defined in §3.4.

9.4.  $\Theta$ -operators on  $q$ -expansions

We continue to assume  $R = E$ . We now describe the effect of  $\Theta$ -operators on  $q$ -expansions. We first assume  $U = U(\mathfrak{n})$  for some  $\mathfrak{n}$  sufficiently small that  $v^l = 1 \pmod p$  for all  $v \in U_{\mathfrak{n},+}$ . By Proposition 9.1.2, we can identify  $\overline{Q}_C^{k,l} = Q_{C,E}^{k,l}$  with

$$\overline{D}^{k,l} \otimes_E \widehat{S}_{C,E} = \left\{ \sum_{m \in (\mathfrak{n}^{-1}\mathfrak{a}\mathfrak{b})_+ \cup \{0\}} (\bar{b} \otimes r_m) q^m \mid r_{v^m} = r_m \text{ for all } v \in U_{\mathfrak{n},+} \right\},$$

where  $\bar{b}$  is any basis for  $\overline{D}^{k,l}$ . In particular, note that  $\overline{Q}_C^{k,l}$  is free over  $\widehat{S}_{C,E}$  for all  $k, l$ .

We now appeal to formula (8.2), and observe that it is compatible with the analogous formula defining a map  $\overline{Q}_C^{k,l} \rightarrow \overline{Q}_C^{k',l'}$ , where  $\text{KS}_{\tau}$  is replaced by the completion of  $j_*\text{KS}_{\tau}$  at  $\overline{C} = C_E$ . Moreover, the formula is valid for any choices of bases  $y_{\tau'}$  for the completions of  $j_*\overline{\omega}_{\tau'}$  (which are invertible thanks to our choice of  $U$ ). In particular, we can choose the  $y_{\tau'}$  of the form  $a_{\tau'} \otimes 1$  where the  $a_{\tau'}$  are bases for  $(\alpha \otimes E)_{\tau'}$  such that  $\iota_{\tau'} \otimes a_{\tau'} = a_{\text{Fr}^{-1} \circ \tau'}^{\otimes p}$  for all  $\tau'$ . This gives  $y^k g^l = \bar{b} \otimes 1$  for some basis  $\bar{b}$  of  $\overline{D}^{k,l}$ , and in view of the  $q$ -expansions of the partial Hasse invariants,  $r_{\tau'} = 1$  for all  $\tau'$ . Thus if  $f$  has  $q$ -expansion  $\sum (\bar{b} \otimes r_m) q^m$  at  $C$ , then we are reduced to computing the image of  $\phi_f = \sum r_m q^m$  under the composite

$$\widehat{S}_{C,E} \xrightarrow{d} (\Omega_{X/U/E}^1)_{\overline{C}}^{\wedge} \rightarrow (j_*\Omega_{Y/U/E}^1)_{\overline{C}}^{\wedge} \rightarrow \overline{Q}_C^{(2,-1)_{\tau}}, \tag{9.2}$$

where  $(2, -1)_{\tau} = (k', l') - (k, l) - (k_{\text{Ha}_{\tau}}, 0)$  and the last map is induced by  $j_*\text{KS}_{\tau}$ .

A computation on the toroidal compactification identifies  $(j_*\Omega_{Y/U/E}^1)_{\overline{C}}^{\wedge}$  with

$$\mathfrak{n}^{-1}\mathfrak{a}\mathfrak{b} \otimes \widehat{S}_{C,E}$$

(in view of our assumption that  $U = U(\mathfrak{n})$  for sufficiently small  $\mathfrak{n}$ ) and the composite of the first two maps of (9.2) with  $q^m \mapsto m \otimes q^m$ . Moreover, identifying  $\overline{Q}_C^{(2,-1)_{\tau}}$  with

$$\overline{D}^{(2,-1)_{\tau}} \otimes_E \widehat{S}_{C,E} = (\mathfrak{d}\mathfrak{a}\mathfrak{b} \otimes_{\mathcal{O}} E)_{\tau} \otimes_E \widehat{S}_{C,E},$$

[41, (1.1.20)] gives that the last map of (9.2) is the inverse of the isomorphism induced by the inclusion  $\mathfrak{b}\alpha\mathfrak{b} \rightarrow \mathfrak{n}^{-1}\alpha\mathfrak{b}$ , followed by projection to the  $\tau$ -component. Therefore the  $q$ -expansion of  $\Theta_\tau(f)$  at  $C$  is given by

$$\sum (\iota_\tau \tau(m) \bar{b} \otimes r_m) q^m.$$

In view of the compatibility of  $q$ -expansions with the morphisms induced by inclusions  $U' \subset U$ , this formula is in fact valid for all sufficiently small  $U$ . We have thus proved

**Proposition 9.4.1.** *If the  $q$ -expansion at  $C$  of  $f \in M_{k,l}(U; E)$  is  $\sum (\bar{b} \otimes r_m) q^m$ , then the  $q$ -expansion at  $C$  of  $\Theta_\tau(f)$  is  $\sum (\iota_\tau \tau(m) \bar{b} \otimes r_m) q^m$ .*

Recall from Lemma 9.2.1 that a form is determined by its  $q$ -expansions. Using also that  $\iota_\tau \tau(m) = (\text{Fr}^{-1} \circ \tau(m))^p$ , we deduce the following (see §9.4 for the definition of  $G_\tau$ ):

**Corollary 9.4.2.** *For all  $\tau, \tau' \in \Sigma$  and  $f \in M_{k,l}(U; E)$ , we have the relations*

- $\Theta_\tau \Theta_{\tau'}(f) = \Theta_{\tau'} \Theta_\tau(f)$ , and
- $G_\tau \Theta_{\text{Fr}^{-1} \circ \tau}^p(f) = \text{Ha}_{\text{Fr}^{-1} \circ \tau}^p \text{Ha}_\tau \Theta_\tau(f)$ .

9.5. Hecke operators on  $q$ -expansions

We now describe the effect of the Hecke operators  $T_v$  on  $q$ -expansions in the case of  $U = U_1(\mathfrak{n})$ . For  $f \in M_{k,l}(U_1(\mathfrak{n}); R)$  and  $m \in J_+^{-1} \cup \{0\}$ , we write  $r_m^J(f)$  for the coefficient of  $q^m$  in the  $J$ -component of its  $q$ -expansion as in (9.1).

**Proposition 9.5.1.** *If  $f \in M_{k,l}(U_1(\mathfrak{n}); R)$ ,  $v \nmid \mathfrak{n}p$  and  $m \in J_+^{-1} \cup \{0\}$ , then*

$$r_m^J(T_v f) = \beta_1^l r_{\beta_1 m}^{J_1}(f) + \text{Nm}_{F/\mathbb{Q}}(v) \beta_2^l r_{\beta_2 m}^{J_2}(S_v f),$$

where the  $J_i$  and  $\beta_i \in F_+$  are such that  $vJ = \beta_1 J_1$  and  $v^{-1}J = \beta_2 J_2$  (and we interpret  $r_{\beta_2 m}$  as 0 if  $\beta_2 m \notin J_2^{-1}$ ).

*Proof.* This is a standard computation which we briefly indicate how to carry out in our context. Let  $U = U_1(\mathfrak{n})$ ,  $g = \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix}$ , denote the rational prime in  $\varpi_v$  by  $r$ , and choose a sufficiently large  $N$  prime to  $pr$ . We may extend scalars so as to assume  $\mu_{Nr}(\overline{\mathbb{Q}}) \subset \mathcal{O}$ .

Note that

$$UgU = U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U = \coprod_{i \in \mathbb{P}^1(O_F/\varpi_v)} g_i U,$$

with  $g_i \in \text{GL}_2(O_{F,\varpi_v})$  defined by  $\begin{pmatrix} \varpi_v & [i] \\ 0 & 1 \end{pmatrix}$  if  $i \in O_F/\varpi_v$ , where  $[i]$  is the Teichmüller (or indeed any) lift of  $i$ , and  $g_\infty = \begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}$ . To define the maps  $[U'g_iU] : M_{k,l}(U; R) \rightarrow M_{k,l}(U'; R)$  (where  $U' = U(rN)$  for example), we may take  $N_1 = rN$ ,  $N_2 = N$  and  $\alpha = 1$  in the notation of §4.

Recall that the  $J$ -component of the  $q$ -expansion of  $T_v f$  is given by its image in  $Q_{C,R}^{k,l}$  where the cusp  $C$  of  $X_U$  is the image of a cusp  $\tilde{C}$  of  $X_{J,rN}$  associated to the Tate semi-HBAV  $T_{\alpha,\mathfrak{b}}$  with  $\alpha = O_F$ ,  $\mathfrak{b} = J^{-1}$ , canonical polarisation data (i.e., associated to the

identity  $\alpha b^{-1} = J$ ), and level structure defined by  $\eta(x, y) = t(y)q^{s(x)}$  for some choice of isomorphisms  $s : O_F/rNO_F \simeq (rNJ)^{-1}/J^{-1}$  and  $t : O_F/rNO_F \simeq \mu_{rN} \otimes \mathfrak{d}^{-1}$ .

Suppose first that  $i \in O_F/\mathfrak{w}_v$ . Choosing  $\beta = \beta_1$  in the definition of  $\tilde{\rho}_{g_i} : Y_{J,rN} \rightarrow Y_{J_1,N}$  and extending to minimal compactifications, we find that  $\tilde{\rho}_{g_i}(\tilde{C}) = \tilde{C}_1$  where  $\tilde{C}_1$  is the cusp of  $X_{J_1,N}$  associated to  $T_{O_F,J_1^{-1}}$  with canonical polarisation data and level  $N$  structure defined by  $(x, y) \mapsto t(ry)q^{\beta_1 r s(x)}$ . Moreover, the induced morphism  $\hat{S}_{\tilde{C}_1} \rightarrow \hat{S}_{\tilde{C}}$  on completions is defined by  $q^m \mapsto \zeta_i(\beta_1^{-1}m)q^{\beta_1^{-1}m}$ , with  $\zeta_i$  running through the distinct homomorphisms  $(vNJ)^{-1}/(NJ)^{-1} \rightarrow \mu_r$  as  $i$  runs through the distinct elements of  $O_F/\mathfrak{w}_v$ , and the pull-back to  $\hat{S}$  of the isogeny denoted  $\pi$  in §4 is just the natural projection  $T_{O_F,J^{-1}} \rightarrow \tilde{\rho}_{g_i}^* T_{O_F,J_1^{-1}}$  induced by the identity on  $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$ . Taking into account the normalisation by  $\|\det(g)_i\| = \text{Nm}_{F/\mathbb{Q}}(v)^{-1}$ , we conclude that  $[U'g_iU]$  is compatible with the morphism  $\mathcal{Q}_{\tilde{C}_1,R}^{k,l} \rightarrow \mathcal{Q}_{\tilde{C},R}^{k,l}$  on  $q$ -expansions defined by

$$\sum_{m \in (NJ_1)_+^{-1} \cup \{0\}} (b \otimes r_m)q^m \mapsto \text{Nm}_{F/\mathbb{Q}}(v)^{-1} \sum_{m \in (vNJ)_+^{-1} \cup \{0\}} (\beta_1^l b \otimes \zeta_i(m)r_{\beta_1 m})q^m.$$

As for  $i = \infty$ , note that  $[U'g_\infty U] = [U'hU]S_v$ , where  $h = \begin{pmatrix} 0 & \mathfrak{w}_v^{-1} \\ 1 & 0 \end{pmatrix}$ . Choosing  $\alpha = r$  and  $\beta = r^2\beta_2$  in the definition of  $\tilde{\rho}_h : Y_{J,rN} \rightarrow Y_{J_2,N}$  and extending to minimal compactifications, we find that  $\tilde{\rho}_h(\tilde{C}) = \tilde{C}_2$  where  $\tilde{C}_2$  is the cusp of  $X_{J_2,N}$  associated to  $T_{O_F,J_2^{-1}}$  with canonical polarisation data and level  $N$  structure defined by  $(x, y) \mapsto t(ry)q^{\beta_2 r s(x)}$ . Moreover, the induced morphism  $\hat{S}_{\tilde{C}_2} \rightarrow \hat{S}_{\tilde{C}}$  on completions is defined by  $q^m \mapsto q^{\beta_2^{-1}m}$  and the pull-back of  $\pi$  to  $\hat{S}$  is the map  $T_{O_F,J^{-1}} \rightarrow \tilde{\rho}_h^* T_{O_F,J_2^{-1}}$  induced by multiplication by  $r$  on  $\mathbb{G}_m \otimes \mathfrak{d}^{-1}$ . Taking into account the normalisation by  $\|\det(h)\| = \text{Nm}_{F/\mathbb{Q}}(v)$ , we conclude that  $[U'hU]$  is compatible with the morphism  $\mathcal{Q}_{\tilde{C}_2,R}^{k,l} \rightarrow \mathcal{Q}_{\tilde{C},R}^{k,l}$  on  $q$ -expansions defined by

$$\sum_{m \in (NJ_2)_+^{-1} \cup \{0\}} (b \otimes r_m)q^m \mapsto \text{Nm}_{F/\mathbb{Q}}(v) \sum_{m \in v(NJ)_+^{-1} \cup \{0\}} (\beta_2^l b \otimes r_{\beta_2 m})q^m.$$

Summing over  $i$  then gives the desired formula. ■

### 9.6. Hecke operators at primes dividing the level

We shall also make use of the operator  $T_v = [U \begin{pmatrix} \mathfrak{w}_v & 0 \\ 0 & 1 \end{pmatrix} U]$  on  $M_{k,l}(U; R)$  for  $U = U_1(\mathfrak{n})$  and  $v \mid \mathfrak{n}$ . Note that the operators  $T_v$  on  $M_{k,l}(U_1(\mathfrak{n}); R)$  for all  $v \nmid p$  commute with each other, as well as with the  $S_v$  for  $v \nmid p\mathfrak{n}$ . The effect of  $T_v$  on  $q$ -expansions for  $v \mid \mathfrak{n}$  is computed exactly as in the proof of Proposition 9.5.1 except for the absence of the coset indexed by  $i = \infty$ :

**Proposition 9.6.1.** *If  $f \in M_{k,l}(U_1(\mathfrak{n}); R)$ ,  $v \mid \mathfrak{n}$  and  $m \in J_+^{-1} \cup \{0\}$ , then*

$$r_m^J(T_v f) = \beta_1^l r_{\beta_1 m}^{J_1}(f),$$

where the  $J_1$  and  $\beta_1 \in F_+$  are such that  $vJ = \beta_1 J_1$ .

9.7. Hecke operators at primes dividing  $p$

We also need the operators  $T_v$  for  $v \mid p$  in the case  $R = E, l_\tau = 0, k_\tau \geq 2$  for all  $\tau$ ; we recall the definition. Again let  $J, J_1, \beta_1$  (in  $F_+$ ) be such that  $vJ = \beta_1 J_1$ . Let  $A_1 = A_{J_1, N}$  denote the universal HBAV over  $\bar{Y}_{J_1, N}$ . Letting  $H$  denote the kernel of  $\text{Ver}_{A_1} : A_1^{(p)} \rightarrow A_1$ , we may decompose  $H = \prod_{w \mid p} H_w$  where each  $H_w$  is a free rank 1 ( $O_F/w$ )-module scheme over  $\bar{Y}_{J_1, N}$  and set  $A'_1 = A_1^{(p)}/H'_v$  where  $H'_v = \prod_{w \neq v} H_w$ . The  $O_F$ -action on  $A_1^{(p)}$ , polarisation  $p\beta_1^{-1}\lambda^{(p)}$  and level  $N$  structure  $p^{-1}\eta^{(p)}$  induce ones on  $A'_1$  making it a  $J$ -polarised HBAV over  $\bar{Y}_{J_1, N}$ , corresponding to a finite, flat morphism

$$\tilde{\rho} : \bar{Y}_{J_1, N} \rightarrow \bar{Y}_{J, N}$$

of degree  $\text{Nm}_{F/\mathbb{Q}}(v)$ . Taking the union over  $J_1$ , the resulting morphism descends, for sufficiently small  $U \supset U(N)$ , to a finite, flat endomorphism of  $\bar{Y}_U$  which we denote by  $\rho$ .

To define  $T_v$ , recall that the Kodaira–Spencer isomorphism (6.2) induces  $\bar{\mathcal{L}}_U^{2,-1} \simeq \bar{\mathcal{K}}_U$ , where  $\bar{\mathcal{K}}_U$  is the dualising sheaf on  $\bar{Y}_U$ , and hence an isomorphism

$$\bar{\mathcal{L}}_U^{k,0} \simeq \bar{\mathcal{K}}_U \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\mathcal{L}}_U^{k-2,1}.$$

Letting  $s : A \rightarrow \bar{Y}_{J, N}$  and  $s_1 : A_1 \rightarrow \bar{Y}_{J_1, N}$  denote the structure morphisms for the universal HBAV's, the isogenies  $\pi : A'_1 \rightarrow A_1$  induced by  $\text{Ver}_{A_1}$  yield morphisms  $s_{1,*}\Omega^1_{A_1/\bar{Y}_{J_1, N}} \rightarrow \tilde{\rho}^*s_*\Omega^1_{A/\bar{Y}_{J, N}}$ , which in turn induce morphisms  $\bar{\mathcal{L}}_{J_1, N}^{k-2,0} \rightarrow \tilde{\rho}^*\bar{\mathcal{L}}_{J, N}^{k-2,0}$  (using that  $k_\tau \geq 2$  for all  $\tau$ ) whose union over  $J_1$  descends to

$$\bar{\mathcal{L}}_U^{k-2,0} \rightarrow \rho^*\bar{\mathcal{L}}_U^{k-2,0}. \tag{9.3}$$

Making use of the canonical trivialisations  $\bar{\mathcal{L}}_{J, N}^{0,1} \simeq \text{Nm}_{F/\mathbb{Q}}(J\delta^{-1}) \otimes \mathcal{O}_{\bar{Y}_{J, N}}$  and  $\bar{\mathcal{L}}_{J_1, N}^{0,1} \simeq \text{Nm}_{F/\mathbb{Q}}(J_1\delta^{-1}) \otimes \mathcal{O}_{\bar{Y}_{J_1, N}}$ , we define  $\bar{\mathcal{L}}_{J_1, N}^{0,1} \xrightarrow{\sim} \tilde{\rho}^*\bar{\mathcal{L}}_{J, N}^{0,1}$  by multiplication by  $\text{Nm}_{F/\mathbb{Q}}(JJ_1^{-1})$ . (Note that this is *not* the morphism induced by  $\pi$ , which is in fact 0.) The union over  $J_1$  then descends to an isomorphism  $\bar{\mathcal{L}}_U^{0,1} \xrightarrow{\sim} \rho^*\bar{\mathcal{L}}_U^{0,1}$ , which we tensor with (9.3) to obtain a morphism  $\bar{\mathcal{L}}_U^{k-2,1} \rightarrow \rho^*\bar{\mathcal{L}}_U^{k-2,1}$ . We then define  $T_v$  as the composite

$$\begin{aligned} H^0(\bar{Y}_U, \bar{\mathcal{K}}_U \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\mathcal{L}}_U^{k-2,1}) &\rightarrow H^0(\bar{Y}_U, \bar{\mathcal{K}}_U \otimes_{\mathcal{O}_{\bar{Y}_U}} \rho^*\bar{\mathcal{L}}_U^{k-2,1}) \\ &\xrightarrow{\sim} H^0(\bar{Y}_U, \rho_*\bar{\mathcal{K}}_U \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\mathcal{L}}_U^{k-2,1}) \rightarrow H^0(\bar{Y}_U, \bar{\mathcal{K}}_U \otimes_{\mathcal{O}_{\bar{Y}_U}} \bar{\mathcal{L}}_U^{k-2,1}), \end{aligned}$$

where the first map is given by the one just defined, the second is the canonical isomorphism, and the third is induced by the trace map  $\rho_*\bar{\mathcal{K}}_U \rightarrow \bar{\mathcal{K}}_U$ .

**Proposition 9.7.1.** *Suppose that  $v \mid p, l = 0$  and  $k_\tau \geq 2$  for all  $\tau \in \Sigma$ . If  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  and  $m \in J_+^{-1} \cup \{0\}$ , then*

$$r_m^J(T_v f) = r_{\beta_1 m}^{J_1}(f),$$

where the  $J_1$  and  $\beta_1 \in F_+$  are such that  $vJ = \beta_1 J_1$ .

