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# Isometries of quadratic spaces

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**Abstract.** Let $k$ be a global field of characteristic not 2, and let $f \in k[X]$ be an irreducible polynomial. We show that a non-degenerate quadratic space has an isometry with minimal polynomial $f$ if and only if such an isometry exists over all the completions of $k$. This gives a partial answer to a question of Milnor.

**Keywords.** Quadratic space, isometry, orthogonal group, minimal polynomial

## Introduction

Let $k$ be a field of characteristic not 2. A *quadratic space* is a non-degenerate symmetric bilinear form $q : V \times V \to k$ defined on a finite-dimensional $k$-vector space $V$, and an *isometry* of $(V, q)$ is an element of $O(q)$, in other words an isomorphism $t : V \to V$ such that $q(tx, ty) = q(x, y)$ for all $x, y \in V$. In [M69], Milnor raised the following question:

**Question 1.** Which quadratic spaces admit an isometry with a given irreducible minimal polynomial?

The case of local fields is covered in [M69], and the present paper gives an answer to Milnor's question for global fields.

Let $q$ be a quadratic space, and let $f \in k[X]$ be an irreducible polynomial. The following Hasse principle is proved in Section 9 (Th. 9.1):

**Theorem.** *Suppose that $k$ is a global field. The quadratic space $q$ has an isometry with minimal polynomial $f$ if and only if such an isometry exists over all the completions of $k$.*

In order to obtain a necessary and sufficient criterion, we need to consider the case of reducible minimal polynomials over local fields and the field of real numbers. This leads to a generalization of the above question. Note that any endomorphism $t : V \to V$ gives rise to a torsion $k[X]$-module. We ask the following:

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**Question 2.** Which quadratic spaces admit an isometry with a given torsion module?

Note that this covers several special cases of interest:

- The "vertical" case: if $M = [k[X]/(f)]^m$ where $f \in k[X]$ is an irreducible polynomial and $m \in \mathbb{N}$, then Question 2 is precisely the question of Milnor mentioned above.
- The "horizontal" case: if $M = k[X]/(f_1 \dots f_r)$ with $f_i \in k[X]$ distinct irreducible polynomials, then Question 2 amounts to asking which orthogonal groups contain a maximal torus of a given type (see for instance [BCM03], [G04], [PR10], [GR13], [F12], [Lee14], [B14]).
- The case of "rational knot modules" (see for instance [Le80]).

Integral analogs of this question arise in connection with algebraic-geometric and arithmetic applications (cf. Gross–McMullen [GM02] and [BMa13]).

Most of the results concern fields of cohomological dimension 1, local and global fields; let us illustrate them by a few examples. Let $M$ be a self-dual torsion $k[X]$-module with characteristic polynomial $F_M \in k[X]$ (see §2), and suppose that $\dim(q) = \dim_k(M)$. We shall see that it is sufficient to answer the question in the case of semisimple modules (cf. Prop. 4.1). We have (see Cor. 6.3):

**Proposition.** *Suppose that $k$ is a field of cohomological dimension $\leq 1$, and $M$ is semisimple. Then the quadratic space $q$ has an isometry with module $M$ if and only if* $\det(q)F_M(1)F_M(-1) \in k^2$.
[Note that if $F_M(1)F_M(-1) = 0$, then this means that any quadratic space $q$ has an isometry with module $M$, provided $\dim(q) = \dim_k(M)$.]

In the case of global fields, we give an answer to Milnor's question. Suppose that $f \in k[X]$ is an irreducible, monic polynomial such that $f(X) = X^{\deg(f)} f(X^{-1})$ and that $f \neq X + 1$. Let $m \in \mathbb{N}$ and set $F = f^m$. Assume that $\dim(q) = \deg(F)$. Then we have (see Cor. 9.2):

**Theorem.** *Suppose that $k$ is a global field. The quadratic space $q$ has an isometry with minimal polynomial $f$ if and only if the signature condition and the hyperbolicity condition are satisfied* (see §9)*, and* $\det(q) = F(1)F(-1)$ *in* $k^*/k^{*2}$.

The paper is structured as follows. The first three sections contain some definitions and basic facts, including some results of Milnor [M69]. Sections 4 and 5 are concerned with isometries with a given module over an arbitrary ground field, and are used throughout the paper. The following sections treat the case of fields of cohomological dimension 1 (§6), local fields (§7), the field of real numbers (§8), and global fields (§9–§12).

## 1. Quadratic spaces, isometries and symmetric polynomials

Let $k$ be a field of characteristic not 2. A *quadratic space* is a pair $(V, q)$, where $V$ is a finite-dimensional $k$-vector space, and $q : V \times V \to k$ is a symmetric bilinear form of non-zero determinant. The *determinant* of $(V, q)$ is denoted by $\det(q)$. Let

$n = \dim(V)$. The *discriminant* of $q$ is by definition $\operatorname{disc}(q) = (-1)^{(n-1)n/2}\det(q)$. Any quadratic space can be diagonalized, in other words there exist $a_1, \dots, a_n \in k^*$ such that $q \simeq \langle a_1, \dots, a_n \rangle$. Let us denote by $\mathrm{Br}(k)$ the Brauer group of $k$, considered as an additive abelian group, and let $\mathrm{Br}_2(k)$ be the subgroup of elements of order $\le 2$ of $\mathrm{Br}(k)$. The *Hasse invariant* of $q$ is by definition $w(q) = \sum_{i<j}(a_i, a_j) \in \mathrm{Br}_2(k)$, where $(a_i, a_j)$ is the class of the quaternion algebra over $k$ determined by $a_i, a_j$. For more information concerning basic results on quadratic spaces, see for instance [O'M73] and [Sch85].

An *isometry* of a quadratic space $(V, q)$ is an isomorphism $t : V \to V$ such that $q(tx, ty) = q(x, y)$ for all $x, y \in V$.

A monic polynomial $f \in k[X]$ is said to be $\epsilon$-*symmetric* for some $\epsilon = \pm 1$ if $f(X) = \epsilon X^{\deg(f)} f(X^{-1})$. We say that $f$ is *symmetric* if it is 1-symmetric. The following is well-known:

**Proposition 1.1.** *The minimal polynomials and characteristic polynomials of isometries of quadratic spaces are $\epsilon$-symmetric, where $\epsilon$ is the constant term of the polynomial.*

*Proof.* Let $(V, q)$ be a quadratic space, and let $t : V \to V$ be an isometry of $q$. By definition, we have $q(tx, y) = q(x, t^{-1}y)$ for all $x, y \in V$. This implies that for any polynomial $p \in k[X]$, we have $q(p(t)x, y) = q(x, p(t^{-1})y)$ for all $x, y \in V$. Let $f \in k[X]$ be the minimal polynomial of $t$. Applying the above equality to $p = f$, we see that the endomorphism $t^{\deg(f)} f(t^{-1})$ annihilates $V$. As $f$ is the minimal polynomial of $t$, this implies that $f$ divides $X^{\deg(f)} f(X^{-1})$, therefore we have $f(X) = \epsilon X^{\deg(f)} f(X^{-1})$ for some $\epsilon = \pm 1$. On the other hand, the coefficient of the leading term of $X^{\deg(f)} f(X^{-1})$ is equal to $f(0)$. Therefore $\epsilon = f(0)$. The statement concerning the characteristic polynomial follows from a straightforward computation (see for instance [Le69, Lemma 7(a)]). □

If $f \in k[X]$ is a monic polynomial such that $f(0) \neq 0$, set

$$f^*(X) = \frac{1}{f(0)} X^{\deg(f)} f(X^{-1}).$$

Note that $f^*$ is also monic, $f^*(0) \neq 0$, and $f^{**} = f$.

**Definition 1.2.** Let $f \in k[X]$ be a monic, $\epsilon$-symmetric polynomial with $\epsilon = \pm 1$. We say that $f$ is of

- *type* 0 if $f$ is a product of powers of $X - 1$ and of $X + 1$;
- *type* 1 if $f$ is a product of powers of monic, symmetric, irreducible polynomials in $k[X]$ of even degree;
- *type* 2 if $f$ is a product of polynomials of the form $gg^*$, where $g \in k[X]$ is monic, irreducible, and $g \neq \pm g^*$.

**Proposition 1.3.** *Every monic, $\epsilon$-symmetric polynomial $F \in k[X]$ is a product of polynomials of type* 0*,* 1 *and* 2*.*

*Proof.* Let $f \in k[X]$ be a monic, irreducible factor of $F$. If $f \neq \pm f^*$, then $f^*$ also divides $F$, hence we get a factor of type 2. Suppose that $f = \pm f^*$. It suffices to show that if $f(X) \neq X-1, X+1$, then $\deg(f)$ is even and $\epsilon = 1$. We have $f(X) = \epsilon X^{\deg f} f(X^{-1})$ for some $\epsilon = \pm 1$. If $\epsilon = -1$, then $f(1) = 0$, hence $f$ is divisible by $X - 1$, and this is impossible as $f$ is supposed to be irreducible and $f(X) \neq X - 1$. Hence $\epsilon = 1$. If $\deg(f)$ is odd, then this implies that $f(-1) = 0$, which contradicts the assumption that $f$ is irreducible and $f(X) \neq X + 1$. Therefore $\deg(f)$ is even. □

We say that a monic, $\epsilon$-symmetric polynomial is *hyperbolic* if all its components of type 0 and 1 are of the form $f^e$ with $e$ even.