*Proof.* Let  $C$  be a cusp at infinity on  $X_U$  where  $U = U_1(\mathfrak{n})$ , so that  $C$  is the image of a cusp  $\tilde{C}$  of  $X_{J,N}$  associated to the Tate semi-HBAV  $T_{\alpha,\mathfrak{b}}$  with  $\alpha = O_F$ ,  $\mathfrak{b} = J^{-1}$ , canonical polarisation data and level structure  $\eta(x, y) = t(y)q^{s(x)}$  for some choice of  $s$  and  $t$ . The morphisms  $\tilde{\rho}$  extend uniquely to morphisms  $\overline{X}_{J_1,N} \rightarrow \overline{X}_{J,N}$ , for which one finds that the fibre over  $\tilde{C}_E$  is supported at  $\tilde{C}_{1,E}$ , where  $\tilde{C}_1$  is the cusp of  $X_{J_1,N}$  associated to  $T_{O_F,J_1^{-1}}$ , with canonical polarisation data and level structure  $\eta(x, y) = t(y)q^{s_1(x)}$  for some choice of  $s_1$ .

Moreover, the corresponding map  $\hat{S}_{\tilde{C},E} \rightarrow \hat{S}_{\tilde{C}_1,E}$  is defined by  $q^m \mapsto q^{\beta_1 m}$ , and the pull-back of the isogeny  $\pi$  to  $\hat{S}_{1,E}$  is the canonical projection  $\tilde{\rho}^* T_{O_F,J^{-1},E} \rightarrow T_{O_F,J_1^{-1},E}$  induced by the identity on  $\mathbb{G}_m \otimes \mathfrak{b}^{-1}$  (where  $\hat{S}_1 = S_N(\{\sigma_{\alpha^{\tilde{C}_1}}\})^\wedge$  is again as in [11, 3.4.2] for a suitable cone decomposition  $\{\sigma_{\alpha^{\tilde{C}_1}}\}$ ). In particular, it follows that the morphism (9.3) is compatible with the canonical trivialisations over  $\hat{S}_{1,E}$ , so the resulting map

$$Q_{C_1,E}^{k-2,1} \rightarrow \hat{S}_{C_1,E} \otimes_{\hat{S}_{C,E}} Q_{C,E}^{k-2,1}$$

is induced by multiplication by  $\text{Nm}_{F/\mathbb{Q}}(JJ_1^{-1})$ .

Identifying the pull-back of  $j_* \mathcal{K}_U$  to  $\hat{S}$  with  $\text{Nm}_{F/\mathbb{Q}}(J)^{-1} \otimes \mathcal{O}_{\hat{S}}$ , and similarly for  $\hat{S}_1$  with  $C$  and  $J$  replaced by  $C_1$  and  $J_1$ , we find that the trace  $\rho_* \overline{\mathcal{K}}_U \rightarrow \overline{\mathcal{K}}_U$  extends (over  $\overline{X}_U$ ) to the map whose pull-back to  $\hat{S}_E$  is defined by

$$b \otimes q^m \mapsto \begin{cases} \text{Nm}_{F/\mathbb{Q}}(J_1 J^{-1}) b \otimes q^{\beta_1^{-1} m} & \text{if } m \in v J_1^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [41, (1.1.20)] that the same holds for the corresponding map  $\rho_* \overline{\mathcal{L}}_U^{2,-1} \rightarrow \overline{\mathcal{L}}_U^{2,-1}$ , so this formula describes the resulting map  $Q_{C_1,E}^{2,-1} \rightarrow Q_{C,E}^{2,-1}$ , giving the proposition. ■

One easily sees directly from the definitions that the  $T_v$  for  $v \mid p$  on  $M_{k,0}(U_1(\mathfrak{n}); E)$  commute with the  $S_v$  for  $v \nmid p \mathfrak{n}$  (assuming  $k_\tau \geq 2$  for all  $\tau$ ), and it follows from Propositions 9.5.1, 9.6.1 and 9.7.1 that they commute with each other as well as with the  $T_v$  for all  $v \nmid p$ . (In fact, one can check directly from the definitions that the  $T_v$  commute with each other and the action of the group  $\{g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \text{GL}_2(O_{F,p})\}$  on  $M_{k,0}(E)$ .)

### 9.8. Partial Frobenius operators

We also define operators  $\Phi_v$  for  $v \mid p$  in the case  $R = E$ ,  $l = 0$ , generalising the classical  $V$ -operator. We maintain the notation from the definition of  $T_v$  in §9.7, except that we no longer assume  $k_\tau \geq 2$  for all  $\tau$ . Writing  $s'_1 : A'_1 \rightarrow \overline{Y}_{J_1,N}$  and  $\tilde{\rho}^* s_* \Omega_{A/\overline{Y}_{J,N}}^1 \simeq (s'_1)^* \Omega_{A'_1/\overline{Y}_{J_1,N}}^1 = \bigoplus \overline{\omega}'_\tau$ , we find the isogenies  $A_1^{(p)} \rightarrow A'_1 \rightarrow A_1$  induce isomorphisms

$$\tilde{\rho}^* \overline{\omega}_\tau \simeq \overline{\omega}'_\tau \simeq \begin{cases} \overline{\omega}_{\text{Fr}^{-1} \circ \tau}^{\otimes p} & \text{if } \tau \in \Sigma_v, \\ \overline{\omega}_\tau & \text{if } \tau \notin \Sigma_v, \end{cases} \tag{9.4}$$

on  $\overline{Y}_{J_1, N}$  whose unions over  $J_1$  descend to  $\overline{Y}_U$ . For  $k \in \mathbb{Z}^\Sigma$ , define  $k'$  by  $k'_\tau = pk_{\text{Fr} \circ \tau}$  if  $\tau \in \Sigma_v$  and  $k'_\tau = k_\tau$  if  $\tau \notin \Sigma_v$ , and  $\Phi_v : M_{k,0}(U; E) \rightarrow M_{k',0}(U; E)$  as the composite

$$M_{k,0}(U; E) = H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0}) \rightarrow H^0(\overline{Y}_U, \rho^* \overline{\mathcal{L}}_U^{k,0}) \rightarrow H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k',0}) = M_{k',0}(U; E),$$

where the first map is pull-back and the second is induced by the above isomorphisms.

It is clear from the definition that  $\Phi_v$  is injective, and straightforward to check the operators  $\Phi_v$  commute with each other and the action of  $\{g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g_p \in \text{GL}_2(\mathcal{O}_{F,p})\}$  on  $M_{k,0}(E)$  and  $M_{k',0}(E)$ . In particular,  $\Phi_v$  commutes with the operators  $T_w$  for all  $w \nmid p$  and  $S_w$  for all  $w \nmid np$ . Its effect on  $q$ -expansions is given by the following:

**Proposition 9.8.1.** *Suppose that  $v \mid p$  and  $l = 0$ . If  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  and  $m \in J_+^{-1} \cup \{0\}$ , then*

$$r_m^J(\Phi_v f) = r_{\beta_2 m}^{J_2}(f),$$

where the  $J_2$  and  $\beta_2 \in F_+$  are such that  $v^{-1}J = \beta_2 J_2$  (interpreting  $r_{\beta_2 m}$  as 0 if  $\beta_2 m \notin J_2^{-1}$ , i.e.,  $m \notin vJ^{-1}$ ).

*Proof.* The completion of  $\rho$  at the cusps is already computed in the course of the proof of Proposition 9.7.1. One also finds that the pull-backs of the isomorphisms of (9.4) to  $\widehat{S}_{1,E}$  are compatible with the canonical trivialisations of the push-forwards of the cotangent bundles of the Tate semi-HBAVs  $T_{\mathcal{O}_F, J^{-1}, E}$  and  $T_{\mathcal{O}_F, J_1^{-1}, E}$ . It follows that the map

$$Q_{C,E}^{k,0} \rightarrow \widehat{S}_{C_1,E} \otimes_{\widehat{S}_{C,E}} Q_{C,E}^{k,0} \simeq Q_{C_1,E}^{k',0} \tag{9.5}$$

induced by  $\Phi_v$  is defined by  $q^m \mapsto q^{\beta_1 m}$ . The desired formula follows on relabelling  $J$  as  $J_2$  and  $J_1$  as  $J$ , and taking  $\beta_2 = \beta_1^{-1}$ . ■

The proposition gives an alternative proof (for  $U = U_1(\mathfrak{n})$ ) that the  $\Phi_v$  commute with each other and with the  $T_w$  for all  $w \nmid p$  (after checking that  $\Phi_v$  commutes with  $S_w$  and applying Proposition 9.5.1 for  $w \nmid np$ , and Proposition 9.6.1 for  $w \mid \mathfrak{n}$ ). Note however that  $\Phi_v$  does not commute with  $T_v$  (when the latter is defined, i.e.,  $k_\tau \geq 2$  for all  $\tau$ ).

Note that it is immediate from Proposition 9.4.1 that the kernel of the operator  $\Theta_\tau$  depends only on the prime  $v$  such that  $\tau \in \Sigma_v$ . Moreover, if  $v(f) = k$  (in the notation of §5), then Theorem 8.2.2 implies that  $k_\tau$  is divisible by  $p$  for all  $\tau \in \Sigma_v$ . We will show in Theorem 9.8.2 that (assuming  $k$  is of this form and  $l = 0$ ) this kernel is in fact the image of  $\Phi_v$ , generalising a result of Katz in [40, Section II].

We need to introduce one more operator: we define  $k^\varphi$  by  $k'_\tau = k_{\text{Fr}^{-1} \circ \tau}$  and we let  $\varphi : M_{k,0}(U; E) \rightarrow M_{k^\varphi,0}(U; E)$  be the composite

$$H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0}) \rightarrow H^0(\overline{Y}_U, \text{Fr}_E^* \overline{\mathcal{L}}_U^{k,0}) \rightarrow H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k^\varphi,0}),$$

where the first map is pull-back by the automorphism induced by  $\text{Fr}_E$  on  $\overline{Y}_U$  and the second is given by the canonical isomorphisms  $\text{Fr}_E^* \overline{\omega}_\tau \simeq \overline{\omega}_{\text{Fr} \circ \tau}$ . (Note that we could similarly define  $\varphi : M_{k,l}(U; E) \rightarrow M_{k^\varphi,l}(U; E)$ .) Its effect on  $q$ -expansions of  $f \in M_{k,0}(U_1(\mathfrak{n}); E)$  is given by  $r_m^J(\varphi f) = \text{Fr}_E(r_m^J(f)) = (r_m^J(f))^p$ .

**Theorem 9.8.2.** *Suppose that  $k \in \mathbb{Z}^\Sigma$ ,  $\mathfrak{n}$  is an ideal of  $O_F$  prime to  $p$ ,  $v$  is a prime dividing  $p$  and  $\tau \in \Sigma_v$ . Then the image of*

$$\Phi_v : M_{k,0}(U_1(\mathfrak{n}); E) \rightarrow M_{k',0}(U_1(\mathfrak{n}); E)$$

is the kernel of  $\Theta_\tau$ .

*Proof.* From Proposition 9.4.1 we see that  $f \in M_{k',0}(U_1(\mathfrak{n}); E)$  is in the kernel of  $\Theta_\tau$  if and only if  $r_m^J(f) = 0$  for all  $m, J$  such that  $m \notin vJ_+^{-1}$ . It is therefore immediate from Proposition 9.8.1 that  $\text{image}(\Phi_v) \subset \ker(\Theta_\tau)$ .

For the opposite inclusion, first note that we can assume  $\mathfrak{n}$  is sufficiently small. For each cusp  $C \in \mathcal{S}$ , let  $N_C$  denote the stalk  $(j_* \overline{\mathcal{L}}_U^{k,0})_{\overline{C}}$ , where as usual  $j$  is the inclusion  $\overline{Y}_U \rightarrow \overline{X}_U$  and  $\overline{C} = C_E$ . Similarly let

$$N'_C = (j_* \overline{\mathcal{L}}_U^{k',0})_{\overline{C}_1} = (j_* \rho_* \overline{\mathcal{L}}_U^{k',0})_{\overline{C}}$$

and consider the  $R_C := \mathcal{O}_{\overline{X}_U, \overline{C}}$ -linear map  $\phi_C : N_C \rightarrow N'_C$  of finitely generated  $R_C$ -modules induced by the morphisms in the definition of  $\Phi_v$ . Letting  $\mathcal{F}_U$  denote the sheaf of total fractions on  $\overline{Y}_U$ , we similarly have a map

$$\tilde{\Phi}_v : H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U) \rightarrow H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k',0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U),$$

and thus a commutative diagram of injective maps

$$\begin{CD} M_{k,0}(U_1(\mathfrak{n}); E) @>>> \bigoplus_{C \in \mathcal{S}} N_C @>>> H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U) \\ @VVV @VVV @VVV \\ M_{k',0}(U_1(\mathfrak{n}); E) @>>> \bigoplus_{C \in \mathcal{S}} N'_C @>>> H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k',0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U) \end{CD}$$

where the horizontal maps are the natural inclusions.

The completion  $\hat{\phi}_C$  of  $\phi_C$  is precisely the  $\hat{S}_{\overline{C}}$ -linear map  $Q_{\overline{C}}^{k,0} \rightarrow Q_{\overline{C}_1}^{k',0}$  of (9.5) (where  $\hat{S}_{\overline{C}}$  acts on the target via the map to  $\hat{S}_{\overline{C}_1}$  induced by  $\rho$ ). If  $f \in \ker(\Theta_\tau)$ , then  $r_m^{J_1}(f) = 0$  for all  $m \notin vJ_1^{-1}$ , so the  $q$ -expansion of  $f$  at  $C_1$  is in the image of  $\hat{\phi}_C$  for each  $C \in \mathcal{S}$ . Since  $\hat{S}_{\overline{C}}$  is faithfully flat over  $R_C$ , it follows that  $f$  is in the image of  $\phi_C$  for each  $C \in \mathcal{S}$ , so there exists  $g \in \bigoplus_C N_C \subset H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U)$  such that  $\tilde{\Phi}_v(g) = f$ .

It remains to prove that  $g \in M_{k,0}(U_1(\mathfrak{n}); E)$ . Since  $\overline{Y}_U$  is smooth and  $\overline{\mathcal{L}}_U^{k,0}$  is invertible, it suffices to prove that  $\text{ord}_z(g) \geq 0$  for all prime divisors  $z$  on  $\overline{Y}_U$ . For this, we note that the map  $\phi$  similarly extends to a map  $\tilde{\varphi}$ , and one checks that

$$\tilde{\varphi} \circ \prod_{w|p} \tilde{\Phi}_w : H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{k,0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U) \rightarrow H^0(\overline{Y}_U, \overline{\mathcal{L}}_U^{pk,0} \otimes_{\mathcal{O}_{\overline{Y}_U}} \mathcal{F}_U)$$



is simply the map  $g \mapsto g^p$ . Therefore

$$g^p = \left( \tilde{\varphi} \circ \prod_{w \neq v} \tilde{\Phi}_w \right) (f) = \left( \varphi \circ \prod_{w \neq v} \Phi_w \right) (f) \in M_{pk,0}(U_1(\mathfrak{n}); E),$$

so that  $p \operatorname{ord}_z(g) = \operatorname{ord}_z(g^p) \geq 0$ , and hence  $\operatorname{ord}_z(g) \geq 0$ . ■

**Remark 9.8.3.** One can also check that the relation  $\operatorname{image}(\Phi_v) = \ker(\Theta_\tau)$  holds for arbitrary  $U$  using the same argument as in the proof of the theorem and a straightforward generalisation of the formula in Proposition 9.8.1 (see the next section for similar computations of the effect of operators on  $q$ -expansions at more general cusps).

### 10. Normalised eigenforms

We will prove that if  $\rho$  is irreducible and geometrically modular of weight  $(k, l)$ , then in fact  $\rho$  is associated to an eigenform  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  for some  $\mathfrak{n}$  prime to  $p$ , allowing us to pin down  $q$ -expansions of forms giving rise to  $\rho$ . We will also use partial  $\Theta$ -operators to study the behaviour of minimal weights as  $l$  varies, and prove that if an eigenform is ordinary at a prime over  $v$ , then so is the associated Galois representation.

#### 10.1. Preliminaries

First note that, by definition, if  $\rho$  is geometrically modular of weight  $(k, l)$ , then  $\rho$  is associated to an eigenform  $f \in M_{k,l}(U(\mathfrak{n}); E)$  for some  $\mathfrak{n}$  prime to  $p$ . One approach to replacing  $U(\mathfrak{n})$  by  $U_1(\mathfrak{n}')$  for some  $\mathfrak{n}'$  would be to use the space  $M_{k,l}(E)$  to associate to  $\rho$  a representation of  $\operatorname{GL}_2(F_v)$  for each  $v \mid \mathfrak{n}$ . One then chooses an irreducible subrepresentation  $\pi_v$ , whose existence is given by [60, II, 5.10], and shows, using the irreducibility of  $\rho$ , that  $\pi_v$  does not factor through  $\det$ . It then follows from [59] that  $\pi_v$  has a vector invariant under  $U_1(v^{c_v})$  for some exponent  $c_v$ , and one can take  $\mathfrak{n}' = \prod_c v^{c_v}$ . We shall instead give a more constructive argument that develops some tools we will need anyway. In particular, we define certain twisting operators on forms of level  $U(\mathfrak{n})$ .

We let  $U = U(\mathfrak{n})$  and index the components of  $Y_U$  by pairs  $(J, w)$  where  $J$ , as usual, runs through strict ideal class representatives, and  $w$  runs through a set  $W \subset (O_F/NO_F)^\times$  of representatives for  $(O_F/\mathfrak{n})^\times/O_{F,+}^\times$ . More precisely, choose as before isomorphisms  $s : O_F/NO_F \simeq (NJ)^{-1}/J^{-1}$  (for each  $J$ ) and  $t : O_F/NO_F \simeq \mu_N \otimes \mathfrak{d}^{-1}$ . Then  $s$  determines an isomorphism  $J/NJ \simeq O_F/NO_F$  whose composite with  $t$  defines a component of  $Y_{J,N}$ , hence of  $Y_U$ , and we associate to  $(J, w)$  the component so defined with  $s$  replaced by  $ws$ . One easily checks that this defines a bijection between  $Z_U(\mathcal{O})$  and the set of such pairs. Moreover, there is a unique cusp on each component of  $X_U$  mapping to a cusp at  $\infty$  on  $X_{U_1(\mathfrak{n})}$ , namely the one associated to the Tate semi-HBAV  $T_{O_F, J^{-1}}$  with canonical polarisation and level  $N$  structure  $(x, y) \mapsto t(y)q^{ws(x)}$ . For  $f \in M_{k,l}(U; \mathbb{R})$  and  $m \in (\mathfrak{n}^{-1}J)_+ \cup \{0\}$ , we write  $r_m^{J,w}(f)$  for the corresponding  $q$ -expansion coefficient of  $f$ .