## 2. Self-dual torsion modules

Let $V$ be a finite-dimensional $k$-vector space, and let $t : V \to V$ be an endomorphism. Then $V$ has a structure of torsion $k[X]$-module obtained by setting $X.v = t(v)$ for all $v \in V$. Let us denote by $M(t)$ the torsion $k[X]$-module associated to the endomorphism $t$. The module $M(t)$ will be called the *module of the endomorphism* $t$.

Any torsion $k[X]$-module is isomorphic to a direct sum of modules of the form $[k[X]/(f)]^m$ for some $f \in k[X]$ and $m \in \mathbb{N}$. If $M$ is a torsion $k[X]$-module, set $F_M = \prod f^m$ for all $f \in k[X]$ and $m \in \mathbb{N}$ as above. We call $F_M$ the *characteristic polynomial* of $M$. Note that if $M = M(t)$ for some endomorphism $t$, then $F_M$ is the characteristic polynomial of $t$.

A torsion $k[X]$-module is said to be of *type* $i$, for $i = 0, 1, 2$, if $M$ is a direct sum of modules of the form $[k[X]/(f)]^m$ where $f \in k[X]$ is of type $i$ and $m \in \mathbb{N}$. It is said to be *self-dual* if it is a direct sum of modules of type 0, 1 and 2.

From now on, *module* will mean a self-dual torsion $k[X]$-module that is finite-dimensional as a $k$-vector space.

A module is said to be *hyperbolic* if all its components of type 0 and type 1 are of the form $[k[X]/(f^e)]^n$ with $e$ even. We will see that any quadratic space having an isometry with hyperbolic module is hyperbolic.

## 3. Primary decomposition and transfer

The aim of this section is to recall some results of Milnor [M69]. Let $(V, q)$ be a quadratic space of dimension $2n$, let $t$ be an isometry of $q$ and let $F$ be the characteristic polynomial of $t$. For each monic, irreducible factor $f$ of $F$, set

$$V_f = \{v \in V \mid f^i(t)(v) = 0 \text{ for some } i \in \mathbb{N}\}.$$

Let $U$ and $W$ be two subspaces of $V$. We say that $U$ and $W$ are *orthogonal* to each other if $q(u, w) = 0$ for all $u \in U$ and $w \in W$. We say that $(V, q)$ is *hyperbolic* if $V$ has a self-orthogonal subspace of dimension $n$.

**Proposition 3.1.** *Let $f$ and $g$ be two monic, irreducible factors of $F$. If $f \neq g^*$, then $V_f$ and $V_g$ are orthogonal to each other.*

*Proof.* See Milnor [M69, Lemma 3.1]. □

**Corollary 3.2.** *If $f$ is not symmetric, then $(V_f \oplus V_{f^*}, q)$ is hyperbolic.*

*Proof.* See [M69, §3, Case 3]. □

**Proposition 3.3.** *We have the orthogonal decomposition*

$$(V, q) \simeq \bigoplus (V_f, q) \oplus H$$

*where the sum is taken over all distinct monic, symmetric and irreducible factors of F, and H is a hyperbolic space. The orthogonal factors are stable by the isometry.*

*Proof.* This follows from Prop. 3.1 and Cor. 3.2. □

**Proposition 3.4.** *Let $f \in k[X]$ be monic, symmetric and irreducible. The quadratic space $(V_f, q)$ decomposes as an orthogonal sum of factors, each having an isometry with module*

$$[k[X]/(f^e)]^m$$

*for some integers $e$ and $m$. If $e$ is even, then this orthogonal factor is hyperbolic. If $e$ is odd, then it is the orthogonal sum of a hyperbolic space and a quadratic space having an isometry with module $[k[X]/(f)]^m$. Conversely, let $Q$ be a quadratic space having an isometry with module $[k[X]/(f)]^m$ and let $e \in \mathbb{N}$. Then there exists a quadratic space having an isometry with module $[k[X]/(f^e)]^m$ that admits $Q$ as an orthogonal summand. Moreover, this quadratic space is the orthogonal sum of $Q$ and of a hyperbolic space if $e$ is odd, and it is hyperbolic if $e$ is even.*

*Proof.* See [M69, Ths. 3.2–3.4]. □

Recall that *module* means a self-dual torsion $k[X]$-module which is finite-dimensional as a $k$-vector space, and see §2 for the definition of a hyperbolic module.

**Corollary 3.5.** *A quadratic space having an isometry with hyperbolic module is hyperbolic.*

*Proof.* This follows from Props. 3.3 and 3.4. □

**Proposition 3.6.** *Let $f \in k[X]$ be a monic, symmetric and irreducible polynomial, and set $K = k[X]/(f)$. Then sending $X$ to $X^{-1}$ induces a $k$-linear involution of $K$ denoted by $x \mapsto \overline{x}$. Let $\ell : K \to k$ be a non-trivial $k$-linear map such that $\ell(x) = \ell(\overline{x})$ for all $x \in K$. Then for every quadratic space $(V, q)$ over $k$ and every isometry having minimal polynomial $f$, there exists a non-degenerate hermitian form $(V, h)$ over $K$ such that for all $x, y \in V$, we have*

$$q(x, y) = \ell(h(x, y)).$$

*Conversely, if $V$ is a finite-dimensional vector space over $K$ and if $h : V \to V$ is a non-degenerate hermitian form, then setting $q(x, y) = \ell(h(x, y))$ for all $x, y \in V$ we obtain a quadratic space $(V, q)$ over $k$ together with an isometry with minimal polynomial $f$.*

*Proof.* This is proved in [M69, Lemmas 1.1 and 1.2] in the case where $f$ is separable and $\ell = \mathrm{Tr}_{K/k}$ is the trace of the extension $K/k$. The proof is the same for any non-trivial linear map $\ell$ with $\ell(x) = \ell(\overline{x})$ for all $x \in K$, as pointed out in [M69, Remark 1.4]. □

**Corollary 3.7.** *Let $M$ be a module. Then there exists a quadratic space $q$ having an isometry $t$ such that $M(t) \simeq M$.*

*Proof.* This follows from Props. 3.3, 3.4 and 3.6. □

This implies the following well-known fact:

**Corollary 3.8.** *Let $F \in k[X]$ be a monic, $\epsilon$-symmetric polynomial. Then there exists a quadratic space having an isometry with characteristic polynomial $F$.*

Note that a new proof of this result, based on Bezoutians, is given in [JRV14, Th. 4.1]. Integral analogs of this question are investigated in [JRV14, §3], as well as in [B84], [BMar94] and [B99].

## 4. Isometries with a given module

We keep the notation of the previous sections. In particular, *module* means a self-dual torsion $k[X]$-module which is a finite-dimensional $k$-vector space.

The aim of this paper is to investigate the following question:

**Question.** Which quadratic spaces admit an isometry with a given module?

This is a generalization of Milnor's question quoted in the introduction. Let us fix some notation. For any module $M$, we have

$$M = M^0 \oplus M^1 \oplus M^2,$$

with $M^i$ of type $i$. Note that $\dim(M^2)$ is even, and let $2m_2 = \dim(M^2)$. Let us write $M^0 = M_+ \oplus M_-$ with $M_+ = [k[X]/(X+1)^{e_+}]^{m_+}$ and $M_- = [k[X]/(X-1)^{e_-}]^{m_-}$ for some $e_+, e_-, m_+, m_- \in \mathbb{N}$. Set $n_+ = e_+m_+$ and $n_- = e_-m_-$. Note that $\dim(M^0) = n_+ + n_-$.

Let us first prove that it is sufficient to consider *semisimple* modules. For any module $M$, let us denote by $\mathrm{rad}(M)$ its radical, and set $\overline{M} = M/\mathrm{rad}(M)$. Any module $M$ is the direct sum of modules of the form $M_{f,e} = [k[X]/(f^e)]^n$ for some $f \in k[X]$ with $f$ irreducible and $e, n \in \mathbb{N}$. Let $M_{\mathrm{odd}}$ be the direct sum of the modules $M_{f,e}$ with $e$ odd and $f$ symmetric.