A computation similar to the proof of Proposition 9.5.1 shows that the effect of  $T_v$  on  $q$ -expansions of forms in  $M_{k,l}(U; R)$  is given by the formula

$$r_m^{J,w}(T_v f) = \beta_1^l r_{\beta_1 m}^{J_1, w_1}(f) + \text{Nm}_{F/\mathbb{Q}}(v) \beta_2^l r_{\beta_2 m}^{J_2, w_2}(S_v f) \tag{10.1}$$

for  $m \in (\mathfrak{n}J)_+^{-1} \cup \{0\}$ , where the  $J_i, \beta_i$  are as before, with  $w_i \in W$  satisfying

$$\begin{aligned} \beta_1 \varpi_v^{-1} w s(1) &\equiv w_1 s_1(1) \pmod{\mathfrak{n}(NJ_1)^{-1}}, \\ \beta_2 \varpi_v w s(1) &\equiv w_2 s_2(1) \pmod{\mathfrak{n}(NJ_2)^{-1}}, \end{aligned}$$

where we view  $\beta_1 \varpi_v^{-1}$  as inducing an isomorphism  $J^{-1} \hat{O}_F \simeq J_1^{-1} \hat{O}_F$ , hence  $(NJ)^{-1}/J^{-1} \simeq (NJ_1^{-1})/J_1^{-1}$ , and similarly  $\beta_2 \varpi_v$  as inducing  $(NJ)^{-1}/J^{-1} \simeq (NJ_2)^{-1}/J_2^{-1}$ . We note the following consequence:

**Lemma 10.1.1.** *If  $f \in M_{k,l}(U(\mathfrak{n}); E)$  is an eigenform for the operators  $T_v$  and  $S_v$  for all but finitely many  $v$ , and the associated Galois representation  $\rho_f$  is absolutely irreducible, then  $r_0^{J,w} = 0$  for all pairs  $(J, w)$ .*

*Proof.*<sup>7</sup> If  $v$  is trivial in the strict class group of conductor  $\mathfrak{n}p$ , then it follows from the definitions that  $S_v$  acts trivially on  $M_{k,l}(U; E)$ . Moreover, in formula (10.1) for such  $v$ , we have  $J = J_1 = J_2, w = w_1 = w_2, \beta_1 \equiv \beta_2 \equiv 1 \pmod{pO_{F,p}}$  and  $\text{Nm}_{F/\mathbb{Q}}(v) \equiv 1 \pmod{p}$ , so that  $r_0^{J,w}(T_v f) = 2r_0^{J,w}(f)$ . Therefore if  $r_0^{J,w}(f) \neq 0$  for some  $(J, w)$ , then  $\rho_f(\text{Frob}_v)$  has characteristic polynomial  $(X - 1)^2$  for such  $v$ . By the Chebotarev Density Theorem (and class field theory) it follows that  $\rho_f(g)$  has characteristic polynomial  $(X - 1)^2$  for all  $g \in G_K$ , where  $K$  is the strict ray class field over  $F$  of conductor  $\mathfrak{n}p$ , so by the Brauer–Nesbitt Theorem,  $\rho_f|_{G_K}$  has trivial semisimplification. Since  $K$  is abelian over  $F$ , this contradicts the absolute irreducibility of  $\rho_f$ . ■

We continue to assume  $U = U(\mathfrak{n})$  and define an action of the group  $(O_F/\mathfrak{n})^\times$  on  $M_{k,l}(U; R)$  via its isomorphism with the subgroup of  $\text{GL}_2(O_F/\mathfrak{n})$  consisting of matrices of the form  $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ . Thus  $a \in (O_F/\mathfrak{n})^\times$  acts on  $M_{k,l}(U; R)$  by the operator  $[UgU]$  for any  $g \in \text{GL}_2(\hat{O}_F)$  congruent to  $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}}$ ; we denote the operator by  $\langle a \rangle$ . It is straightforward to check that its effect on  $q$ -expansions is given by the formula

$$r_m^{J,w}(\langle a \rangle f) = \nu^l r_{\nu m}^{J, w'}(f), \tag{10.2}$$

where  $\nu \in O_{F,+}^\times$  and  $w' \in W$  are such that  $\nu w \equiv aw' \pmod{\mathfrak{n}}$ .

We also define the operator  $T_v = [U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U]$  on  $M_{k,l}(U; R)$  for  $v \mid \mathfrak{n}$  and a choice of uniformiser  $\varpi_v$  for  $F_v$ . Another computation similar to Proposition 9.5.1 (or more precisely, Proposition 9.6.1) shows that its effect on  $q$ -expansions is given by

$$r_m^{J,w}(T_v f) = \beta_1^l r_{\beta_1 m}^{J_1, w_1}(f) \tag{10.3}$$

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<sup>7</sup>Alternatively, this can be proved by revisiting the construction in Theorem 6.1.1 and observing that if  $r_0^{J,w} \neq 0$  for some  $(J, w)$ , then the lift  $\tilde{f}$  is non-cuspidal. One then deduces that the Galois representation  $\rho_{\tilde{f}}$  is reducible, and hence so is  $\rho_f$  (possibly after extending scalars in the case  $p = 2$ ).

with notation as in (10.1). Note that  $T_v$  depends on the choice of  $\varpi_v$ : replacing  $\varpi_v$  by  $u\varpi_v$  for  $u \in O_{F,v}^\times$  replaces  $T_v$  with  $(\pi(u))T_v$  where  $\pi$  is the natural map  $O_{F,v}^\times \rightarrow (O_F/\mathfrak{n})^\times$ . We see directly from the definitions that the operators  $T_v$  for  $v \mid \mathfrak{n}$  commute with the  $T_v$  and  $S_v$  for  $v \nmid p\mathfrak{n}$  (and each other), as well as the action of  $(O_F/\mathfrak{n})^\times$ .

Suppose that  $\xi : (O_F/\mathfrak{n})^\times \rightarrow R^\times$  is a character of conductor  $\mathfrak{m} \mid \mathfrak{n}$ . Choose an element  $c \in O_{F,\mathfrak{n}} = \prod_{v \mid \mathfrak{n}} O_{F,v}$  generating  $\mathfrak{m}\mathfrak{n}^{-1}O_{F,\mathfrak{n}}$ , and define a *twisting operator* on  $M_{k,l}(U; R)$  by the formula

$$\Theta_\xi = \sum_{b \in (O_F/\mathfrak{m})^\times} \xi(b)^{-1}[Ug_bU] = \sum_{b \in (O_F/\mathfrak{m})^\times} \xi(b)^{-1}g_b, \tag{10.4}$$

where  $g_b \equiv \begin{pmatrix} 1 & bc \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{n}}$ . The operator  $\Theta_\xi$  commutes with the operators  $T_v$  and  $S_v$  for  $v \nmid \mathfrak{n}p$ , and it is straightforward to check that

$$\langle a \rangle \circ \Theta_\xi = \xi(a)\Theta_\xi \circ \langle a \rangle. \tag{10.5}$$

(Note also the dependence on  $c$ : replacing  $c$  by  $uc$  for  $u \in O_{F,\mathfrak{n}}^\times$  replaces  $\Theta_\xi$  by  $\xi(u)\Theta_\xi$ .) One finds the effect on  $q$ -expansions is given by

$$r_m^{J,w}(\Theta_\xi(f)) = G_J(\xi, w^{-1}cm)r_m^{J,w}(f), \tag{10.6}$$

where  $G_J(\xi, m) = \sum_{b \in (O_F/\mathfrak{m})^\times} \xi(b)^{-1}\zeta(-bm)$  for  $m \in (\mathfrak{m}J)_+^{-1} \cup \{0\}$ . (Recall that  $\zeta$  is the homomorphism  $(NJ)^{-1}/J^{-1} \rightarrow \mu_N$  induced by the trace and our choices of  $s$  and  $t$ ; see the discussion before Proposition 9.1.2.) Standard results on Gauss sums show that  $G_J(\xi, am) = \xi(a)G_J(\xi, m)$  for all  $a \in O_F$ ,  $m \in (\mathfrak{m}J)^{-1}$ , where  $\xi$  is viewed as a function  $O_F \rightarrow R$  by setting  $\xi(a) = 0$  for  $a$  not prime to  $\mathfrak{m}$ . One deduces that if  $m$  generates  $(\mathfrak{m}J)^{-1}/J^{-1}$ , then

$$G_J(\xi, m)G_J(\xi^{-1}, -m) = \text{Nm}_{F/\mathbb{Q}}(\mathfrak{m})$$

(in particular,  $G_J(\xi, m) \in R^\times$ ), and otherwise  $G_J(\xi, m) = 0$ .

### 10.2. Eigenforms of level $U_1(\mathfrak{n})$

**Lemma 10.2.1.** *If  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible and geometrically modular of weight  $(k, l)$ , then  $\rho$  arises from an eigenform of weight  $(k, l)$  and level  $U_1(\mathfrak{n})$  for some  $\mathfrak{n}$  prime to  $p$ ; i.e., there exist  $\mathfrak{n}$  prime to  $p$ , a field  $E$  and an eigenform  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  for  $S_v$  and  $T_v$  for all  $v \nmid \mathfrak{n}p$  such that  $\rho \simeq \rho_f$ .*

*Proof.* By assumption, there exist  $\mathfrak{n}$  (prime to  $p$ ),  $E$  and  $f \in M_{k,l}(U(\mathfrak{n}); E)$ , an eigenform for all  $S_v$  and  $T_v$  with  $v \nmid \mathfrak{n}p$ , such that  $\rho \simeq \rho_f$ . Since the action of  $(O_F/\mathfrak{n})^\times$  commutes with the operators  $S_v$  and  $T_v$ , we can further assume that  $f$  is an eigenform for this action, i.e., there is a character  $\xi : (O_F/\mathfrak{n})^\times \rightarrow E^\times$  (enlarging  $E$  if necessary) such that  $\langle a \rangle f = \xi(a)f$  for all  $a \in (O_F/\mathfrak{n})^\times$ .

By Lemmas 9.2.1 and 10.1.1, we must have  $r_m^{J,w}(f) \neq 0$  for some  $J$ , some (hence all by (10.2))  $w \in W$  and some  $m \in (\mathfrak{n}J)_+^{-1}$  (i.e.,  $m \neq 0$ ). Letting  $e_v = \text{ord}_v(\mathfrak{m}\mathfrak{n}J)$  for  $v \mid \mathfrak{n}$

and  $f' = \prod_{v|\mathfrak{n}} T_v^{e_v} f$ , formula (10.3) implies that  $r_m^{J',w'}(f') \neq 0$  for some  $(J', w')$  and  $m' \in (\mathfrak{n}J')_+^{-1}$  with  $m' \mathfrak{n}J'$  prime to  $\mathfrak{n}$  (choose  $J'$  equivalent to  $\prod_{v|\mathfrak{n}} v^{-e_v} J$  and let  $m' = \prod_{v|\mathfrak{n}} \beta_{1,v}^{-e_v} m$ ). Replacing  $f$  by  $f'$ , we now have  $\langle a \rangle f = \xi(a) f$  for all  $a \in (O_F/\mathfrak{n})^\times$ , and  $r_m^{J,w}(f) \neq 0$  for some  $(J, w)$  and  $m \in (\mathfrak{n}J)_+^{-1}$  generating  $(\mathfrak{n}J)^{-1}/J^{-1}$ .

Now replace  $f$  by  $\Theta_{\xi^{-1}}(f)$  where  $\Theta_{\xi^{-1}}$  is the twisting operator associated to  $\xi^{-1}$  as defined in (10.4). Since  $cm$  generates  $(\mathfrak{m}J)^{-1}/J^{-1}$ , we have  $G_J(\xi^{-1}, cm) \neq 0$ , so formula (10.6) shows that  $f \neq 0$ . By formula (10.5),  $f$  is invariant under the action of  $(O_F/\mathfrak{n})^\times$ , hence under the action of the open compact subgroup

$$U' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(\mathfrak{n}) \mid b \in \mathfrak{n} \widehat{O}_F \right\}.$$

Now let

$$g = \prod_{v|\mathfrak{n}} \begin{pmatrix} \varpi_v^{-e_v} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{where } e_v = \text{ord}_v(\mathfrak{n}).$$

Then  $g^{-1}U_1(\mathfrak{n}^2)g \subset U'$ , so the lemma follows with  $f$  replaced by  $[U_1(\mathfrak{n}^2)gU']f$  and  $\mathfrak{n}$  replaced by  $\mathfrak{n}^2$ . ■

### 10.3. Twisting eigenforms

Suppose now that  $k, l, l' \in \mathbb{Z}^\Sigma$ , and that  $\mathfrak{m}, \mathfrak{n}$  are ideals of  $O_F$  with  $\mathfrak{m} \mid \mathfrak{n}$  and  $\mathfrak{n}$  prime to  $p$ , and let  $V_{\mathfrak{m}} \subset \widehat{O}_F^\times$  denote the kernel of the natural projection to  $(O_F/\mathfrak{m})^\times$ .

**Definition 10.3.1.** We say a character

$$\xi : \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\} / V_{\mathfrak{m}} \rightarrow E^\times$$

is a *character of weight  $l'$*  if  $\xi(\alpha) = \bar{\alpha}^{l'}$  for all  $\alpha \in F_+^\times \cap O_{F,p}^\times$ .

Suppose that  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  and  $\xi$  is a character of weight  $l'$  and conductor  $\mathfrak{m}$ . Recall from §4.6 that we can associate to  $\xi$  a form  $e_\xi \in M_{0,l'}(U(\mathfrak{m}); E)$ , and hence a form  $e_\xi \otimes f \in M_{k,l+l'}(U(\mathfrak{n}); E)$ . Choosing  $c = (\varpi_v^{e_v - d_v})_v$  where  $d_v = \text{ord}_v m$  and  $e_v = \text{ord}_v \mathfrak{n}$ , and applying the following (normalised) composite of operators from the proof of Lemma 10.2.1:

$$\text{Nm}_{F/\mathbb{Q}}(\mathfrak{n})^{-1} [U_1(\mathfrak{n}^2)gU'] \circ \Theta_{\xi^{-1}} \circ \prod_{v|\mathfrak{n}} T_v^{e_v}$$

to  $e_\xi \otimes f$  then yields a form in  $M_{k,l+l'}(U_1(\mathfrak{n}^2); E)$  which we denote  $f'_\xi$ . It is straightforward to check that in fact

$$f'_\xi = e_\xi \otimes \sum_{b \in (O_F/\mathfrak{m})^\times} \xi(b) \begin{pmatrix} 1 & \tilde{b}c' \\ 0 & 1 \end{pmatrix} f \in M_{k,l+l'}(U_1(\mathfrak{m}\mathfrak{n}); E),$$

where  $\tilde{b}$  is any lift of  $b$  to  $O_{F,\mathfrak{n}}^\times$  and  $c' = (\varpi_v^{-d_v})_v$ .

We now relate the  $q$ -expansions of  $f$  and  $f'_\xi$ . Firstly, the form  $e_\xi$  has constant  $q$ -expansions satisfying the formula

$$\xi(a)r_0^{J_0, w_0}(e_\xi) = \|a\|^{-1}r_0^{J_1, w_1}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}e_\xi\right) = \bar{\beta}_1^\ell r_0^{J_1, w_1}(e_\xi)$$

for  $a \in (\mathbb{A}_F^\infty)^\times$ ,  $\beta_1 \in F_+^\times$ ,  $w_0, w_1 \in W$  such that  $a_p \in O_{F,p}^\times$ ,  $\beta_1 J_1 = (a)J$  and

$$\beta_1 a^{-1} w_0 s_0(1) \equiv w_1 s_1(1) \pmod{\mathfrak{m}(NJ_1)^{-1}}.$$

Assume for simplicity that  $1 \in W$ ,  $O_F$  is chosen as the representative for the trivial ideal class,  $s_0(1) = N^{-1}$  for  $J_0 = O_F$ , and  $e_\xi$  is normalised so that  $r_0^{O_F, 1}(e_\xi) = 1$ . We then have

$$r_0^{J,w}(e_\xi) = \xi(tw^{-1}),$$

where  $t$  is chosen so that  $J = (t)$  and  $t^{-1} \equiv s(1) \pmod{\mathfrak{m}(NJ)^{-1}}$ . Applying (10.3) and (10.6) with  $l$  replaced by  $l + l'$ , and

$$r_m^J([U_1(\mathfrak{n}^2)gU']f) = \text{Nm}_{F/\mathbb{Q}}(\mathfrak{n})\bar{\beta}_2^{-l+l'} r_{\beta_2 m}^{J_2, w_2}(f)$$

for  $f \in M_{k, l+l'}(U'; E)$  (with  $g$  and  $U'$  as in the proof of Lemma 10.2.1),  $m \in J_+^{-1} \cup \{0\}$ ,  $\mathfrak{n}^{-1}J = \beta_2 J_2$  and  $(\beta_2 \prod_{v|\mathfrak{n}} \varpi_v^{e_v})s(1) \equiv w_2 s_2(1) \pmod{\mathfrak{m}NJ_2^{-1}}$  then gives the formula<sup>8</sup>

$$r_m^J(f'_\xi) = \xi(t)G_J(\xi^{-1}, c'm)r_m^J(f).$$

Let  $\xi_{\mathfrak{m}}^{-1}$  denote the character of  $(O_F/\mathfrak{m})^\times$  induced by  $\xi$ , extended to a map  $\hat{O}_F \rightarrow O_F/\mathfrak{m} \rightarrow E$  by setting  $\xi_{\mathfrak{m}}^{-1}(a) = 0$  if  $(a)$  is not prime to  $\mathfrak{m}$ . We then have

$$G_J(\xi^{-1}, c'm) = G_{O_F}(\xi^{-1}, c'tm) = \xi_{\mathfrak{m}}^{-1}(tm)G_{O_F}(\xi^{-1}, c),$$

so setting

$$f_\xi = G_{O_F}(\xi^{-1}, c')^{-1} f'_\xi \tag{10.7}$$

gives  $f_\xi \in M_{k, l+l'}(U_1(\mathfrak{m}\mathfrak{n}^2); E)$  satisfying

$$r_m^J(f_\xi) = \xi(t)\xi_{\mathfrak{m}}^{-1}(tm)r_m^J(f). \tag{10.8}$$

Note that this is independent of the choice of  $t$  such that  $J = (t)$ . Furthermore, if we choose  $t$  so that  $t_p = 1$ , then  $\xi(t) = \xi'(t)$  where  $\xi' : \mathbb{A}_F^\times/F^\times F_{\infty,+}^\times V_{\mathfrak{m}p} \rightarrow E^\times$  is the character in the proof of Theorem 6.1.1. (Recall that  $\xi'$  is defined by  $\xi'(\alpha z a) = \xi(a)\bar{a}_p^{-l'}$  for  $\alpha \in F^\times$ ,  $z \in F_{\infty,+}^\times$  and  $a \in (\mathbb{A}_F^\infty)^\times$  with  $a_p \in O_{F,p}^\times$ .) Since  $\xi'(m) = 1$ , we then have

$$\xi(t)\xi_{\mathfrak{m}}^{-1}(tm) = \begin{cases} \xi'((tm)^{(\mathfrak{m})}) & \text{if } (tm) \text{ is prime to } \mathfrak{m}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $(tm)^{(\mathfrak{m})}$  denotes the projection of  $tm$  to the components prime to  $\mathfrak{m}$ .

We record the above construction:

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<sup>8</sup>This also follows more directly from the alternative description of  $f'_\xi$  and a formula analogous to (10.6).

**Lemma 10.3.2.** *If  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  and  $\xi$  is a character of weight  $l'$  and conductor  $\mathfrak{m}$ , then  $f_\xi \in M_{k,l+l'}(U_1(\mathfrak{n}\mathfrak{m}^2); E)$  has  $q$ -expansion coefficients defined by (10.8). In particular, if  $r_m^J(f) \neq 0$  for some  $m \in J_+^{-1}$  with  $mJ$  prime to  $\mathfrak{m}$ , then  $f_\xi \neq 0$ , in which case if  $f$  is an eigenform, then so is  $f_\xi$ , and  $\rho_{f_\xi} \simeq \rho_{\xi'} \otimes \rho_f$ .*

10.4.  $\Theta$ -operators on eigenforms

Recall from Corollary 8.2.4 that  $\Theta_\tau$  defines a map

$$M_{k,l}(U_1(\mathfrak{n}); E) \rightarrow M_{k',l'}(U_1(\mathfrak{n}); E),$$

where  $k'$  and  $l'$  are defined in Definition 8.2.1 (in particular  $l'_\tau = l_\tau - \delta_{\tau,\tau'}$ ). Moreover,  $\Theta_\tau$  commutes with the operators  $T_v$  (for all  $v \nmid p$ ) and  $S_v$  (for all  $v \nmid \mathfrak{n}p$ ).

**Lemma 10.4.1.** *With notation as in Lemma 10.2.1, we can take the eigenform  $f$  in the conclusion so that  $\Theta_\tau(f) \neq 0$  for all  $\tau \in \Sigma$ .*

*Proof.* Let  $v$  be a prime dividing  $p$ , and suppose that  $\tau \in \Sigma_v$ . Let  $f$  be an eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$  giving rise to  $\rho$ , and let  $m, J$  be such that  $r_m^J(f) \neq 0$  (so  $m \in J_+^{-1}$ ). We wish to prove that we can choose  $f$  with  $r_m^J(f) \neq 0$  for some  $m' \notin vJ_+^{-1}$ , so that  $f \notin \ker(\Theta_\tau)$ .

By Chevalley’s Theorem, we can (enlarging  $E$  if necessary) choose a character  $\xi$  of weight  $-l$  and conductor  $\mathfrak{m}$  for some  $\mathfrak{m}$  prime to  $pmJ$ . Thus  $f_\xi \in M_{k,0}(U_1(\mathfrak{n}'); E)$  where  $\mathfrak{n}' = \mathfrak{n}\mathfrak{m}^2$ ,  $r_m^J(f_\xi) \neq 0$ , and  $f_\xi$  is an eigenform giving rise to  $\rho' = \rho_{\xi'} \otimes \rho$ .

For eigenforms  $g \in M_{k,0}(U_1(\mathfrak{n}'); E)$  giving rise to  $\rho'$ , define  $\delta_v(g)$  to be the least  $d \geq 0$  such that  $r_m^J(g) \neq 0$  for some  $m, J$  such that  $m \notin v^d J^{-1}$ . Thus  $\delta_v(g) = 0$  if and only if  $g \notin \ker(\Theta_\tau)$ . We claim that if  $\delta_v(g) > 0$ , then  $\rho'$  arises from some  $h$  with  $\delta_v(h) = \delta_v(g) - 1$ ; moreover, if  $r_m^J(g) = 0$  for all  $mJ$  not prime to  $\mathfrak{m}$ , then the same is true for  $h$ .