**Proposition 4.1.** *Let $q$ be a quadratic space and let $M$ be a module. Then $q$ has an isometry with module $M$ if and only if $q$ is isomorphic to the orthogonal sum of a quadratic space $\overline{q}$ with module $\overline{M}_{\mathrm{odd}}$ and of a hyperbolic space.*

*Proof.* Suppose that $q$ has an isometry with module $M$. Then by Props. 3.3 and 3.4 and Cor. 3.5, the quadratic space $q$ is isomorphic to the orthogonal sum of a quadratic space $q_{\mathrm{odd}}$ having an isometry with module $M_{\mathrm{odd}}$ and of a hyperbolic space. Moreover, by Prop. 3.4 the quadratic space $q_{\mathrm{odd}}$ is isomorphic to the orthogonal sum of quadratic spaces $q_{f,e}$ having isometries with modules $M_{f,e}$. Further, Prop. 3.4 also implies that $q_{f,e} \simeq \overline{q}_{f,e} \oplus H_{f,e}$, where $H_{f,e}$ is a hyperbolic space and $\overline{q}_{f,e}$ has an isometry with module $\overline{M}_{f,e}$. Note that $\overline{M}_{\mathrm{odd}}$ is the direct sum of the modules $\overline{M}_{f,e}$ for $e$ odd and $f$ symmetric. Let $\overline{q}$ be the orthogonal sum of the quadratic spaces $\overline{q}_{f,e}$. Then $\overline{q}$ has an isometry with module $\overline{M}_{\mathrm{odd}}$, and $q$ is the orthogonal sum of $\overline{q}$ and of a hyperbolic space.

Conversely, suppose that $q \simeq \overline{q} \oplus H$, where $\overline{q}$ is a quadratic space having an isometry with module $\overline{M}$, and $H$ is a hyperbolic space. Then $\overline{q}$ is the orthogonal sum of quadratic spaces $\overline{q}_{f,e}$ having isometries with modules $\overline{M}_{f,e}$. By Prop. 3.4 we get quadratic spaces $q_{f,e} \simeq \overline{q}_{f,e} \oplus H_{f,e}$ having isometries with modules $M_{f,e}$, and $q$ is the orthogonal sum of the spaces $q_{f,e}$ and of a hyperbolic space. Hence $q$ has an isometry with module $M$. $\square$

Recall that the *Witt index* of a quadratic space $q$ is the number of hyperbolic planes contained in the Witt decomposition of $q$.

**Lemma 4.2.** *If $q$ is a quadratic space having an isometry with module $M$, then the Witt index of $q$ is $\geq m_2$.*

*Proof.* Indeed, by Cor. 3.5, any quadratic space having an isometry with a module of type 2 is hyperbolic.

For the remainder of this section, let us assume that $M$ is a *semisimple* module. Then the converse also holds if $M^1 = 0$:

**Proposition 4.3.** *Let $q$ be a quadratic space such that* $\dim(q) = \dim(M)$. *Suppose that $M^1 = 0$. Then $q$ has an isometry with module $M$ if and only if the Witt index of $q$ is at least $m_2$.*

*Proof.* We already know that if $q$ has an isometry with module $M$, then the Witt index of $q$ is at least $m_2$. Conversely, suppose that the Witt index of $q$ is at least $m_2$, and write $q$ as the orthogonal sum of a quadratic space $(V_0, q_0)$ and a hyperbolic form of dimension $2m_2$. Then $\dim(V_0) = \dim(M^0) = n_+ + n_-$. Let us decompose $(V_0, q_0)$ as an orthogonal sum of $(V_+, q_+)$ and $(V_-, q_-)$ with $\dim(V_+) = n_+$, $\dim(V_-) = n_-$, and let $t : V_0 \to V_0$ be defined by $t(x) = -x$ if $x \in V_+$ and $t(x) = x$ if $x \in V_-$. Then $t$ is an isometry of $q_0$, hence we obtain an isometry of $q$ with module $M$. $\square$

This proposition has some useful consequences. In order to state the first one, we need the notion of $u$-invariant of a field. Recall that a quadratic space $(V, q)$ is said to be *isotropic* if there exists a non-zero $v \in V$ with $q(v, v) = 0$, and *anisotropic* otherwise. The *$u$-invariant* of $k$, denoted by $u(k)$, is the largest dimension of an anisotropic quadratic form over $k$.

**Corollary 4.4.** *Let $q$ be a quadratic space with* $\dim(q) = \dim(M)$. *Suppose that* $u(k) \leq \dim(M^0)$. *Then $q$ has an isometry with module $M$.*

*Proof.* Let $q_1$ be a quadratic space having an isometry with module $M^1$ (cf. Cor. 3.7). As $u(k) \leq \dim(M^0)$, we have $q \oplus (-q_1) \simeq q_0 \oplus H$, where $H$ is hyperbolic and $q_0$ is a quadratic space with $\dim(q_0) = \dim(M^0)$. Let $q_2$ be the hyperbolic space of dimension $2m_2$. Then the quadratic space $q_0 \oplus q_2$ has dimension $\dim(M^0 \oplus M^2)$ and Witt index $\geq m_2$. Therefore by Prop. 4.3 the quadratic space $q_0 \oplus q_2$ has an isometry with module $M^0 \oplus M^2$; hence $q_0 \oplus q_1 \oplus q_2$ has an isometry with module $M$. We have $q \oplus q_2 \oplus q_1 \oplus (-q_1) \simeq q_0 \oplus q_1 \oplus q_2 \oplus H$. Since $q_1 \oplus (-q_1)$ and $q_2$ are hyperbolic, and $\dim(q) = \dim(M) = \dim(q_0 \oplus q_1 \oplus q_2)$, Witt cancellation implies that $q \simeq q_0 \oplus q_1 \oplus q_2$. Therefore $q$ has an isometry with module $M$. $\square$

Let us recall that two quadratic spaces $q$ and $q'$ are *Witt-equivalent* if there exist hyperbolic spaces $H$ and $H'$ such that $q \oplus H \simeq q' \oplus H'$.

**Corollary 4.5.** *Suppose that $M^1 = 0$, and let $q$ be a quadratic space with $\dim(q) = \dim(M^0)$. Then any quadratic space of dimension equal to $\dim(M)$ and Witt-equivalent to $q$ has an isometry with module $M$.*

*Proof.* Indeed, as $M^1 = 0$ we have $M = M^0 \oplus M^2$, hence $\dim(M) = \dim(M^0) + 2m_2$. Let $q'$ be a quadratic space with $\dim(q) = \dim(M)$ and Witt-equivalent to $q$. Then the Witt index of $q'$ is at least $m_2$, hence by Prop. 4.3 the quadratic space $q'$ has an isometry with module $M$. □

The next corollary will be used several times.

**Corollary 4.6.** *Suppose that $M^0 \neq 0$, and let $d \in k^*$. Then there exists a quadratic space $q$ having an isometry with module $M$ and determinant $d$.*

*Proof.* We have $M = M^0 \oplus M^1 \oplus M^2$. If $M^1 \neq 0$, let $q_1$ be a quadratic space having an isometry with module $M^1$ (cf. Cor. 3.7), and let $d_1 = \det(q_1)$. If $M^1 = 0$, set $d_1 = 1$.

Since $M^0 \neq 0$, there exists a quadratic space $q_0$ with determinant $dd_1(-1)^{m_2}$ and $\dim(q_0) = \dim(M^0)$. Let $H$ be the hyperbolic form of dimension $2m_2 = \dim(M^2)$, and set $q_2 = q_0 \oplus H$. Then $\dim(q_2) = \dim(M^0 \oplus M^2)$ and $\det(q_2) = dd_1$. Moreover, $q_2$ is Witt-equivalent to $q_0$. Therefore Cor. 4.5 implies that the quadratic space $q_0$ has an isometry with module $M^0 \oplus M^2$. Set $q = q_1 \oplus q_2$. Then $\det(q) = d$ in $k^*/k^{*2}$, and $q$ has an isometry with module $M$. □

## 5. Determinants and values of the characteristic polynomial

We have a relationship between the determinant of a quadratic space and the values of the characteristic polynomials of its isometries:

**Proposition 5.1.** *Let $(V, q)$ be a quadratic space, and let $F \in k[X]$ be the characteristic polynomial of an isometry $t$ of $q$. Then*

$$\det(q)F(1)F(-1) \in k^2.$$

*Proof.* Let us define $q' : V \times V \to k$ by $q'(x, y) = q(x, (t - t^{-1})(y))$. Then $q'$ is skew-symmetric, hence $\det(q') \in k^2$. On the other hand, we have

$$\det(q') = \det(q)\det(t)\det(t + 1)\det(t - 1) = \det(q)\det(t)F(1)F(-1).$$

If $F(1)F(-1) = 0$, then the statement is clear, so we can assume that $F(1)F(-1) \neq 0$. It is easy to see that $F(1) \neq 0$ implies that $F(X) = X^{\deg(F)}F(X^{-1})$, and $F(-1) \neq 0$ implies that $\deg(F)$ is even. Hence $\det(t) = 1$, and so $\det(q)F(1)F(-1) \in k^2$, as stated. □

The following corollary is well-known (see for instance [Le69, Lemma 7(c)], or [GM02, appendix]):

**Corollary 5.2.** *Let $q$ be a quadratic space, and let $F \in k[X]$ be the characteristic polynomial of an isometry of $q$. Suppose that $F(1)F(-1) \neq 0$. Then*

$$\det(q) = F(1)F(-1) \quad \text{in } k^*/k^{*2}.$$

*Proof.* This is an immediate consequence of Prop. 5.1. □

The following lemma will be useful in the next sections.