To prove the claim, recall that if  $g \in \ker(\Theta_\tau)$  and  $k' = \nu(g)$ , then  $p \mid k'_\tau$  for all  $\tau \in \Sigma_v$ . (Recall that  $\nu(g)$  is defined in §5.2 and  $\nu(g) \in \mathbb{Z}_{\geq 0}$  by [19].) Writing  $g = g' \prod_{\tau' \in \Sigma} \text{Ha}_\tau^{n'_{\tau'}}$  for some  $g' \in M_{k',0}(U_1(\mathfrak{n}'); E)$  and  $n' \in \mathbb{Z}_{\geq 0}^\Sigma$ , we have  $g' \in \ker(\Theta_\tau)$ . By Theorem 9.8.2, we have  $g' = \Phi_v(g'')$  for some  $g'' \in M_{k'',0}(U_1(\mathfrak{n}'); E)$  where  $k''_{\tau'} = k'_{\tau'}$  for  $\tau' \notin \Sigma_v$ , and  $k''_{\tau'} = p^{-1}k'_{\mathbb{F}_r^{-1} \circ \tau'}$  for  $\tau' \in \Sigma_v$ . Now

$$h := g'' \prod_{\tau' \in \Sigma_v} \text{Ha}_\tau^{k''_{\tau'}} \prod_{\tau' \in \Sigma} \text{Ha}_\tau^{n'_{\tau'}}$$

is an eigenform in  $M_{k,0}(U_1(\mathfrak{n}'); E)$  giving rise to  $\rho'$ , and Proposition 9.8.1 immediately gives that  $\delta_v(h) = \delta_v(g) - 1$ , and if  $r_m^J(g) = 0$  for all  $mJ$  not prime to  $\mathfrak{m}$ , then the same is true for  $h$ .

Starting with  $f_\xi$  and applying the claim inductively, we conclude that  $\rho'$  arises from an eigenform  $g \in M_{k,0}(U_1(\mathfrak{n}'); E)$  such that  $r_m^J(g) \neq 0$  for some  $m, J$  with  $mJ$  prime to  $v\mathfrak{m}$ . Therefore  $g_{\xi^{-1}}$  is an eigenform in  $M_{k,l}(U_1(\mathfrak{n}'\mathfrak{m}^2); E)$  giving rise to  $\rho$ , and  $g_{\xi^{-1}} \notin \ker(\Theta_\tau)$ .

An elementary linear algebra argument then shows that, after possibly further shrinking  $\mathfrak{n}$  and enlarging  $E$ , there is an eigenform  $f$  which satisfies the conclusion simultaneously for all  $\tau \in \Sigma$ . ■

We now have the following immediate consequences of Theorem 8.2.2:

**Theorem 10.4.2.** *Suppose that  $\rho$  is irreducible and geometrically modular of weight  $(k, l)$ . Then  $\rho$  is geometrically modular of weight  $(k', l')$ , and furthermore of weight  $(k' - k_{\text{Ha}_\tau}, l')$  if  $p \nmid k_\tau$  (where  $k'$  is as in Definition 8.2.1).*

**Corollary 10.4.3.** *Suppose that  $\rho$  is irreducible and  $l, l' \in \mathbb{Z}^\Sigma$  with  $l'_\tau = l_\tau - \delta_{\tau, \tau'}$ . Suppose further that there exist  $k = k_{\min}(\rho, l)$  and  $k_{\min}(\rho, l')$  as in part (1) of Conjecture 7.3.1. Then*

$$k_{\min}(\rho, l') \leq_{\text{Ha}} \begin{cases} k' & \text{if } p \nmid k_\tau, \\ k' - k_{\text{Ha}_\tau} & \text{if } p \mid k_\tau. \end{cases}$$

**Remark 10.4.4.** We remark that we expect equality to hold in the corollary in the case that  $p \nmid k_\tau$ . We caution however that the analogous strengthening of Theorem 8.2.2 is false: i.e., it is possible for  $\Theta_\tau(f)$  to be divisible by  $\text{Ha}_{\tau'}$  for some  $\tau' \neq \tau$  even if  $p \nmid k_\tau$  and  $f$  is not divisible by  $\text{Ha}_{\tau'}$ .

**Corollary 10.4.5.** *Suppose that  $\rho$  is irreducible and geometrically modular of some weight  $(k_0, l_0)$ . Then for every  $l \in \mathbb{Z}^\Sigma$ , there exist  $k \in \mathbb{Z}^\Sigma$  such that  $\rho$  is geometrically modular of weight  $(k, l)$ .*

*Proof.* Note that if  $\rho$  is geometrically modular of some weight  $(k_0, l_0)$ , then multiplying by the constant section  $e_1$  of weight  $(0, n(p - 1))$ , we can replace  $l_0$  by  $l_0 + n(p - 1)$  for any  $n \in \mathbb{Z}$  and hence assume  $l_{0, \tau} \geq l_\tau$  for all  $\tau \in \Sigma$ . The corollary then follows from Theorem 10.4.2 by induction on  $\sum_\tau (l_{0, \tau} - l_\tau)$ . ■

### 10.5. Normalised eigenforms

We continue to assume for simplicity that  $J = O_F$  is chosen as an ideal class representative.

**Definition 10.5.1.** Suppose that  $(k, l)$  is an algebraic weight (i.e.,  $k_\tau \geq 2$  for all  $\tau \in \Sigma$ ). We say that  $f \in M_{k, l}(U_1(\mathfrak{n}); E)$  is a *normalised eigenform* if

- $r_1^{O_F}(f) = 1$ ;
- $f$  is an eigenform for  $T_v$  for all  $v \nmid p$ , and for  $S_v$  for all  $v \nmid \mathfrak{n}p$ ;
- $f_\xi \in M_{k, 0}(U_1(\mathfrak{n}m^2); E')$  is an eigenform for  $T_v$  for all  $v \mid p$  and all characters

$$\xi : \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F, p}^\times\} / V_m \rightarrow (E')^\times$$

of weight  $-l$ , conductor  $\mathfrak{m}$  prime to  $p$ , and values in extensions  $E'$  of  $E$ .

It is straightforward to check that if  $f_\xi$  is an eigenform for  $T_v$  (where  $v \mid p$  and  $\xi$  has weight  $-l$  and conductor prime to  $p$ ), then so is  $f_{\xi_1, \xi_2}$  for any characters  $\xi_1, \xi_2$  such that  $\xi_1 \xi_2 = \xi$  (where the  $\xi_i$  have conductors  $\mathfrak{m}_i$  prime to  $p$  and weights  $-l_i$  such that  $l = l_1 + l_2$ ). In particular, it follows that if  $f$  is a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$ , then  $f_{\xi_1}$  is a normalised eigenform in  $M_{k,l+l_1}(U_1(\mathfrak{n}\mathfrak{m}_1^2); E)$  (enlarging  $E$  if necessary).

We have the following strengthening of Lemma 10.4.1 for algebraic weights:

**Proposition 10.5.2.** *If  $\rho$  is irreducible and geometrically modular of weight  $(k, l)$  with  $k_\tau \geq 2$  for all  $\tau$ , then  $\rho$  arises from a normalised eigenform of weight  $(k, l)$  and level  $U_1(\mathfrak{n})$  for some  $\mathfrak{n}$  prime to  $p$ .*

*Proof.* Suppose first that  $l = 0$ . Then Lemma 10.2.1 implies that  $\rho$  arises from an eigenform  $f \in M_{k,0}(U_1(\mathfrak{n}); E)$  for some  $\mathfrak{n}$  prime to  $p$  (and some  $E$ ). Recall that in this case ( $l = 0$  and all  $k_\tau \geq 2$ ), we have defined Hecke operators  $T_v$  for all primes  $v \mid \mathfrak{n}p$ , commuting with each other and with the operators  $T_v$  and  $S_v$  for  $v \nmid \mathfrak{n}p$ , so we may further assume that  $f$  is an eigenform of  $T_v$  for all  $v$  and of  $S_v$  for all  $v \nmid \mathfrak{n}p$ . It suffices to prove that  $r_1^{O_F}(f) \neq 0$ .

Suppose that  $r_1^{O_F}(f) = 0$ ; we will show that  $f = 0$ , yielding a contradiction. Recall from Lemma 10.1.1 that the absolute irreducibility of  $\rho \simeq \rho_f$  implies that  $r_0^J(f) = 0$  for all  $J$ . We will prove that  $r_m^J(f) = 0$  for all  $J$  and  $m \in J_+^{-1}$  by induction on  $n = \text{Nm}_{F/\mathbb{Q}}(mJ)$ .

If  $n = 1$ , then  $mJ = O_F$ , so  $J = O_F$ ,  $m \in O_{F,+}^\times$ , and  $r_m^{O_F}(f) = r_1^{O_F}(f) = 0$ .

Now suppose that  $n > 1$  and  $r_m^J(f) = 0$  for all  $m, J$  with  $\text{Nm}_{F/\mathbb{Q}}(mJ) < n$ , and let  $m_1, J_1$  be such that  $\text{Nm}_{F/\mathbb{Q}}(m_1 J_1) = n$ . Let  $v$  be any prime dividing  $m_1 J_1$ . If  $v^2 \nmid m_1 J_1$  or  $v \mid \mathfrak{n}p$ , then Propositions 9.5.1, 9.6.1 and 9.7.1 give

$$r_{m_1}^{J_1}(f) = r_m^J(T_v f) = a_v r_m^J(f),$$

where  $m_1 J_1 = v m J$  and  $a_v$  is the eigenvalue of  $T_v$  on  $f$ . We have  $r_m^J(f) = 0$  by the induction hypothesis, and hence  $r_{m_1}^{J_1}(f) = 0$ . If  $v^2 \mid m_1 J_1$  and  $v \nmid \mathfrak{n}p$ , then Proposition 9.5.1 gives

$$r_{m_1}^{J_1}(f) = r_m^J(T_v f) - \text{Nm}_{F/\mathbb{Q}}(v) r_{m_2}^{J_2}(S_v f) = a_v r_m^J(f) - d_v \text{Nm}_{F/\mathbb{Q}}(v) r_{m_2}^{J_2}(f),$$

where  $m_1 J_1 = v m J = v^2 m_2 J_2$  and  $a_v$  (resp.  $d_v$ ) is the eigenvalue of  $T_v$  (resp.  $S_v$ ) on  $f$ . By the induction hypothesis, we have  $r_m^J(f) = r_{m_2}^{J_2}(f) = 0$ , so again  $r_{m_1}^{J_1}(f) = 0$ . This completes the proof of the proposition in the case  $l = 0$ .

Now consider the case of arbitrary  $l$ . Let  $\mathfrak{m}$  (prime to  $p$ ) be such that there is a character

$$\xi : \{a \in (\mathbb{A}_F^\infty)^\times \mid a_p \in O_{F,p}^\times\} / V_{\mathfrak{m}} \rightarrow E^\times$$

of conductor  $\mathfrak{m}$  satisfying  $\xi(\alpha) = \bar{\alpha}^{-l}$  for all  $\alpha \in F_+^\times \cap O_{F,p}^\times$ . Then  $\rho_{\xi'} \otimes \rho$  is geometrically modular of weight  $(k, 0)$ , and therefore arises from a normalised eigenform  $f \in M_{k,0}(U_1(\mathfrak{n}); E)$  for some  $\mathfrak{n}$  prime to  $p$ . Furthermore, we may assume  $\mathfrak{m} \mid \mathfrak{n}$  (for example by replacing  $\mathfrak{n}$  by  $\mathfrak{n}\mathfrak{m}$ ). Then  $f_{\xi^{-1}}$  is a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{m}\mathfrak{n}); E)$  giving rise to  $\rho$ . ■



10.6. Stabilised eigenforms

We assume for the rest of the section that the weight  $(k, l)$  is algebraic.

**Definition 10.6.1.** We say that a normalised eigenform  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  is *stabilised* if  $r_m^J(f) = 0$  for all  $(m, J)$  such that  $m \in J_+^{-1}$  and  $mJ$  is not prime to  $\mathfrak{n}$ . Note that this is equivalent to the condition that  $T_v f = 0$  for all  $v \mid \mathfrak{n}$ .

**Lemma 10.6.2.** *If  $\rho$  arises from a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{m}); E)$ , and  $\mathfrak{n} \subset \mathfrak{m}$  is an ideal prime to  $p$ , then  $\rho$  arises from a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$  (enlarging  $E$  if necessary). Moreover, if  $\mathfrak{m}$  and  $\mathfrak{n}$  satisfy*

- $\text{ord}_v(\mathfrak{n}\mathfrak{m}^{-1}) \geq 1$  for all  $v \mid \mathfrak{m}$ ,
- $\text{ord}_v(\mathfrak{n}\mathfrak{m}^{-1}) \neq 1$  for all  $v \nmid \mathfrak{m}$ ,

*then  $\rho$  arises from a stabilised eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$ .*

*Proof.* The first assertion immediately reduces to the case  $\mathfrak{n} = \mathfrak{m}v$  where  $v$  is a prime not dividing  $\mathfrak{m}p$ . Suppose that  $f \in M_{k,l}(U_1(\mathfrak{m}); E)$  is a normalised eigenform giving rise to  $\rho$ , and let  $\alpha \in E$  (enlarging  $E$  is necessary) be a root of  $X^2 - a_v X + d_v \text{Nm}_{F/\mathbb{Q}}(v)$ , the characteristic polynomial of  $\rho(\text{Frob}_v)$ , so  $a_v$  (resp.  $d_v$ ) is the eigenvalue of  $T_v$  (resp.  $S_v$ ) on  $f$ . A standard calculation then shows that

$$f' = f - \text{Nm}_{F/\mathbb{Q}}(v)^{-1} \begin{pmatrix} \varpi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} \alpha f$$

is a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$ . Moreover,  $f'$  has the same eigenvalues as  $f$ , except that  $T_v f' = (a_v - \alpha) f'$ , and therefore  $\rho_{f'} \simeq \rho_f$ .

In view of the first assertion, the second immediately reduces to the case  $\mathfrak{n} = \mathfrak{m} \prod_{v \mid \mathfrak{m}} v$ . So suppose that  $f \in M_{k,l}(U_1(\mathfrak{m}); E)$  is a normalised eigenform giving rise to  $\rho$ , and for each  $v \mid \mathfrak{m}$ , let  $\beta_v$  be the eigenvalue of  $T_v$  on  $f$ . A similar standard calculation then shows that

$$f' = \prod_{v \mid \mathfrak{m}} (1 - \text{Nm}_{F/\mathbb{Q}}(v)^{-1} \begin{pmatrix} \varpi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} \beta_v) f$$

is a normalised eigenform in  $M_{k,l}(U_1(\mathfrak{n}); E)$ . Moreover,  $f'$  has the same eigenvalues as  $f$ , except that  $T_v f' = 0$  for all  $v \mid \mathfrak{n}$ . Therefore  $f'$  is stabilised and gives rise to  $\rho$ . ■

**Remark 10.6.3.** A more careful analysis easily shows that the first assertion of the lemma requires at most a quadratic extension of  $E$ , and the second holds over the original field  $E$ .

**Definition 10.6.4.** We say that a stabilised eigenform  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  is *strongly stabilised* if  $r_m^J(f) = 0$  for all  $(m, J)$  such that  $m \in J_+^{-1} \cup \{0\}$  and  $mJ$  is not prime to  $p$ .<sup>9</sup>

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<sup>9</sup>Note that our conventions allow a stabilised eigenform to have  $r_0^J(f) \neq 0$  in the case  $\mathfrak{n} = O_F$ , but a strongly stabilised eigenform necessarily has  $r_0^J(f) = 0$ .

Thus a stabilised eigenform is strongly stabilised if and only if  $T_v f_\xi = 0$  for all  $v \mid p$  and characters  $\xi$  of weight  $-l$ . (Note that given  $m \in J_+^{-1} \cup \{0\}$ , we can always choose  $\xi$  of weight  $-l$  and conductor prime to  $mJ$  unless  $m = 0$  and  $\bar{v}^l \neq 1$  for some  $v \in O_{F,+}^\times$ , in which case we automatically have  $r_m^J(f) = 0$ .)

**Lemma 10.6.5.** *There is at most one strongly stabilised eigenform  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  giving rise to  $\rho$ .*

*Proof.* If  $\rho$  arises from  $f$ , then  $T_v f = a_v f$  and  $S_v f = d_v f$  for all  $v \nmid \mathfrak{n}p$ , where  $a_v = \text{tr}(\rho(\text{Frob}_v))$  and  $d_v = \text{Nm}_{F/\mathbb{Q}}(v)^{-1} \det(\rho(\text{Frob}_v))$ .

Suppose then that  $f$  and  $f'$  are strongly stabilised eigenforms giving rise to  $\rho$ , and let  $f'' = f - f'$ . It suffices to prove that  $r_m^J(f'') = 0$  for all  $(m, J)$  with  $m \in J_+^{-1} \cup \{0\}$ . Since  $f$  and  $f'$  are strongly stabilised, we have  $r_m^J(f'') = 0$  whenever  $mJ$  is not prime to  $\mathfrak{n}p$ , so we can assume  $mJ$  is prime to  $\mathfrak{n}p$ . We then proceed as in the proof of Proposition 10.5.2 by induction on  $n = \text{Nm}_{F/\mathbb{Q}}(mJ)$ .

If  $n = 1$ , then  $mJ = O_F$ , so  $J = O_F$ ,  $m \in O_{F,+}^\times$ , and  $r_m^{O_F}(f'') = m^{-1} r_1^{O_F}(f'') = 0$  since  $r_1^{O_F}(f) = r_1^{O_F}(f') = 1$ .

Now suppose that  $n > 1$  and  $r_m^J(f'') = 0$  for all  $m, J$  with  $\text{Nm}_{F/\mathbb{Q}}(mJ) < n$ , and let  $m_1, J_1$  be such that  $m_1 J_1$  is prime to  $\mathfrak{n}p$  and  $\text{Nm}_{F/\mathbb{Q}}(m_1 J_1) = n$ . Let  $v$  be any prime dividing  $m_1 J_1$ . If  $v^2 \nmid m_1 J_1$ , then Proposition 9.5.1 gives

$$r_{m_1}^{J_1}(f'') = m_1^{-1} m^l a_v r_m^J(f''),$$

where  $m_1 J_1 = vmJ$ . so the induction hypothesis implies that  $r_{m_1}^{J_1}(f'') = 0$ . If  $v \nmid \mathfrak{n}p$ , then we get instead

$$r_{m_1}^{J_1}(f'') = m_1^{-1} m^l a_v r_m^J(f'') - m_1^{-l} m_2^l d_v \text{Nm}_{F/\mathbb{Q}}(v) r_{m_2}^{J_2}(f''),$$

where  $m_1 J_1 = vmJ = v^2 m_2 J_2$ , and again the induction hypothesis implies that  $r_{m_1}^{J_1}(f'') = 0$ . ■

**Remark 10.6.6.** Note that if  $f$  is a normalised (resp. stabilised, strongly stabilised) eigenform, then the same is true for both  $\text{Ha}_\tau f$  and  $\Theta_\tau f$  for any  $\tau$  (assuming  $k_\tau \geq 3$  if  $\tau \neq \text{Fr} \circ \tau$  in the case of  $\text{Ha}_\tau f$ ).

**Remark 10.6.7.** If  $\rho$  is geometrically modular of weight  $(k, l)$ , then it does not necessarily arise from a strongly stabilised eigenform of weight  $(k, l)$  (for any level  $\mathfrak{n}$ ); for example, there may be a prime  $v \mid p$  such that  $r_m^J(f) \neq 0$  whenever  $mJ = v$ . We do however have the following two ways of establishing the existence of strongly stabilised eigenforms. One is to apply partial  $\Theta$ -operators to a stabilised eigenform (hence changing the weight); the other is to use Theorem 10.7.1 below, or more precisely its corollary.

### 10.7. Ordinarity

The forthcoming Theorem 10.7.1 can be viewed as stating that if an eigenform is ordinary in a suitable sense, then so is the associated Galois representation. For the proof, we need

to verify certain compatibility properties for the operators  $T_v$  for  $v \mid p$  (assuming  $k$  is algebraic and  $l = 0$ ), which we shall do using their effect on  $q$ -expansions at more general cusps than the ones used above.