**Lemma 5.3.** *Let $M$ be a semisimple module, and let $d \in k^*$ with $dF_M(1)F_M(-1) \in k^2$. Then there exists a quadratic space $q$ of determinant $d$ having an isometry with module $M$.*

*Proof.* Suppose first that $F_M(1)F_M(-1) \neq 0$; then the hypothesis implies that $d = F_M(1)F_M(-1)$ in $k^*/k^{*2}$. Let $q$ be any quadratic space with module $M$ (cf. Cor. 3.7). By Cor. 5.2 we have $\det(q) = F_M(1)F_M(-1)$ in $k^*/k^{*2}$, hence $\det(q) = d$ in $k^*/k^{*2}$. Suppose now that $F_M(1)F_M(-1) = 0$, and note that this implies that $M^0 \neq 0$. By Cor. 4.6, there exists a quadratic space $q'$ with determinant $d$ having an isometry of module $M$, and this completes the proof of the lemma. □

## 6. Fields with $I(k)^2 = 0$

We keep the notation introduced in §4. In particular, *module* means a self-dual torsion $k[X]$-module that is a finite-dimensional $k$-vector space. Recall that by Prop. 4.1 it is sufficient to consider the case of *semisimple* modules. Let $W(k)$ be the Witt ring of $k$, and let $I(k)$ be the fundamental ideal of $W(k)$. Let $q$ be a quadratic space, and let $M$ be a semisimple module such that $\dim(q) = \dim(M)$.

**Proposition 6.1.** *Suppose that $I(k)^2 = 0$. Then the quadratic space $q$ has an isometry with module $M$ if and only if*

$$\det(q)F_M(1)F_M(-1) \in k^2.$$

*Proof.* The condition is necessary by Prop. 5.1. Let us show that it is sufficient. By Lemma 5.3 there exists a quadratic space $q'$ having an isometry with module $M$ and such that $\det(q') = \det(q)$. Then $q$ and $q'$ have the same dimension and determinant. As $I(k)^2 = 0$, this implies that they are isomorphic, therefore $q$ has an isometry with module $M$. □

**Corollary 6.2.** *Suppose that $I(k)^2 = 0$ and $F_M(1)F_M(-1) \neq 0$. Then the quadratic space $q$ has an isometry with module $M$ if and only if*

$$\det(q) = F_M(1)F_M(-1) \quad \text{in } k^*/k^{*2}.$$

*Proof.* This follows from Prop. 6.1. □

Let $k_s$ be a separable closure of $k$, and set $\Gamma_k = \mathrm{Gal}(k_s/k)$. We say that the 2-cohomological dimension of $k$, denoted by $\mathrm{cd}_2(k)$, is at most 1 if $H^r(\Gamma_k, A) = 0$ for all finite 2-primary $\Gamma_k$-modules $A$ and for all $r > 1$.

**Corollary 6.3.** *Suppose that $\mathrm{cd}_2(k) \leq 1$. Then the quadratic space $q$ has an isometry with module $M$ if and only if*

$$\det(q)F_M(1)F_M(-1) \in k^2.$$

*If moreover $F_M(1)F_M(-1) \neq 0$, then $q$ has an isometry with module $M$ if and only if*

$$\det(q) = F_M(1)F_M(-1) \quad \text{in } k^*/k^{*2}.$$

*Proof.* It is well-known that if $\mathrm{cd}_2(k) \leq 1$, then $I(k)^2 = 0$, hence the corollary follows from Prop. 6.1 and Cor. 6.2. □

## 7. Local fields

We keep the notation of §4. In particular, *module* means a self-dual torsion $k[X]$-module that is a finite-dimensional $k$-vector space. For any module $M$, we have $M = M^0 \oplus M^1 \oplus M^2$, where $M^i$ is of type $i$. Let us suppose that $M$ is *semisimple* (this is possible by Prop. 4.1). Note that if $M^1 = 0$, then a quadratic space has an isometry with module $M$ if and only if its Witt index is $\geq m_2$, where $2m_2 = \dim(M^2)$ (cf. Prop. 4.3). Therefore from now on we can restrict ourselves to modules $M$ with $M^1 \neq 0$.

Suppose that $k$ is a local field. Let $q$ be a quadratic space, and let $M$ be a module with $M_1 \neq 0$. Suppose that $\dim(q) = \dim(M)$.