Fix a set  $\{\alpha\}$  of ideal class representatives and a set  $\{g\}$  of coset representatives for  $P_{\mathfrak{n}} \backslash \mathrm{SL}_2(O_F/\mathfrak{n})$ . For consistency with previous computations, choose  $\alpha = O_F$  and  $g = 1$  for the trivial classes. Also fix choices of  $t : O_F/NO_F \simeq \mu_N \otimes (\alpha\mathfrak{d})^{-1}$  for each  $\alpha$  (independent of  $J$ ) and  $s : O_F/NO_F \simeq (NJ)^{-1}\alpha/J^{-1}\alpha$  for each  $\alpha, J$ . The cusps of  $X_U$  for  $U = U(\mathfrak{n})$  are then in bijection with the quadruples  $(J, \alpha, w, g)$ , where the corresponding cusp is the one associated to the Tate semi-HBAV  $T_{\alpha, \mathfrak{b}}$ , where  $\mathfrak{b} = \alpha J^{-1}$ , with canonical polarisation and level structure  $\eta_w \circ r_{g^{-1}}$ , where  $\eta_w(x, y) = t(y)q^{ws(x)}$ . Then for  $v \mid p$ , we find (by the same proof as for Proposition 9.7.1) that the effect on  $q$ -expansions of the action of  $T_v$  on  $M_{k,0}(U; E)$  is given by

$$r_m^C(T_v f) = r_{\beta_1 m}^{C_1}(f), \tag{10.9}$$

where the  $q$ -expansion coefficients lie in  $\overline{D}^{k,0}$ , and if  $C$  is the cusp corresponding to  $(J, \alpha, w, g)$ , then  $C_1$  corresponds to  $(J_1, \alpha, w_1, g)$  for  $J_1, w_1, \beta_1 \in F_+$  such that  $vJ = \beta_1 J_1$  and  $\beta_1 \varpi_v^{-1} w s(1) \equiv w_1 s_1(1) \pmod{N^{-1}\mathfrak{n}\mathfrak{b}_1}$  (where  $\mathfrak{b}_1 = \alpha J_1^{-1}$  and  $s_1$  is the chosen isomorphism).

Finally, we need to consider the action of  $T_v$  on  $M_{k,0}(U'; L)$  for  $v \mid p$ , where  $U' = U \cap U_1(p)$  and  $k \in \mathbb{Z}_{\geq 2}$ . Note that this may be defined in the usual way as the operator  $[U'(\begin{smallmatrix} \varpi_v & 0 \\ 0 & 1 \end{smallmatrix})U']$  on forms in characteristic zero, making it compatible with the action of  $T_v$  on the space  $A_{k,0}(U')$  of automorphic forms. Recall from §6.4 that  $X_{U'}$  denotes the minimal compactification of  $Y_{U'}$ , and that its cusps are in bijection with triples  $(C, \mathfrak{f}, P)$  where  $C$  is a cusp of  $X_U$ ,  $\mathfrak{f}$  is an ideal such that  $pO_F \subset \mathfrak{f} \subset O_F$  and  $P$  is a generator of  $\mathfrak{b}/\mathfrak{b}\mathfrak{f}$ , and the corresponding cusp may be identified with the  $\mathcal{O}$ -scheme  $\mathrm{Spec} \mathcal{O}'_{\mathfrak{f}}$  representing generators of  $\mu_p \otimes \mathfrak{f}(\alpha\mathfrak{d})^{-1}/p(\alpha\mathfrak{d})^{-1}$  (where  $\alpha$  and  $\mathfrak{b}$  are as in the description of  $C$ ). We only need to consider those cusps for which  $\mathfrak{f} = O_F$ ; for each cusp  $C$  of  $X_U$ , we write  $C'$  for the unique such cusp of  $X_{U'}$  lying over it. We assume  $L$  contains the  $p$ th roots of unity, so that the components of  $C'_L$  are copies of  $\mathrm{Spec} L$  in bijection with the generators  $\zeta_p$  of  $\mu_p(L) \otimes (\alpha\mathfrak{d})^{-1}$ . We may then compute the effect of  $T_v$  on the completion at each component of  $C'_L$  exactly as in Proposition 9.6.1 (see also (10.3)) to conclude that if  $f \in M_{k,0}(U'; L)$ , then

$$r_m^{C'}(T_v f) = r_{\beta_1 m}^{C'_1}(f), \tag{10.10}$$

where the notation is as in (10.9), except that the  $q$ -expansion coefficients lie in the fibre of  $j'_* \mathcal{L}_{U'}^{k,0}$  at  $C'_L$ , which we may identify with  $\bigoplus_{\zeta_p} (D^{k,0} \otimes_{\mathcal{O}} L)$  (where  $\zeta_p$  runs over generators of  $\mu_p(L) \otimes (\alpha\mathfrak{d})^{-1}$ ).

**Theorem 10.7.1.** *Suppose that  $k \in \mathbb{Z}^{\Sigma}$  with  $k_{\tau} \geq 2$  for all  $\tau$ ,  $U$  is an open compact subgroup of  $\mathrm{GL}_2(\widehat{O}_F)$  containing  $\mathrm{GL}_2(O_{F,p})$ ,  $Q$  is a finite set of primes containing all  $v \mid p$  and all  $v$  such that  $\mathrm{GL}_2(O_{F,v}) \not\subset U$ , and  $v_0$  is a prime over  $p$ . Suppose that*

$f \in M_{k,0}(U; E)$  is an eigenform for  $T_v$  and  $S_v$  for all  $v \notin Q$  and  $T_{v_0}f = a_{v_0}f$  for some  $a_{v_0} \neq 0$ . Then (possibly after enlarging  $E$  and semisimplifying  $\rho_f$ )

$$\rho_f|_{G_{F_{v_0}}} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix},$$

where  $\chi_1$  is an unramified character,  $\chi_1(\text{Frob}_{v_0}) = a_{v_0}$ , and  $\chi_2|_{I_{F_{v_0}}} = \prod_{\tau \in \Sigma_{v_0}} \epsilon_{\tau}^{1-k_{\tau}}$ .

*Proof.* We may assume that  $U = U(\mathfrak{n})$  for some sufficiently small  $\mathfrak{n}$  prime to  $p$  and that  $\mathcal{O}$  is sufficiently large; in particular, we assume  $\mu_{Np}(\mathbb{Q}) \subset \mathcal{O}$  for some  $N \in \mathfrak{n}$ .

Recall that the proof of Theorem 6.1.1 in §6.5 yields injections

$$M_{k,-1}(U; E) \rightarrow M_{k',-1}(U'; E) \rightarrow M_{m+2,-1}(U'; \mathcal{O}) \otimes_{\mathcal{O}} E,$$

which are compatible with  $T_v$  and  $S_v$  for  $v \nmid \mathfrak{n}p$ , where  $U = U(\mathfrak{n})$ ,  $U' = U(\mathfrak{n}) \cap U_1(p)$ ,  $k'$  is nearly parallel and  $m$  is a (sufficiently large) positive integer. Tensoring with the (pull-backs of the) canonical section  $e_1 \in H^0(Y_U, \mathcal{L}_U^{0,1})$ , we may replace  $l = -1$  by  $l = 0$ . Since the first injection is defined by multiplication by partial Hasse invariants, which have  $q$ -expansions equal to 1 at every cusp, we see from (10.9) that it is also compatible with  $T_v$  for  $v \mid p$ . We may therefore replace  $k$  by  $k'$  and assume that  $k$  is nearly parallel.

Recall that for a cusp  $C$  of  $X_U$ , we let  $C'$  denote the unique cusp of  $X_{U'}$  with  $\bar{f} = \mathcal{O}_F$ . For  $\mathcal{O}$ -algebras  $R$ , let  $\mathcal{Q}_{C',R}^{m+2,0}$  denote the completion at  $C'_R$  of  $j'_*(\mathcal{K}_{U',R} \otimes_{\mathcal{O}_{Y_{U',R}}} \mathcal{L}_{U',R}^{m,1})$  (in the notation of §6, and as usual omitting subscripts if  $R = \mathcal{O}$  and using  $\bar{\cdot}$  in the case  $R = E$ ). From the description of  $j'_*\mathcal{K}_{U'}$  in §6.5, we see that  $\mathcal{Q}_{C',R}^{m+2,0}$  is canonically isomorphic to

$$D^{m+2,0} \otimes_{\mathcal{O}} \text{Hom}_{\widehat{S}_{C,R}}(\widehat{S}_{C',R}, \widehat{S}_{C,R}) \simeq \text{Hom}_{\mathcal{O}}(\mathcal{O}'_{\mathcal{O}_F}, D^{m+2,0}) \otimes_{\mathcal{O}} \widehat{S}_{C,R}$$

as a module over  $\widehat{S}_{C',R} \simeq \mathcal{O}'_{\mathcal{O}_F} \otimes_{\mathcal{O}} \widehat{S}_{C,R}$ .

Letting  $\mathcal{S}$  denote the set of cusps of  $X_U$ , we have natural  $q$ -expansion maps

$$M_{m+2,0}(U'; R) \rightarrow \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{C',R}^{m+2,0},$$

which are injective if  $R = L$  (and hence  $R = \mathcal{O}$ ) since  $\bigsqcup_{C \in \mathcal{S}} C'_L$  includes cusps on every connected component of  $X_{U',L}$ . We define  $\widetilde{M}_{m+2,0}(U'; \mathcal{O})$  to be the preimage of  $\bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{C',L}^{m+2,0}$  in  $M_{m+2,0}(U'; L)$  under the  $q$ -expansion map to  $\bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{C',L}^{m+2,0}$ . We thus have an inclusion  $M_{m+2,0}(U'; \mathcal{O}) \subset \widetilde{M}_{m+2,0}(U'; \mathcal{O})$  with finite index, so in particular  $\widetilde{M}_{m+2,0}(U'; \mathcal{O})$  is finitely generated over  $\mathcal{O}$ .

We see directly from the definition that, for  $v \nmid p\mathfrak{n}$ , the Hecke operators  $T_v$  and  $S_v$  on  $M_{m+2,0}(U'; \mathcal{O})$  also act on the modules  $\mathcal{Q}_{C',L}^{m+2,0}$  compatibly with the  $q$ -expansion map, from which it follows that the operators preserve  $\widetilde{M}_{m+2,0}(U'; \mathcal{O})$ . Furthermore, from (10.10) and the fact that the isomorphism

$$\mathcal{Q}_{C',L}^{m+2,0} \simeq D^{m+2,0} \otimes_{\mathcal{O}} \widehat{S}_{C',L}$$

induced by the Kodaira–Spencer isomorphism  $\mathcal{K}_{U',L} \simeq \mathcal{L}_{U',L}^{2,-1}$  is the same as the one induced by the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}'_{O_F}, L) \simeq \bigoplus_{\xi_p} L \simeq \mathcal{O}'_{O_F} \otimes_{\mathcal{O}} L,$$

we see that  $\tilde{M}_{m+2,0}(U'; \mathcal{O})$  is also stable under  $T_v$  for  $v \mid p$ , with the action on  $q$ -expansions being defined by the same formula, but with coefficients in the fibre  $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}'_{O_F}, D^{m+2,0})$  (and a reconciliation of the duplicate use of  $m$ ).

Now consider the commutative diagram

$$\begin{array}{ccccc} M_{k,0}(U; E) & \longrightarrow & M_{m+2,0}(U'; \mathcal{O}) \otimes_{\mathcal{O}} E & \longrightarrow & \tilde{M}_{m+2,0}(U'; \mathcal{O}) \otimes_{\mathcal{O}} E \\ \downarrow & & & & \downarrow \\ \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}}^{k,0} & \longrightarrow & \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}'}^{m+2,0} & \longleftarrow & \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}'}^{m+2,0} \otimes_{\mathcal{O}} E \end{array} \tag{10.11}$$

The proof of Theorem 6.1.1 shows that the first map on the bottom row is injective and the second is an isomorphism. Since the left vertical arrow is injective by the  $q$ -expansion principle, it follows that the top composite is also injective. Furthermore, the right vertical arrow is injective since  $\tilde{M}_{m+2,0}(U'; \mathcal{O}) \rightarrow \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}'}^{m+2,0}$  is injective with torsion-free cokernel (by construction).

We already saw in the proof of Theorem 6.1.1 that the map

$$M_{k,0}(U; E) \rightarrow M_{m+2,0}(U'; \mathcal{O}) \otimes_{\mathcal{O}} E$$

is compatible with the operators  $T_v$  and  $S_v$  for  $v \nmid pn$ , and it follows that the same holds for the composite on the top row of (10.11). We claim that this composite is also compatible with the operators  $T_v$  for  $v \mid p$ . To see this, note that we have actions of these operators on  $\bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}}^{k,0}$  and  $\bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}'}^{m+2,0} \simeq \bigoplus_{C \in \mathcal{S}} \mathcal{Q}_{\bar{C}'}^{m+2,0} \otimes_{\mathcal{O}} E$  which are compatible with the vertical maps of (10.11). Since these maps are injective, the claim will follow from the compatibility of the bottom row of (10.11) with these operators. The desired compatibility then follows from the fact that the first map on the bottom row is induced by the  $\mathcal{O}_S$ -dual of the pull-back to  $S = \mathrm{Spec} \hat{S}_{\bar{C}} - \bar{C}$  of the isomorphism  $\bar{\pi}_* i_* \mathcal{O}_{Y_U^\mu} \simeq \bigoplus_{\kappa} \bar{\mathcal{L}}_U^{\kappa,0}$  constructed in the proof of Theorem 6.1.1, which is given with respect to the canonical trivialisations over  $S$  by an isomorphism

$$\mathcal{O}'_{O_F} \otimes_{\mathcal{O}} E \simeq \bigoplus_{\kappa} \bar{D}^{\kappa,0}.$$

We have now shown that the top row of (10.11) defines an injective homomorphism

$$M_{k,0}(U; E) \rightarrow \tilde{M}_{m+2,0}(U'; \mathcal{O}) \otimes_{\mathcal{O}} E,$$

compatible with the operators  $T_v$  for  $v \nmid n$  and  $S_v$  for  $v \nmid np$ . It is therefore a homomorphism of  $\mathbb{T}$ -modules, where  $\mathbb{T}$  is the (commutative)  $\mathcal{O}$ -algebra of endomorphisms of  $\tilde{M}_{m+2,0}(U'; \mathcal{O})$  generated by  $T_{v_0}$  and the operators  $T_v$  and  $S_v$  for  $v \notin Q$ . The same

(standard) argument as at the end of the proof of Theorem 6.1.1 now shows that (after enlarging  $\mathcal{O}$ ,  $L$  and  $E$  if necessary), there is an eigenform  $\tilde{f} \in M_{m+2,0}(U'; L)$  for the operators  $T \in \mathbb{T}$  such that the eigenvalues are lifts of the corresponding ones for  $f$ . In particular,  $\bar{\rho}_{\tilde{f}}$  and  $\rho_f$  have isomorphic semisimplifications, and  $T_{v_0}\tilde{f} = \tilde{a}_{v_0}\tilde{f}$  for some  $\tilde{a}_{v_0} \in \mathcal{O}^\times$ .

We now deduce that  $\rho_f|_{G_{F_{v_0}}}$  has the desired form from the analogous fact for the characteristic zero modular Galois representation  $\rho_{\tilde{f}}$ , which is a special case of local-global compatibility at  $v_0$  for the corresponding automorphic and Galois representations. More precisely, suppose first that  $\tilde{f}$  is cuspidal and view it as a vector fixed by  $U'$  in the associated automorphic representation  $\Pi$ , so  $\tilde{a}_{v_0}$  is an eigenvalue for  $T_{v_0}$  on  $\Pi_{v_0}^{U'_{v_0}}$ , where  $\Pi_{v_0}$  is the local factor of  $\Pi$  at  $v_0$  and  $U'_{v_0} = U_1(v_0) \cap \text{GL}_2(\mathcal{O}_{F_{v_0}})$ . We may assume for simplicity that  $m > 0$ , so that  $\Pi_{v_0}$  must be a principal series representation<sup>10</sup> of the form  $I(\psi_1|\cdot|^{1/2}, \psi_2|\cdot|^{1/2})$  where  $\psi_1, \psi_2$  are characters  $F_{v_0}^\times \rightarrow \overline{\mathbb{Q}}^\times$  such that  $\psi_1$  is unramified with  $\psi_1(\varpi_{v_0}) = \tilde{a}_{v_0}$  (and  $\psi_2$  is at most tamely ramified with  $\psi_2(\varpi_{v_0}) \text{Nm}_{F/\mathbb{Q}}(v_0)^{-m-1} \in \mathcal{O}^\times$ ). The main theorem of [57] (adapted to our conventions) then implies that  $\rho_{\tilde{f}}|_{G_{F_{v_0}}}$  is potentially crystalline with labelled Hodge–Tate weights  $(m + 1, 0)$  and associated Weil–Deligne representation  $\psi_1 \oplus \psi_2$  (writing  $\psi_i$  also for the representations of  $W_{F_{v_0}}$  to which they correspond by local class field theory). A standard exercise in  $p$ -adic Hodge theory then shows that  $\rho_{\tilde{f}}|_{G_{F_{v_0}}}$  must be of the form

$$\begin{pmatrix} \tilde{\chi}_1 & * \\ 0 & \tilde{\chi}_2 \end{pmatrix}$$

for some  $\tilde{\chi}_1, \tilde{\chi}_2 : G_{F_{v_0}} \rightarrow L^\times$  with  $\tilde{\chi}_1$  unramified and  $\tilde{\chi}_1(\text{Frob}_{v_0}) = \tilde{a}_{v_0}$  (and  $\tilde{\chi}_2\chi_{\text{cyc}}^{m+1}$  at most tamely ramified). The theorem then follows in this case from the fact that  $\rho_f$  is (up to semisimplification) the reduction mod  $\pi$  of  $\rho_{\tilde{f}}$ , together with the description of  $\det(\rho_f)$  in Remark 6.5.1.

Suppose on the other hand that  $\tilde{f}$  is not cuspidal, in which case its eigenvalues for  $T_v$  and  $S_v$  for  $v \notin Q$  are the same as those of an Eisenstein series associated to a pair of Hecke characters  $\psi_1, \psi_2$  such that  $\psi_1(x) = 1$  and  $\psi_2(x) = x^{-m-1}$  for  $x \in F_{\infty,+}^\times$ . Moreover  $\tilde{a}_{v_0} = \psi_i(\varpi_{v_0})$  for some  $i$  such that  $\psi_i$  is unramified at  $v_0$ , and we must have  $i = 1$  since  $\tilde{a}_{v_0} \in \mathcal{O}^\times$ . In this case the (semisimplification of the) associated Galois representation  $\rho_{\tilde{f}}$  is  $\tilde{\chi}_1 \oplus \tilde{\chi}_2$ , where  $\tilde{\chi}_1$  (resp.  $\tilde{\chi}_2\chi_{\text{cyc}}^{m+1}$ ) is the character associated to  $\psi_1$  (resp.  $\psi_2|\cdot|^{m+1}$ ) by class field theory. The theorem thus follows as before on reduction mod  $\pi$ . ■

**Corollary 10.7.2.** *Let  $f \in M_{k,l}(U_1(\mathfrak{n}); E)$  be a normalised eigenform and  $v_0$  a prime of  $F$  over  $p$ . If  $T_{v_0}f_\xi \neq 0$  for some character  $\xi$  of weight  $-l$ , then (possibly after enlarging  $E$  and semisimplifying  $\rho_f$ )*

$$\rho_f|_{G_{F_{v_0}}} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

<sup>10</sup>Permitting  $m = 0$  would allow the possibility that  $\Pi_{v_0}$  be an unramified twist of the Steinberg representation, which could anyway have been treated similarly.

for some characters  $\chi_1, \chi_2 : G_{F_{v_0}} \rightarrow E^\times$  such that  $\chi_1|_{I_{F_{v_0}}} = \prod_{\tau \in \Sigma_{v_0}} \epsilon_\tau^{-l_\tau}$  and  $\chi_2|_{I_{F_{v_0}}} = \prod_{\tau \in \Sigma_{v_0}} \epsilon_\tau^{1-k_\tau-l_\tau}$ .

*Proof.* Since  $\rho_{f_\xi} \simeq \rho_f \otimes \rho_{\xi'}$  and  $\rho_{\xi'}|_{I_{F_{v_0}}} = \prod_{\tau \in \Sigma_{v_0}} \epsilon_\tau^{l_\tau}$ , we may reduce to the case  $l = 0$  and  $f = f_\xi$ , which is immediate from Theorem 10.7.1. ■

### 11. The inert quadratic case

We now specialise to the inert quadratic case, with a focus on non-algebraic weights, and in particular the case of “partial weight 1” since it exhibits phenomena not present in the classical case. We provide evidence and an approach to Conjectures 7.3.1 and 7.3.2 by deducing results in this setting from ones in the case of algebraic weights.

#### 11.1. Notation

For the rest of the paper, we let  $F$  be a real quadratic field in which  $p$  is inert, and we let  $\mathfrak{p} = p\mathcal{O}_F$  and  $K = F_{\mathfrak{p}}$ , so  $K$  is the unramified quadratic extension of  $\mathbb{Q}_p$ . Fix an embedding  $\tau_0 : F \rightarrow \overline{\mathbb{Q}}_p$  and write  $\Sigma = \{\tau_0, \tau_1\}$ . We identify  $\Sigma$  with  $\Sigma_K$  and hence with the set of embeddings  $\mathcal{O}_F/p\mathcal{O}_F \rightarrow \overline{\mathbb{F}}_p$ . We shall write weights  $k \in \mathbb{Z}^\Sigma$  in the form  $(k_0, k_1)$  where  $k_i = k_{\tau_i}$  for  $i = 0, 1$ . Recall that our conventions for Hodge–Tate types and weights of crystalline lifts of two-dimensional representations are given in §7.2.

#### 11.2. $p$ -adic Hodge theory lemmas

Let  $\chi : G_K \rightarrow \overline{\mathbb{F}}_p^\times$  be a character such that  $\chi|_{I_K} = \epsilon_{\tau_0}^i$  with  $1 \leq i \leq p - 1$ . Then the  $\overline{\mathbb{F}}_p$ -vector space  $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$  is two-dimensional, and we recall from [5, §3] the definition of a certain one-dimensional subspace. Note that  $\chi|_{I_K} = \epsilon_{\tau_0}^{i-1} \epsilon_{\tau_1}^p$ , so  $\chi$  has a crystalline lift  $\tilde{\chi} : G_K \rightarrow \mathcal{O}^\times$  with Hodge–Tate type  $(1 - i, -p) \in \mathbb{Z}^\Sigma$  (where  $\mathcal{O}$  is assumed to be sufficiently large that  $\chi$  takes values in  $E^\times$ ). Such lifts are unique up to twist by unramified characters with trivial reduction, and we choose<sup>11</sup> the one such that if  $g$  corresponds via local class field theory to  $p \in K^\times$ , then  $\tilde{\chi}(g)$  is the Teichmüller lift of  $\chi(g)$ . A standard argument shows that the space  $H_f^1(G_K, L(\tilde{\chi}))$  classifying crystalline extensions is one-dimensional over  $L$ , with pre-image  $V_{\tilde{\chi}} \subset H^1(G_K, \mathcal{O}(\tilde{\chi}))$  free of rank 1 over  $\mathcal{O}$ . We then define  $V_\chi = V_{\tilde{\chi}} \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ . Similarly,  $\chi$  has a crystalline lift  $\tilde{\chi}' : G_K \rightarrow \mathcal{O}^\times$  with Hodge–Tate type  $(-i, 0)$ , unique up to unramified twist, and we choose the one sending  $g$  (corresponding to  $p$ ) to the Teichmüller lift of  $\chi(g)$ . We again find that  $H_f^1(G_K, L(\tilde{\chi}'))$  is one-dimensional, with pre-image  $V_{\tilde{\chi}'}$  free of rank 1 over  $\mathcal{O}$ , and we define  $V'_\chi = V_{\tilde{\chi}'} \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$ .

<sup>11</sup>By [12, Remark 7.13], or more generally [34, proof of Thm. 9.1], the subspace  $V_\chi$  turns out to be independent of the choice of unramified twist, but we fix it for clarity and consistency with [5]. Similarly the proof of Lemma 11.2.1 below shows the same holds for  $V'_\chi$ .

**Lemma 11.2.1.** *With the above notation,  $V_\chi = V'_\chi$ .*

*Proof.* We use the description of  $V_\chi$  obtained in [12] together with a similar analysis of  $V'_\chi$ . All references in this proof are to [12].

In the notation of [12], the  $(\phi, \Gamma)$ -module corresponding to  $E(\chi)$  (the one-dimensional  $E$ -vector space  $E(\chi)$  equipped with  $G_K$ -action by  $\chi$ ) has the form  $M_{C\bar{c}}$  with

$$\bar{c} = (p - 1 - i, p - 1),$$

and Proposition 5.11 (for  $p > 2$ ), Proposition 6.11 (for  $p = 2$ ) and Theorem 7.12 show that  $V_\chi$  is the subspace corresponding to the span of the class  $[B_0] \in \text{Ext}^1(M_{\bar{0}}, M_{C\bar{c}})$ .

We may analyze  $V'_\chi$  similarly as follows. We can write the  $(\phi, \Gamma)$ -module corresponding to  $E(\chi^{-1})$  in the form  $M_{A\bar{a}}$  where  $\bar{a} = (i, 0)$  and  $A = C^{-1}$ , and consider the subspace of bounded extensions

$$\text{Ext}^1_{\text{bdd}}(M_{A\bar{a}}, M_{\bar{0}}) \subset \text{Ext}^1(M_{A\bar{a}}, M_{\bar{0}})$$

defined exactly in Definition 5.1 (dropping the assumption that one of  $a_i$  or  $b_i$  is non-zero for each  $i$ ). As in §5.1, we have an isomorphism

$$\iota : \text{Ext}^1(M_{A\bar{a}}, M_{\bar{0}}) \simeq \text{Ext}^1(M_{\bar{0}}, M_{C\bar{c}}).$$

A straightforward adaptation of part (2) of the proof of Proposition 5.11<sup>12</sup> then shows that the image of  $\text{Ext}^1_{\text{bdd}}(M_{A\bar{a}}, M_{\bar{0}})$  under  $\iota$  is again spanned by  $[B_0]$ .

By the same argument as in the proof of Theorem 7.8 (with the appeal to Lemma 7.6 replaced by a direct application of Proposition 7.4), one finds that  $V'_\chi$  is contained in (the extension of scalars to  $\overline{\mathbb{F}}_p$  of) the image of  $\text{Ext}^1_{\text{bdd}}(M_{A\bar{a}}, M_{\bar{0}})$ . Therefore  $V'_\chi \subset V_\chi$ , and equality follows on comparing dimensions. ■

**Remark 11.2.2.** In [62, Chapter 5], Wiersema gives an alternative proof of the preceding lemma by first generalising methods and results of [34, 35] to the setting of two-dimensional crystalline representations of  $G_K$  whose  $\tau$ -labelled weights  $w_{1,\tau}, w_{2,\tau}$  need not be distinct. One can then appeal to those results instead of the ones in [12] (see [62, Lemma 5.3.2]). Similarly the following lemma can be proved using those results instead of Fontaine–Laffaille theory (see [62, Lemma 5.4.7]).

**Lemma 11.2.3.** *Suppose that  $2 \leq k_0 \leq p$ . A representation  $\sigma : G_K \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$  if and only if either*

- $\sigma \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$  with  $\chi_1$  unramified,  $\chi_2|_{I_K} = \epsilon_{\tau_0}^{1-k_0}$  and associated extension class in  $V_{\chi_1\chi_2^{-1}}$ , or
- $\sigma \simeq \text{Ind}_{G_{K'}}^{G_K} \xi$  where  $K'$  is the unramified quadratic extension of  $K$  and  $\xi|_{I_{K'}} = \epsilon_{\tau'_0}^{1-k_0}$  for some extension  $\tau'_0$  of  $\tau_0$  to the residue field of  $K'$ .

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<sup>12</sup>Strictly speaking, this is Proposition 6.11 in the case  $p = 2$ , but the proof there is omitted since it is essentially the same as that of Proposition 5.11, using the cocycles constructed in §6.3.



*Proof.* For the “if” direction, in the first case, let  $\tilde{\chi}_1$  be an unramified lift of  $\chi_1$  and let  $\tilde{\chi}_2 = \tilde{\chi}_1(\tilde{\chi}')^{-1}$ , where  $\chi = \chi_1\chi_2^{-1}$ . By Lemma 11.2.1 and the definition of  $V'_{\chi}$ , the representation  $\chi_2^{-1} \otimes \sigma$  is isomorphic to the reduction of an  $\mathcal{O}[G_K]$ -module  $T$  associated to an extension class

$$0 \rightarrow \mathcal{O}(\tilde{\chi}') \rightarrow T \rightarrow \mathcal{O} \rightarrow 0$$

such that  $T \otimes_{\mathcal{O}} L$  is crystalline. It follows that  $\sigma$  has a crystalline lift  $T \otimes_{\mathcal{O}} L(\tilde{\chi}_2)$  with  $\tau_0$ -labelled weights  $(k_0 - 1, 0)$  and  $\tau_1$ -labelled weights  $(0, 0)$ , as required.

In the second case, note that the character  $\xi$  has a crystalline lift  $\tilde{\xi}$  of Hodge–Tate type  $(k_0 - 1, 0, 0, 0)$  (where the first coordinate corresponds to  $\tau'_0 \in \Sigma_{K'}$ ), so that  $\text{Ind}_{G_{K'}}^{G_K} \tilde{\xi}$  is a crystalline lift of  $\sigma$  with the required labelled weights.

The other direction can be proved as follows using Fontaine–Laffaille theory. We follow the notation of [28]; in particular, see §0.9 for the definition of the category  $\underline{\text{MF}}_{\text{tor}}^{f, p'}$ , but we take (their)  $E$  to be  $\mathbb{Q}_p$  and consider objects with an action of (our)  $E$ . The results of [28, §§7–8] imply that there is an object  $M$  of the category  $\underline{\text{MF}}_{\text{tor}}^{f, p'}$  and an embedding  $E \rightarrow \text{End}(M)$  (for large enough  $E$ ) such that

$$\sigma \simeq \text{Hom}_E(\underline{U}_S(M), \overline{\mathbb{F}}_p)$$

as representations of  $G_K$ ; moreover, decomposing  $M = M_0 \oplus M_1$  (according to the idempotents of  $O_K \otimes E$  corresponding to  $\tau_0, \tau_1$ ), each component is two-dimensional over  $E$  and

$$\text{Fil}^j M = \begin{cases} M & \text{if } j \leq 0, \\ Ex_0 & \text{if } 0 < j < k_0, \\ 0 & \text{if } j \geq k_0, \end{cases}$$

for some non-zero  $x_0 \in M_0$ .

We claim that bases  $(x_i, y_i)$  for  $M_i$  over  $E$  can be chosen so that  $\text{Fil}^1 M = Ex_0$  as above and the  $O_K \otimes E$ -linear morphisms  $\phi^j : \text{Fr}^* \text{Fil}^j M \rightarrow M$  are defined so that  $\phi^{k_0-1}x_0 = x_1, \phi^0y_0 = y_1$ , and one of the following holds:

- $\phi^0x_1 = ax_0 + by_0, \phi^0y_1 = cy_0$  for some  $a, c \in E^\times, b \in \{0, 1\}$ ,
- $\phi^0x_1 = y_0, \phi^0y_1 = ax_0$  for some  $a \in E^\times$ .

Indeed, choose any basis  $x_0$  for  $\text{Fil}^1 M$  and let  $x_1 = \phi^{k_0-1}x_0$ . If  $\phi^0x_1 \in \text{Fil}^1 M$ , then we are in the first case with  $b = 0$ . Otherwise let  $z_0 = \phi^0x_1, y_1 = \phi^0z_0$ , write  $\phi^0y_1 = \alpha x_0 + \beta z_0$  for some  $\alpha, \beta \in E$ , and note that  $\alpha \neq 0$ . If  $\beta = 0$ , then let  $y_0 = z_0$ , giving the second case; otherwise let  $y_0 = z_0 + \beta^{-1}\alpha x_0$ , giving the first case with  $b = 1$ .

In the first case,  $M$  is reducible (as an object of  $\underline{\text{MF}}_{\text{tor}}^{f, p'}$  with  $E$ -action), fitting in an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ , where  $M' = Ey_0 \oplus Ey_1$ . It follows that  $\sigma$  has the form  $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$  where  $\chi_1$  (resp.  $\chi_2$ ) is obtained by applying the functor  $\text{Hom}_E(\underline{U}_S(\cdot), \overline{\mathbb{F}}_p)$  to  $M'$  (resp.  $M''$ ). Moreover,  $\chi_1$  (resp.  $\chi_2$ ) has a crystalline lift of Hodge–Tate type  $(0, 0)$  (resp.  $(k_0 - 1, 0)$ ) and the subspace of  $H^1(G_K, \overline{\mathbb{F}}_p(\chi_1\chi_2^{-1}))$  obtained from such extensions is one-dimensional. Therefore  $\chi_1$  is

unramified,  $\chi_2|_{I_K} = \epsilon_{\tau_0}^{1-k_0}$ , and since the subspace must contain  $V'_{\chi_1\chi_2^{-1}} = V_{\chi_1\chi_2^{-1}}$ , these subspaces in fact coincide, so  $\sigma$  has the required form.

In the second case, consider  $\sigma|_{G_{K'}}$ , which (in view of the compatibility noted at the end of §3 of [28]) is isomorphic to  $\text{Hom}_E(\underline{U}'_S(M'), \overline{\mathbb{F}}_p)$ , where  $M' = M \otimes_{O_K} O_{K'}$  and  $\underline{U}'_S$  is the Fontaine–Laffaille functor on the category  $\underline{\text{MF}}_{\text{tor}}^{f,p'}$  defined with  $K$  replaced by  $K'$ . Assuming  $E$  is chosen sufficiently large (in particular containing the residue field of  $K'$ ), we may decompose  $M' = \bigoplus M'_i$  according to the embeddings  $\tau'_i = \text{Fr}^i \circ \tau'_0$  for  $i = 0, 1, 2, 3$ , where  $\tau'_0$  is a choice of extension of  $\tau_0$ . Writing  $x'_i, x'_{i+2}$  (resp.  $y'_i, y'_{i+2}$ ) for the image of  $x_i \otimes 1$  (resp.  $y_i \otimes 1$ ) in the corresponding component, observe that  $M'$  decomposes as

$$(Ex'_0 \oplus Ey'_1 \oplus Ey'_2 \oplus Ex'_3) \oplus (Ey'_0 \oplus Ex'_1 \oplus Ex'_2 \oplus Ey'_3).$$

It follows that  $\sigma|_{G_{K'}} \simeq \xi \oplus \xi'$  where the character  $\xi$  has a crystalline lift of Hodge–Tate type  $(k_0 - 1, 0, 0, 0)$ , so that  $\xi|_{I_{K'}} = \epsilon_{\tau'_0}^{1-k_0}$  (note that similarly  $\xi' = \epsilon_{\tau'_2}^{1-k_0}$ ) and  $\sigma$  has the required form. ■

**Remark 11.2.4.** For completeness, note that  $\sigma$  has a crystalline lift of weight  $((1, 1), (0, 0))$  if and only if it is unramified.

**Remark 11.2.5.** The non-semisimple representations of  $G_K$  occurring in the statement of the lemma are precisely those which are gently (but not tamely) ramified in the terminology of [17, §3.3]. This is a special case of [17, Conjecture 7.2], proved in [7].

**Lemma 11.2.6.** *Suppose that  $2 \leq k_0 \leq p$ . A representation  $\sigma : G_K \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$  if and only if all of the following hold:*

- (1)  $\sigma$  has a crystalline lift of weight  $((k_0 - 1, p + 1), (0, 0))$  if  $k_0 > 2$ , and of weight  $((p + 1, p), (0, 0))$  if  $k_0 = 2$ ;
- (2)  $\sigma$  has a crystalline lift of weight  $((k_0 + 1, p + 1), (-1, 0))$ ;
- (3)  $\sigma$  is not of the form  $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$  where  $\chi_1|_{I_K} = \epsilon_{\tau_0}$ .

*Proof.* We firstly prove the “only if” direction. Suppose that  $\sigma$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ .

First consider the case that  $\sigma$  is reducible, so by Lemma 11.2.3, it is an unramified twist of a representation of the form  $\begin{pmatrix} 1 & * \\ 0 & \chi^{-1} \end{pmatrix}$  with  $\chi|_{I_K} = \epsilon_{\tau_0}^{k_0-1}$  and the associated extension class in  $V_\chi$ .

For (1), note that  $\chi|_{I_K} = \epsilon_{\tau_0}^{k_0-2} \epsilon_{\tau_1}^p$  (resp.  $\epsilon_{\tau_0}^p \epsilon_{\tau_1}^{p-1}$ ) if  $k_0 > 2$  (resp.  $k_0 = 2$ ), so that  $\chi$  has a crystalline lift  $\tilde{\chi}''$  of Hodge–Tate type  $(2 - k_0, -p)$  (resp.  $(-p, 1 - p)$ ). Since  $H_f^1(G_K, L(\tilde{\chi}'')) = H^1(G_K, L(\tilde{\chi}''))$  and the map  $H^1(G_K, \mathcal{O}(\tilde{\chi}'')) \rightarrow H^1(G_K, E(\chi))$  is surjective, it follows as in the proof of Lemma 11.2.3 that  $\sigma$  has a crystalline lift of the required weight.

For (2), we instead write  $\chi|_{I_K} = \epsilon_{\tau_0}^{k_0} \epsilon_{\tau_1}^{-p}$  and use the lift  $\tilde{\chi}$  in the definition of  $V_\chi$ . Since the extension class associated to  $\sigma$  lies in  $V_\chi$ , it follows that  $\sigma$  has a crystalline lift with  $\tau_0$ -labelled weights  $(k_0, 0)$  and  $\tau_1$ -labelled weights  $(0, -p)$ . Twisting by a crystalline

character of Hodge–Tate type  $(-1, p)$  and trivial reduction, we conclude that  $\sigma$  has a crystalline lift of the required weight.

Finally, (3) is clear since  $\epsilon_{\tau_0}$  is not  $\epsilon_{\tau_0}^{1-k_0}$  or the trivial character.

Now suppose that  $\sigma$  is irreducible, so  $\sigma \simeq \text{Ind}_{G_{K'}}^{G_K} \xi$  where  $\xi|_{I_{K'}} = \epsilon_{\tau'_0}^{1-k_0}$  for some extension  $\tau'_0$  of  $\tau_0$ . Writing  $\xi|_{I'_K} = \epsilon_{\tau'_0}^{2-k_0} \epsilon_{\tau'_3}^{-p}$  (resp.  $\epsilon_{\tau'_2}^{-p} \epsilon_{\tau'_3}^{1-p}$ ) if  $k_0 > 2$  (resp.  $k_0 = 2$ ), we see that  $\xi$  has a lift  $\tilde{\xi}$  of Hodge–Tate type  $(k_0 - 2, 0, 0, p)$  (resp.  $(0, 0, p, p - 1)$ ), and  $\text{Ind}_{G_{K'}}^{G_K} \tilde{\xi}$  is a crystalline lift of  $\sigma$  of the required weight for 1).

For (2), we proceed similarly by writing  $\xi|_{I_{K'}} = \epsilon_{\tau'_0}^{1-k_0} \epsilon_{\tau'_1}^{-p} \epsilon_{\tau'_2}$  to see that  $\xi$  has a crystalline lift of Hodge–Tate type  $(k_0 - 1, p, -1, 0)$  whose induction to  $G_K$  has the required weight.

Finally, (3) is clear since  $\sigma$  is irreducible.

We now prove the “if direction”. Suppose that (1)–(3) all hold. We will use the results of [34] and their extension to  $p = 2$  in [61], which show that if  $\sigma$  has a crystalline lift of weight  $(k, l)$  with  $2 \leq k_\tau \leq p + 1$  for all  $\tau$ , then  $\sigma$  is of the form prescribed in [5] for the corresponding Serre weight (i.e.,  $W^{\text{cris}}(\sigma) \subset W^{\text{explicit}}(\sigma)$ ) in the terminology of [35], but note that the conventions for Hodge–Tate weights in [34, 35] are opposite to ours).

First suppose that  $\sigma$  is reducible and write  $\sigma \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ . From condition (1) and [34, Thm. 9.1] (extended to  $p = 2$  in [61]), it follows that if  $k_0 > 2$ , then  $\sigma$  is isomorphic to the reduction of a lattice in a crystalline representation of the form

$$\begin{pmatrix} \tilde{\chi}_1 & * \\ 0 & \tilde{\chi}_2 \end{pmatrix},$$

where  $\tilde{\chi}_1$  and  $\tilde{\chi}_2$  are of Hodge–Tate types  $(k_0 - 2, p)$  and  $(0, 0)$  (in either order) or of Hodge–Tate types  $(k_0 - 2, 0)$  and  $(0, p)$  (again in either order). Furthermore, in the first case, if  $\tilde{\chi}_1$  has Hodge–Tate type  $(k_0 - 2, p)$ , then the representation is necessarily decomposable, so we may exchange  $\tilde{\chi}_1$  and  $\tilde{\chi}_2$ . We therefore conclude that  $\sigma|_{I_K}$  is of the form

$$\begin{pmatrix} 1 & * \\ 0 & \epsilon_{\tau_0}^{1-k_0} \end{pmatrix}, \begin{pmatrix} \epsilon_{\tau_0}^{-1} & * \\ 0 & \epsilon_{\tau_0}^{2-k_0} \end{pmatrix} \text{ or } \begin{pmatrix} \epsilon_{\tau_0}^{2-k_0} & * \\ 0 & \epsilon_{\tau_0}^{-1} \end{pmatrix}.$$

If  $k_0 = 2$ , then the same reasoning shows that  $\sigma|_{I_K}$  is of the form

$$\begin{pmatrix} 1 & * \\ 0 & \epsilon_{\tau_0}^{-1} \end{pmatrix}, \begin{pmatrix} \epsilon_{\tau_0}^{p-1} & * \\ 0 & \epsilon_{\tau_0}^{-p} \end{pmatrix} \text{ or } \begin{pmatrix} \epsilon_{\tau_0}^{-p} & * \\ 0 & \epsilon_{\tau_0}^{p-1} \end{pmatrix}.$$

Similarly, from condition (2), we find that  $\sigma|_{I_K}$  is of the form

$$\begin{pmatrix} \epsilon_{\tau_0} & * \\ 0 & \epsilon_{\tau_0}^{-k_0} \end{pmatrix}, \begin{pmatrix} 1 & * \\ 0 & \epsilon_{\tau_0}^{1-k_0} \end{pmatrix} \text{ or } \begin{pmatrix} \epsilon_{\tau_0}^{1-k_0} & * \\ 0 & 1 \end{pmatrix}.$$

Moreover, in the second case, the associated extension class lies in  $V_\chi$  (where we exchange  $\chi_1$  and  $\chi_2$  if necessary if  $\sigma$  splits, and use the fact that  $V_\chi$  is independent of the

choice of unramified twist in its definition). That  $\sigma$  has the required form is then immediate on comparing the possibilities resulting from (1) and (2), taking (3) into account in the case  $k_0 = p = 2$ , and applying Lemma 11.2.3.

Finally, suppose that  $\sigma$  is irreducible. Then condition (1) and [34, Thm. 10.1] (extended to  $p = 2$ ) imply that  $\sigma \simeq \text{Ind}_{G_{K'}}^{G_K} \xi$  for some  $\xi$  with  $\xi|_{I_{K'}}$  of the form  $\epsilon_{\tau'_0}^{2-k_0} \epsilon_{\tau'_3}^{-p} = \epsilon_{\tau'_0}^{1-k_0}$  (if the balanced subset  $J$  in [34, Thm. 10.1] is  $\{\tau'_0, \tau'_3\}$  or its complement) or  $\epsilon_{\tau'_0}^{2-k_0} \epsilon_{\tau'_1}^{-p} = \epsilon_{\tau'_0}^{2-k_0-p^2}$  (if  $J = \{\tau'_0, \tau'_1\}$  or its complement), with the latter possibility replaced by  $\epsilon_{\tau'_0}^{p-p^2-p^3}$  if  $k_0 = 2$ . Similarly condition (2) implies that  $\sigma \simeq \text{Ind}_{G_{K'}}^{G_K} \xi'$  for some  $\xi'$  with  $\xi'|_{I_{K'}}$  of the form  $\epsilon_{\tau'_0}^{1-k_0}$  or  $\epsilon_{\tau'_0}^{p^2-k_0}$ . Since neither  $\epsilon_{\tau'_0}^{p^2-k_0}$  nor its conjugate  $\epsilon_{\tau'_0}^{1-p^2k_0}$  agrees with any of the possibilities resulting from condition (1), we deduce that  $\xi|_{I_{K'}} = \epsilon_{\tau'_0}^{1-k_0}$ , and the desired conclusion again follows from Lemma 11.2.3. ■

**Remark 11.2.7.** Note that we only needed to use condition (3) in the case  $k_0 = p = 2$ , so it is otherwise implied by (1) and (2).

### 11.3. Weight shifting

We now prove an analogue of Lemma 11.2.6 in the context of geometric modularity. (See [62, Chapter 4] for generalisations to the setting of arbitrary totally real  $F$  in which  $p$  is unramified.)

**Lemma 11.3.1.** *Suppose that  $2 \leq k_0 \leq p$  and that  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible. If  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ , then*

- (1)  $\rho$  is geometrically modular of weight  $((k_0 - 1, p + 1), (0, 0))$  if  $k_0 > 2$ , and of weight  $((p + 1, p), (0, 0))$  if  $k_0 = 2$ , and
- (2)  $\rho$  is geometrically modular of weight  $((k_0 + 1, p + 1), (-1, 0))$ .

Moreover, the converse holds if we assume in addition that

- (3)  $\rho|_{G_K}$  is not of the form  $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$  where  $\chi_1|_{I_K} = \epsilon_{\tau_0}$ .

*Proof.* Suppose first that  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ , i.e.  $\rho$  is equivalent to  $\rho_f$  for some eigenform  $f \in M_{(k_0,1),(0,0)}(U; E)$ . Multiplying  $f$  by  $\text{Ha}_{\tau_0}$  (resp.  $\text{Ha}_{\tau_0} \text{Ha}_{\tau_1}$ ) if  $k_0 > 2$  (resp.  $k_0 = 2$ ) yields an eigenform giving rise to  $\rho$  of the weight required for (1). Conclusion (2) is immediate from Theorem 10.4.2.

Conversely, suppose (1)–(3) all hold. First consider the case  $k_0 > 2$ . By Proposition 10.5.2, hypotheses 1) and 2) imply that  $\rho$  arises from normalised eigenforms in  $M_{(k_0-1,p+1),(0,0)}(U_1(\mathfrak{m}_1); E)$  and  $M_{(k_0+1,p+1),(-1,0)}(U_1(\mathfrak{m}_2); E)$  for some ideals  $\mathfrak{m}_1, \mathfrak{m}_2$  prime to  $p$  (and sufficiently large  $E$ ). We may then choose  $\mathfrak{n}$  satisfying the conditions in Lemma 10.6.2 with  $\mathfrak{n} = \mathfrak{m}_i$  for  $i = 1, 2$  (for example take  $\mathfrak{n} = \mathfrak{m}_1^2 \mathfrak{m}_2^2$ ) to deduce that  $\rho$  arises from stabilised eigenforms in  $f_1 \in M_{(k_0-1,p+1),(0,0)}(U_1(\mathfrak{n}); E)$  and

$f_2 \in M_{(k_0+1,p+1),(-1,0)}(U_1(\mathfrak{n}); E)$ . By Proposition 9.4.1,  $\Theta_{\tau_0}(f_1)$  is a strongly stabilised eigenform in  $M_{(k_0,2p+1),(-1,0)}(U_1(\mathfrak{n}); E)$ . By Corollary 10.7.2 and hypothesis (3), so is  $\text{Ha}_{\tau_0} f_2$ . Lemma 10.6.5 then implies that  $\Theta_{\tau_0}(f_1) = \text{Ha}_{\tau_0} f_2$ , and now it follows from Theorem 8.2.2 that  $f_1 = \text{Ha}_{\tau_0} f$  for some  $f \in M_{(k_0,1),(0,0)}(U_1(\mathfrak{n}); E)$ , so  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ .

The case  $k_0 = 2$  is similar, but instead one has  $f_1 \in M_{(p+1,p),(0,0)}(U_1(\mathfrak{n}); E)$ , and obtains  $f \in M_{(p+2,0),(0,0)}(U_1(\mathfrak{n}); E)$ . Theorem 1.1 of [19] now implies that  $f$  is divisible by  $\text{Ha}_{\tau_1}$ , so that  $\rho$  is geometrically modular of weight  $((2, 1), (0, 0))$ . ■

11.4. Geometric modularity in partial weight 1

**Theorem 11.4.1.** *Suppose that  $2 \leq k_0 \leq p$  and that  $\rho : G_F \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible and modular. Suppose that [5, Conjecture 3.14] and Conjecture 7.5.2 hold for  $\rho$ . Then  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$  if and only if  $\rho|_{G_K}$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ .*

*Proof.* Suppose first that  $\rho|_{G_K}$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ . Then Lemma 11.2.6 implies that  $\rho|_{G_K}$  has crystalline lifts of weight  $((k_0 - 1, p + 1), (0, 0))$  (resp.  $((p + 1, p), (0, 0))$ ) if  $k_0 > 2$  (resp.  $k_0 = 2$ ) and  $((k_0 + 1, p + 1), (-1, 0))$ , and that  $\rho|_{G_K}$  has no subrepresentation on which  $I_K$  acts as  $\epsilon_{\tau_0}$ . Conjecture 3.14 of [5] then implies that  $\rho$  is algebraically modular of the two indicated weights, and then Conjecture 7.5.2 implies it is geometrically modular of those weights. It then follows from Lemma 11.3.1 that  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ .

Now suppose that  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ . Reversing the argument above then shows that  $\rho|_{G_K}$  has crystalline lifts of weight  $((k_0 - 1, p + 1), (0, 0))$  (resp.  $((p + 1, p), (0, 0))$ ) if  $k_0 > 2$  (resp.  $k_0 = 2$ ) and  $((k_0 + 1, p + 1), (-1, 0))$ . If  $p > 2$ , then as noted in Remark 11.2.7, this already implies that  $\rho|_{G_K}$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ . To conclude, we can assume  $k_0 = p = 2$ , and we just need to rule out the possibility that  $\rho|_{G_K} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$  where  $\chi_1|_{I_K} = \epsilon_{\tau_0}$ . We do this by an ad hoc argument.

It is more convenient to work with  $\rho' = \rho \otimes \rho_{\xi'}$ , where  $\xi$  is a character of weight  $(1, 0)$ . Then  $\rho'$  is geometrically modular of weight  $((2, 1), (1, 0))$ , Conjecture 7.5.2 holds as well for  $\rho'$ , and we assume for the sake of contradiction that  $\rho'|_{G_K}$  has an unramified subrepresentation. We let  $v_0 = 2O_F$ .

By Lemma 10.2.1,  $\rho$  arises from an eigenform  $f_0 \in M_{(2,1),(1,0)}(U_1(\mathfrak{m}_1); E)$  for some  $\mathfrak{m}_1$  and  $E$ . Moreover, by Lemma 10.4.1, we can assume that  $\Theta_{\tau_1}(f_0) \neq 0$ , i.e.  $r_m^J(f_0) \neq 0$  for some  $m, J$  such that  $m \notin 2J^{-1}$ . The same argument as in the proof of Proposition 10.5.2 then shows that we may assume  $f_0$  satisfies the first two conditions in the definition of a normalised eigenform. (With regard to the third condition, note that we have not defined  $T_{v_0}$  in this context.) Therefore  $\Theta_{\tau_0}(f_0)$  is a normalised eigenform in  $M_{(3,3),(0,0)}(U_1(\mathfrak{m}_1); E)$ ; note that it is an eigenform for  $T_{v_0}$  since  $r_m^J(\Theta_{\tau_1}(f_0)) = 0$  if  $m \in 2J^{-1}$ . By Theorem 8.2.2, we have  $v(\Theta_{\tau_0}(f_0)) \leq_{\text{Ha}} (4, 1)$  (where the notation is as in §5), and [19, Thm. 1.1] then implies that  $v(\Theta_{\tau_0}(f_0)) \leq_{\text{Ha}} (2, 2)$ . We may therefore

write  $\Theta_{\tau_0}(f_0) = \text{Ha}_{\tau_0} \text{Ha}_{\tau_1} f_1$  for a normalised eigenform  $f_1 \in M_{(2,2),(0,0)}(U_1(\mathfrak{m}_1); E)$  with  $r_m^J(f_1) = 0$  for all  $m \in 2J^{-1}$ .

We have shown in particular that  $\rho'$  is geometrically modular of weight  $((2, 2), (0, 0))$ , hence algebraically modular of weight  $((2, 2), (0, 0))$  by our supposition of Conjecture 7.5.2. Therefore (for example by [5, Prop. 2.5]),  $\rho' \simeq \bar{\rho}_{\tilde{f}}$  for a characteristic zero eigenform  $\tilde{f}$  of weight  $((2, 2), (0, 0))$ ; we may further assume that  $\tilde{f}$  is a newform in  $M_{(2,2),(0,0)}(U_1(\mathfrak{m}_2); \mathcal{O})$  for some  $\mathfrak{m}_2$ , enlarging  $L$  if necessary, so it is a normalised eigenform for  $T_v$  for all primes  $v$ , and for  $S_v$  for all  $v \nmid \mathfrak{m}_2$ . By local-global compatibility,  $\rho_{\tilde{f}}|_{G_K}$  is crystalline with  $\tau_i$ -labelled weights  $(1, 0)$  for  $i = 0, 1$ , and the characteristic polynomial of  $\phi^2$  on  $D_{\text{cris}}(\rho_{\tilde{f}}|_{G_K})$  is  $X^2 - \tilde{a}X + 4\tilde{d}$  where  $\tilde{a}$  is the eigenvalue of  $T_{v_0}$  on  $\tilde{f}$  and  $\tilde{d} \in \mathcal{O}^\times$  is the eigenvalue of  $S_{v_0}$ . Using for example that  $\rho_{\tilde{f}}|_{G_K}$  is dual to a representation arising from a 2-divisible group over  $O_K$ , we see from the form of  $\rho'|_{G_K}$  that

$$\rho_{\tilde{f}}|_{G_K} \simeq \begin{pmatrix} \tilde{\chi}_1 & * \\ 0 & \tilde{\chi}_2 \end{pmatrix}$$

with  $\tilde{\chi}_1$  unramified and  $\tilde{\chi}_1(\text{Frob}_{v_0}) = \tilde{a} \in \mathcal{O}^\times$  (and  $\chi_{\text{cyc}}\tilde{\chi}_2$  is unramified with  $\chi_{\text{cyc}}\tilde{\chi}_2(\text{Frob}_{v_0}) = \tilde{a}^{-1}\tilde{d}$ ). The reduction of  $\tilde{f}$  is thus a normalised eigenform  $f_2 \in M_{(2,2),(0,0)}(U_1(\mathfrak{m}_2); E)$  giving rise to  $\rho'$ , with the property that the eigenvalue of  $T_{v_0}$  on  $f_2$  is non-zero.

As in the proof of Lemma 11.3.1, we can choose  $\mathfrak{n}$  so that the conditions in Lemma 10.6.2 are satisfied for  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$ , and the proof of the lemma then yields eigenforms  $g_1, g_2 \in M_{(2,2),(0,0)}(U_1(\mathfrak{n}); E)$  such that  $g_1$  is strongly stabilised, whereas  $g_2$  is stabilised and satisfies  $T_{v_0}g_2 = ag_2$  for some  $a \in E^\times$ . Now consider the form  $f_3 = a^{-1}(g_2 - g_1)$ ; its  $q$ -expansion coefficients are given by  $r_m^J(f_3) = 0$  unless  $m \in 2J_+^{-1}$ , in which case  $r_m^J(f_3) = r_m^J(g_2)$ . In particular,  $f_3 \in \ker(\Theta_{\tau_0})$ , so  $f_3 = \Phi_{v_0}(g_3)$  for some  $g_3 \in M_{(1,1),(0,0)}(U_1(\mathfrak{n}); E)$ . By Proposition 9.8.1, we have  $r_m^J(g_3) = r_m^J(g_2)$  for all  $m, J$ , so  $g_2 = \text{Ha}_{\tau_0} \text{Ha}_{\tau_1} g_3$ .

Furthermore, note that  $\nu(g_3) = (1, 1)$ ; otherwise [19, Corollary 1.2] would force  $\nu(g_3) = (0, 0)$ , making  $g_3$  locally constant and contradicting the irreducibility of  $\rho$ . Now consider  $\Theta_{\tau_1}(g_3) \in M_{(3,2),(0,-1)}(U_1(\mathfrak{n}); E)$ . By Theorem 8.2.2,  $\Theta_{\tau_1}(g_3)$  is not divisible by  $\text{Ha}_{\tau_1}$ . We claim that  $\Theta_{\tau_1}(g_3)$  is not divisible by  $\text{Ha}_{\tau_0}$  either. Indeed, if it were, then we would have  $\nu(\Theta_{\tau_1}(g_3)) \leq_{\text{Ha}} (4, 0)$ , and [19, Thm. 1.1] would imply divisibility by  $\text{Ha}_{\tau_1}$ . Therefore Theorem 8.2.2 implies that  $\Theta_{\tau_0}\Theta_{\tau_1}(g_3)$  is not divisible by  $\text{Ha}_{\tau_0}$  (and in fact a similar argument gives  $\nu(\Theta_{\tau_0}\Theta_{\tau_1}(g_3)) = (4, 4)$ ). Note that  $\Theta_{\tau_0}\Theta_{\tau_1}(g_3) \in M_{(4,4),(-1,-1)}(U_1(\mathfrak{n}); E)$  is a strongly stabilised eigenform giving rise to  $\rho$ . However so is  $e_1 \text{Ha}_{\tau_0}^2 \text{Ha}_{\tau_1}^2 g_1$ , where  $e_1$  is the constant section in  $M_{(0,0),(-1,-1)}(U_1(\mathfrak{n}); E)$  with value 1. We therefore conclude that  $\Theta_{\tau_0}\Theta_{\tau_1}(g_3) = e_1 \text{Ha}_{\tau_0}^2 \text{Ha}_{\tau_1}^2 g_1$  is divisible by  $\text{Ha}_{\tau_0}$ , yielding the desired contradiction. ■

**Remark 11.4.2.** Note that the theorem holds just as well for weights of the form  $((k_0, 1), l)$  and  $((1, k_0), l)$  for any  $l \in \mathbb{Z}^\Sigma$ .

Recall from Proposition 7.5.4 that one direction of Conjecture 7.5.2 holds if  $k$  is paritious in the sense of Definition 3.2.1. Recall also that [5, Conjecture 3.14] has been proved under mild technical hypotheses by Gee and collaborators (see especially [33, 34]), with an alternative to part due to Newton [50]. In particular, it holds under the assumptions that  $p > 2$ ,  $\rho|_{G_F(\zeta_p)}$  is irreducible, and if  $p = 5$ , then  $\rho|_{G_F(\zeta_5)}$  does not have projective image isomorphic to  $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_5)$ . It might be possible to treat this exceptional case with  $p = 5$  using the same methods along with [3, Thm. 3.2.1], but we only need to do this in a particular instance in order to obtain one direction of Theorem 11.4.1 for odd  $k_0$  unconditionally.

**Theorem 11.4.3.** *Suppose that  $3 \leq k_0 \leq p$ ,  $k_0$  is odd and  $\rho : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)$  is irreducible and modular. If  $\rho|_{G_K}$  has a crystalline lift of weight  $((k_0, 1), (0, 0))$ , then  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ .*

*Proof.* We first show that the local condition at  $p$  implies that  $\rho|_{G_F(\zeta_p)}$  is irreducible. Indeed, if it is not, then  $\rho$  is induced from  $G_{F'}$  for a quadratic extension  $F'/F$  which is ramified at  $p$ , and hence  $\rho|_{G_K}$  is induced from  $G_{K'}$  for a ramified quadratic extension  $K'/K$ . This in turns implies that  $\rho|_{I_K} \simeq \chi_1 \oplus \chi_2$  for some characters  $\chi_1, \chi_2$  such that  $\chi_1\chi_2^{-1}$  is quadratic. However, the explicit description of the possibilities for  $\rho|_{I_K}$  from Lemma 11.2.3 shows that  $\chi_1\chi_2^{-1}$  would have the form  $\epsilon_{\tau_0}^{\pm(k_0-1)}$  or  $\epsilon_{\tau_0'}^{\pm(k_0-1)(p^2-1)}$ , which gives a contradiction since such a character has order  $(p^2 - 1)/i$  or  $(p^2 + 1)/i$  for some  $i \leq p - 1$ .

We may therefore apply [35, Thm. A] to conclude that  $\rho$  is algebraically modular of weights  $((k_0 - 1, p + 1), (0, 0))$  and  $((k_0 + 1, p + 1), (-1, 0))$ , unless  $p = 5$  and  $\rho|_{G_F(\zeta_5)}$  has projective image isomorphic to  $\mathrm{PSL}_2(\mathbb{F}_5)$ . Aside from this exceptional case, it follows from Proposition 7.5.4 that  $\rho$  is geometrically modular of weights  $((k_0 - 1, p + 1), (0, 0))$  and  $((k_0 + 1, p + 1), (-1, 0))$ , and then from Lemma 11.3.1 that  $\rho$  is geometrically modular of weight  $((k_0, 1), (0, 0))$ .

Suppose then that  $p = 5$  and  $\rho|_{G_F(\zeta_5)}$  has projective image isomorphic to  $\mathrm{PSL}_2(\mathbb{F}_5)$ , so that of  $\rho$  is isomorphic to  $\mathrm{PSL}_2(\mathbb{F}_5)$  or  $\mathrm{PGL}_2(\mathbb{F}_5)$ . Again using the explicit descriptions in Lemma 11.2.3, we see this is only possible if  $k_0 = 5$  and  $\rho|_{G_K} \simeq \chi_1 \oplus \chi_2$  where  $\chi_1$  is unramified and  $\chi_2|_{I_K} = \epsilon_{\tau_0}^{-4}$  has order 6. In this case the conjectural set of Serre weights for  $\rho^\vee = \mathrm{Hom}_{\overline{\mathbb{F}}_5}(\rho, \overline{\mathbb{F}}_5)$  (with the notation of §7) is

$$\{V_{(4,6),(0,0)}, V_{(2,2),(-1,0)}, V_{(6,6),(-1,0)}, V_{(6,4),(4,0)}\}.$$

In particular, if  $\xi$  is a character of weight  $(1, 0)$ , then  $(\rho \otimes \rho_{\xi'})^\vee|_{G_K}$  has a Barsotti–Tate lift (necessarily non-ordinary), and the argument of [30, §3.1] (using the method of Khare–Wintenberger [43]) then shows that  $(\rho \otimes \rho_{\xi'})^\vee$  is modular of weight  $V_{(2,2),(0,0)}$ , from which it follows that  $\rho$  is algebraically modular of weight  $((2, 2), (-1, 0))$ .

Similarly  $\rho^\vee|_{G_K}$  has a potentially Barsotti–Tate lift of type  $[\epsilon_{\tau_0}^2 \epsilon_{\tau_1}^4] \oplus 1$ , so the same argument (but now using [3, Thm. 3.2.1] for the existence of ordinary lifts) shows that  $\rho^\vee$  is modular of some weight in the set of Jordan–Hölder constituents  $\mathrm{Ind}_B^{\mathrm{GL}_2(O_F/p)} \psi$

where

$$\psi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \tau_0(a)^2 \tau_1(a)^4,$$

namely  $\{V_{(4,6),(0,0)}, V_{(3,5),(3,0)}, V_{(4,2),(2,4)}\}$ . Therefore  $\rho$  is algebraically modular of weight  $((4, 6), (0, 0))$ ,  $((3, 5), (3, 0))$  or  $((4, 2), (2, 4))$ . Since  $\rho|_{G_K}$  has no crystalline lifts of weight  $((3, 5), (3, 0))$  or  $((4, 2), (2, 4))$  (by [34, Thm. 2.12], but in fact already by [28]), these possibilities are ruled out by local-global compatibility and the discussion before Proposition 7.5.4. Therefore  $\rho$  is algebraically modular of weight  $((4, 6), (0, 0))$ .

We have now shown that  $\rho$  is algebraically modular of weights  $((4, 6), (0, 0))$  and  $((2, 2), (-1, 0))$ , so also geometrically modular of these weights by Proposition 7.5.4. Therefore  $\rho$  is also geometrically modular of weight  $((6, 6), (-1, 0))$ , and it follows from Lemma 11.3.1 that  $\rho$  is geometrically modular of weight  $((5, 1), (0, 0))$ , as required. ■

**Remark 11.4.4.** Again the theorem holds also for weights of the form  $((k_0, 1), l)$  and  $((1, k_0), l)$  for any  $l \in \mathbb{Z}^\Sigma$ .

**Remark 11.4.5.** We remark that the assumption that  $F$  is unramified at  $p$  ensures that the conditions at  $p = 5$  in the modularity lifting theorems of [3, 29, 47] are satisfied. If  $p = 5$  is ramified in  $F$ , then a more exceptional case can arise, and is treated in work of Khare and Thorne [45].

### 11.5. An example

Consider the Galois representation defined in [17, §9, Example IIIb<sub>1</sub>], so  $F = \mathbb{Q}(\sqrt{5})$ ,  $p = 3$  and  $\rho : G_F \rightarrow \mathrm{GL}_2(\mathbb{F}_9)$  is absolutely irreducible and has the property that  $\rho|_{G_K} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_0 \end{pmatrix}$  where  $\chi_i|_{I_K} = \epsilon_{\tau_i}$  for appropriately chosen  $\tau_i : O_F/p \simeq \mathbb{F}_9$ . Setting  $\chi = \chi_1 \chi_0^{-1}$ , we have  $\chi|_{I_K} = \epsilon_{\tau_0}^{-2}$ , and the discussion in [17] shows that the associated extension class lies in the line  $V_\chi$  of Lemma 11.2.1. It follows that  $\rho$  has a crystalline lift of weight  $((3, 1), (0, -1))$ .

The modularity of  $\rho$  is strongly indicated by the data exhibited in [17, §10.4]. In particular, it follows from the explicit computations described there that there is an eigenform  $f \in M_{(2,4),(0,-1)}(U_1(\mathfrak{n}); \mathbb{F}_9)$  with  $\mathfrak{n} = (10\sqrt{5})$  whose eigenvalue for  $T_v$  coincides with  $\mathrm{tr}\rho(\mathrm{Frob}_v)$  for all  $v \nmid 30$  such that  $\mathrm{Nm}_{F/\mathbb{Q}}(v) < 100$ , and whose eigenvalue for  $S_v$  is  $1 = \mathrm{Nm}_{F/\mathbb{Q}}(v)^{-1} \det \rho(\mathrm{Frob}_v)$  for all  $v \nmid 30$ . We assume for the rest of the discussion that it is indeed the case that  $\rho_f \simeq \rho$ . It then follows from Theorem 11.4.1 that  $\rho$  is geometrically modular of weight  $((3, 1), (0, -1))$ , i.e.,  $\rho \simeq \rho_g$  for some eigenform  $g \in M_{(3,1),(0,-1)}(U_1(\mathfrak{n}); \mathbb{F}_9)$

Consider also the form  $g_\xi$  for a character  $\xi$  of conductor  $(\sqrt{5})$  and weight  $(0, 2)$ , in the sense of Definition 10.3.1. (There are two such characters, both of order 4, differing by the quadratic character corresponding to the extension  $F(\mu_5)$ .) Then we have  $g_\xi \in M_{(3,1),(0,1)}(U_1(\mathfrak{n}); \mathbb{F}_9)$ , and as the weight  $((3, 1), (0, 1))$  is paritious (in the sense of Definition 3.2.1), it is natural to ask whether  $g_\xi$  lifts to a characteristic zero eigenform of partial weight 1.



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## References

- [1] Andreatta, F., Goren, E. Z.: Hilbert modular forms: mod  $p$  and  $p$ -adic aspects. *Mem. Amer. Math. Soc.* **173**, no. 819, vi+100 pp. (2005) Zbl [1071.11023](#) MR [2110225](#)
- [2] Ash, A., Sinnott, W.: An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod  $p$  cohomology of  $GL(n, \mathbf{Z})$ . *Duke Math. J.* **105**, 1–24 (2000) Zbl [1015.11018](#) MR [1788040](#)
- [3] Barnet-Lamb, T., Gee, T., Geraghty, D.: Congruences between Hilbert modular forms: constructing ordinary lifts, II. *Math. Res. Lett.* **20**, 67–72 (2013) Zbl [1298.11038](#) MR [3126722](#)
- [4] Breuil, C., Mézard, A.: Multiplicités modulaires et représentations de  $GL_2(\mathbf{Z}_p)$  et de  $\text{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$  en  $l = p$ . *Duke Math. J.* **115**, 205–310 (2002) Zbl [1042.11030](#) MR [1944572](#)
- [5] Buzzard, K., Diamond, F., Jarvis, F.: On Serre's conjecture for mod  $\ell$  Galois representations over totally real fields. *Duke Math. J.* **155**, 105–161 (2010) Zbl [1227.11070](#) MR [2730374](#)
- [6] Calegari, F.: Is Serre's conjecture still open? Blogpost, <https://galoisrepresentations.wordpress.com/2014/08/10/is-serres-conjecture-still-open>
- [7] Calegari, F., Emerton, M., Gee, T., Mavrides, L.: Explicit Serre weights for two-dimensional Galois representations. *Compos. Math.* **153**, 1893–1907 (2017) Zbl [1417.11096](#) MR [3705280](#)
- [8] Calegari, F., Geraghty, D.: Modularity lifting beyond the Taylor–Wiles method. *Invent. Math.* **211**, 297–433 (2018) Zbl [1476.11078](#) MR [3742760](#)
- [9] Carayol, H.: Sur les représentations  $l$ -adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)* **19**, 409–468 (1986) Zbl [0616.10025](#) MR [870690](#)
- [10] Chai, C.-L.: Compactification of Siegel Moduli Schemes. *London Math. Soc. Lecture Note Ser.* 107, Cambridge Univ. Press, Cambridge (1985) Zbl [0578.14009](#) MR [853543](#)
- [11] Chai, C.-L.: Arithmetic minimal compactification of the Hilbert–Blumenthal moduli spaces. *Ann. of Math. (2)* **131**, 541–554 (1990) Zbl [0754.14030](#) MR [1053489](#)
- [12] Chang, S., Diamond, F.: Extensions of rank one  $(\phi, \Gamma)$ -modules and crystalline representations. *Compos. Math.* **147**, 375–427 (2011) Zbl [1235.11105](#) MR [2776609](#)
- [13] Coleman, R. F., Voloch, J. F.: Companion forms and Kodaira–Spencer theory. *Invent. Math.* **110**, 263–281 (1992) Zbl [0770.11024](#) MR [1185584](#)
- [14] Conrad, B.: Lifting global representations with local properties. Preprint, <http://math.stanford.edu/~conrad/papers/locchar.pdf>

- [15] Conrad, B.: Grothendieck Duality and Base Change. Lecture Notes in Math. 1750, Springer, Berlin (2000) Zbl [0992.14001](#) MR [1804902](#)
- [16] Deligne, P., Pappas, G.: Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. Compos. Math. **90**, 59–79 (1994) Zbl [0826.14027](#) MR [1266495](#)
- [17] Dembélé, L., Diamond, F., Roberts, D. P.: Serre weights and wild ramification in two-dimensional Galois representations. Forum Math. Sigma **4**, art. e33, 49 pp. (2016) Zbl [1417.11099](#) MR [3589336](#)
- [18] Diamond, F.: Geometric weight-shifting operators on Hilbert modular forms in characteristic  $p$ . J. Inst. Math. Jussieu (online, 2021)
- [19] Diamond, F., Kassaei, P. L.: Minimal weights of Hilbert modular forms in characteristic  $p$ . Compos. Math. **153**, 1769–1778 (2017) Zbl [1391.11073](#) MR [3668413](#)
- [20] Diamond, F., Kassaei, P. L., Sasaki, S.: A mod  $p$  Jacquet–Langlands relation and Serre filtration via the geometry of Hilbert modular varieties: Splicing and dicing. Astérisque, to appear
- [21] Dimitrov, M.: Compactifications arithmétiques des variétés de Hilbert et formes modulaires de Hilbert pour  $\Gamma_1(c, n)$ . In: Geometric Aspects of Dwork Theory, Vol. I, II, de Gruyter, Berlin, 527–554 (2004) Zbl [1076.14029](#) MR [2099078](#)
- [22] Dimitrov, M., Tilouine, J.: Variétés et formes modulaires de Hilbert arithmétiques pour  $\Gamma_1(c, n)$ . In: Geometric Aspects of Dwork Theory, Vol. I, II, de Gruyter, Berlin, 555–614 (2004) Zbl [1127.11036](#) MR [2099080](#)
- [23] Dimitrov, M., Wiese, G.: Unramifiedness of Galois representations attached to Hilbert modular forms mod  $p$  of weight 1. J. Inst. Math. Jussieu **19**, 281–306 (2020) Zbl [1437.11081](#) MR [4079146](#)
- [24] Edixhoven, B.: The weight in Serre’s conjectures on modular forms. Invent. Math. **109**, 563–594 (1992) Zbl [0777.11013](#) MR [1176206](#)
- [25] Emerton, M., Reduzzi, D., Xiao, L.: Unramifiedness of Galois representations arising from Hilbert modular surfaces. Forum Math. Sigma **5**, art. e29, 70 pp. (2017) Zbl [1433.11070](#) MR [3725733](#)
- [26] Emerton, M., Reduzzi, D. A., Xiao, L.: Galois representations and torsion in the coherent cohomology of Hilbert modular varieties. J. Reine Angew. Math. **726**, 93–127 (2017) Zbl [1430.11081](#) MR [3641654](#)
- [27] Fontaine, J.-M.: Le corps des périodes  $p$ -adiques. Astérisque **223**, 59–111 (1994) Zbl [0940.14012](#) MR [1293971](#)
- [28] Fontaine, J.-M., Laffaille, G.: Construction de représentations  $p$ -adiques. Ann. Sci. École Norm. Sup. (4) **15**, 547–608 (1982) Zbl [0579.14037](#) MR [707328](#)
- [29] Gee, T.: A modularity lifting theorem for weight two Hilbert modular forms. Math. Res. Lett. **13**, 805–811 (2006) Zbl [1185.11030](#) MR [2280776](#)
- [30] Gee, T.: Automorphic lifts of prescribed types. Math. Ann. **350**, 107–144 (2011) Zbl [1276.11085](#) MR [2785764](#)
- [31] Gee, T., Herzig, F., Savitt, D.: General Serre weight conjectures. J. Eur. Math. Soc. **20**, 2859–2949 (2018) Zbl [1456.11093](#) MR [3871496](#)
- [32] Gee, T., Kassaei, P.: Companion forms in parallel weight one. Compos. Math. **149**, 903–913 (2013) Zbl [1329.11039](#) MR [3077656](#)
- [33] Gee, T., Kisin, M.: The Breuil–Mézard conjecture for potentially Barsotti–Tate representations. Forum Math. Pi **2**, art. e1, 56 pp. (2014) Zbl [1408.11033](#) MR [3292675](#)
- [34] Gee, T., Liu, T., Savitt, D.: The Buzzard–Diamond–Jarvis conjecture for unitary groups. J. Amer. Math. Soc. **27**, 389–435 (2014) Zbl [1288.11045](#) MR [3164985](#)
- [35] Gee, T., Liu, T., Savitt, D.: The weight part of Serre’s conjecture for  $GL(2)$ . Forum Math. Pi **3**, art. e2, 52 pp. (2015) Zbl [1317.11045](#) MR [3324938](#)
- [36] Goldring, W., Koskivirta, J.-S.: Strata Hasse invariants, Hecke algebras and Galois representations. Invent. Math. **217**, 887–984 (2019) Zbl [1431.11071](#) MR [3989256](#)

- [37] Gross, B. H.: A tameness criterion for Galois representations associated to modular forms (mod  $p$ ). *Duke Math. J.* **61**, 445–517 (1990) Zbl [0743.11030](#) MR [1074305](#)
- [38] Grothendieck, A.: *Revêtements étales et groupe fondamental* (SGA 1). *Lecture Notes in Math.* 224, Springer (1971) Zbl [0234.14002](#) MR [0217087](#)
- [39] Illusie, L.: Déformations de groupes de Barsotti–Tate (d’après A. Grothendieck). *Astérisque* **127**, 151–198 (1985) Zbl [1182.14050](#) MR [801922](#)
- [40] Katz, N. M.: A result on modular forms in characteristic  $p$ . In: *Modular Functions of One Variable, V* (Bonn, 1976), *Lecture Notes in Math.* 601, Springer, Berlin, 53–61 (1977) Zbl [0392.10026](#) MR [0463169](#)
- [41] Katz, N. M.:  $p$ -adic  $L$ -functions for CM fields. *Invent. Math.* **49**, 199–297 (1978) Zbl [0417.12003](#) MR [513095](#)
- [42] Katz, N. M., Mazur, B.: *Arithmetic Moduli of Elliptic Curves*. *Ann. of Math. Stud.* 108, Princeton Univ. Press, Princeton, NJ (1985) Zbl [0576.14026](#) MR [772569](#)
- [43] Khare, C., Wintenberger, J.-P.: Serre’s modularity conjecture. I. *Invent. Math.* **178**, 485–504 (2009) Zbl [1304.11041](#) MR [2551763](#)
- [44] Khare, C., Wintenberger, J.-P.: Serre’s modularity conjecture. II. *Invent. Math.* **178**, 505–586 (2009) Zbl [1304.11041](#) MR [2551764](#)
- [45] Khare, C. B., Thorne, J. A.: Automorphy of some residually  $S_5$  Galois representations. *Math. Z.* **286**, 399–429 (2017) Zbl [1420.11081](#) MR [3648503](#)
- [46] Kisin, M.: Modularity of 2-adic Barsotti–Tate representations. *Invent. Math.* **178**, 587–634 (2009) Zbl [1304.11043](#) MR [2551765](#)
- [47] Kisin, M.: Moduli of finite flat group schemes, and modularity. *Ann. of Math. (2)* **170**, 1085–1180 (2009) Zbl [1201.14034](#) MR [2600871](#)
- [48] Koblitz, N.:  $p$ -adic variation of the zeta-function over families of varieties defined over finite fields. *Compos. Math.* **31**, 119–218 (1975) Zbl [0332.14008](#) MR [414557](#)
- [49] Lan, K.-W.: Higher Koecher’s principle. *Math. Res. Lett.* **23**, 163–199 (2016) Zbl [1416.11097](#) MR [3512882](#)
- [50] Newton, J.: Serre weights and Shimura curves. *Proc. London Math. Soc. (3)* **108**, 1471–1500 (2014) Zbl [1334.11047](#) MR [3218316](#)
- [51] Pappas, G.: Arithmetic models for Hilbert modular varieties. *Compos. Math.* **98**, 43–76 (1995) Zbl [0890.14010](#) MR [1353285](#)
- [52] Pappas, G., Rapoport, M.: Local models in the ramified case. II. Splitting models. *Duke Math. J.* **127**, 193–250 (2005) Zbl [1126.14028](#) MR [2130412](#)
- [53] Rapoport, M.: Compactifications de l’espace de modules de Hilbert–Blumenthal. *Compos. Math.* **36**, 255–335 (1978) Zbl [0386.14006](#) MR [515050](#)
- [54] Sasaki, S.: Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representations, and weight one forms. *Invent. Math.* **215**, 171–264 (2019) Zbl [1443.11113](#) MR [3904451](#)
- [55] Schein, M. M.: Weights in Serre’s conjecture for Hilbert modular forms: the ramified case. *Israel J. Math.* **166**, 369–391 (2008) Zbl [1197.11063](#) MR [2430440](#)
- [56] Serre, J.-P.: Sur les représentations modulaires de degré 2 de  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ . *Duke Math. J.* **54**, 179–230 (1987) Zbl [0641.10026](#) MR [885783](#)
- [57] Skinner, C.: A note on the  $p$ -adic Galois representations attached to Hilbert modular forms. *Doc. Math.* **14**, 241–258 (2009) Zbl [1177.11045](#) MR [2538615](#)
- [58] Taylor, R.: On Galois representations associated to Hilbert modular forms. *Invent. Math.* **98**, 265–280 (1989) Zbl [0705.11031](#) MR [1016264](#)
- [59] Vignéras, M.-F.: Représentations modulaires de  $\text{GL}(2, F)$  en caractéristique  $l$ ,  $F$  corps  $p$ -adique,  $p \neq l$ . *Compos. Math.* **72**, 33–66 (1989) Zbl [0706.22014](#) MR [1026328](#)
- [60] Vignéras, M.-F.: Représentations  $l$ -modulaires d’un groupe réductif  $p$ -adique avec  $l \neq p$ . *Progr. Math.* 137, Birkhäuser Boston, Boston, MA (1996) Zbl [0859.22001](#) MR [1395151](#)

- 
- [61] Wang, X.: Weight elimination in two dimensions when  $p = 2$ . arXiv:[1711.09035](https://arxiv.org/abs/1711.09035) (2017)
- [62] Wiersema, H.: Minimal weights of mod  $p$  Galois representations. Ph.D. thesis, King's College London (2021)