**Theorem 7.1.** *The quadratic space $q$ has an isometry with module $M$ if and only if*

$$\det(q)F_M(1)F_M(-1) \in k^2.$$

The proof of Th. 7.1 uses the following result of Milnor. Let $K$ be an extension of $k$ of finite degree endowed with a non-trivial $k$-linear involution $x \mapsto \overline{x}$. Let $\ell : K \to k$ be a non-trivial linear form such that $\ell(x) = \ell(\overline{x})$ for all $x \in K$. For any non-degenerate hermitian form $h : V \times V \to K$, let us denote by $q_h : V \times V \to k$ the quadratic space defined by $q_h(x, y) = \ell(h(x, y))$ for all $x, y \in V$. We have

**Theorem 7.2.** *If the hermitian spaces $h$ and $h'$ have the same dimension but different determinants, then the quadratic spaces $q_h$ and $q_{h'}$ have the same dimension and determinant but different Hasse invariants.*

*Proof.* See [M69, Th. 2.7]. □

*Proof of Theorem 7.1.* If $q$ has an isometry with module $M$, then by Prop. 5.1 we have $\det(q)F_M(1)F_M(-1) \in k^2$.

Conversely, suppose that

$$\det(q)F_M(1)F_M(-1) \in k^2.$$

By Lemma 5.3, there exists a quadratic space $q'$ having an isometry with module $M$ such that $\det(q') = \det(q)$. It is well-known that two quadratic spaces over a local field are isomorphic if and only if they have the same dimension, determinant and Hasse–Witt invariant. Therefore if the Hasse–Witt invariants of $q$ and $q'$ are equal, then $q \simeq q'$, hence we are done.

Suppose that this is not the case. As $M^1 \neq 0$, there exists a monic, symmetric, irreducible polynomial $f \in k[X]$ of even degree such that for some $n \in \mathbb{N}$ and for some odd integer $e$, the module $M_f = [k[X]/(f^e)]^n$ is a direct summand of $M^1$. Set $M = M_f \oplus \tilde{M}$. By Props. 3.3 and 3.4, we have an orthogonal decomposition $q' \simeq q_f \oplus \tilde{q}$, where $q_f$ has an isometry with module $M_f$ and $\tilde{q}$ has an isometry with module $\tilde{M}$.

Set $K = k[X]/(f)$, and consider the $k$-linear involution of $K$ induced by $X \mapsto X^{-1}$. Let $E$ be the fixed field of this involution. Set $V = K^n$. Then by Props. 3.4 and 3.6, there exists a hermitian form $h : V \times V \to K$ such that the quadratic space $q_f$ has an orthogonal decomposition $q_f \simeq q_h \oplus H$, where $H$ is a hyperbolic space and $q_h : V \times V \to k$ is defined by

$$q_h(x, y) = \ell(h(x, y)).$$

Let $\alpha_1, \dots, \alpha_n \in E^*$ be such that $h \simeq \langle \alpha_1, \dots, \alpha_n \rangle$. Let us denote by $N_{K/E} : K \to E$ the norm map, and let $\alpha \in E^*$ be such that $\alpha \notin N_{K/E}(K^*)$. Let $h' : V \times V \to K$ be the hermitian form defined by $h' = \langle \alpha\alpha_1, \dots, \alpha_n \rangle$. Let us define $q_{h'} : V \times V \to k$ by

$$q_{h'}(x, y) = \ell(h'(x, y)).$$

Then $h$ and $h'$ have the same dimension but different determinants. Therefore by Th. 7.2, the quadratic forms $q_h$ and $q_{h'}$ have the same dimension and determinant but different Hasse–Witt invariants.

Set $q'_f = q_{h'} \oplus H$. By Prop. 3.4, the quadratic space $q'_f$ has an isometry with module $M_f$. Let $q'' = q'_f \oplus \tilde{q}$. Then $q''$ has an isometry with module $M$, and the quadratic spaces $q$ and $q''$ have equal dimension, determinant and Hasse–Witt invariants. Therefore $q \simeq q''$, hence $q$ has an isometry with module $M$ as claimed. □

Recall that we are assuming that $M^1 \neq 0$. The following corollary shows that if in addition $M^0 \neq 0$, then any quadratic form of dimension $\dim(M)$ has an isometry with module $M$.

**Corollary 7.3.** *Suppose that $M^0 \neq 0$. Then any quadratic space of dimension $\dim(M)$ has an isometry with module $M$.*

*Proof.* Let $q$ be a quadratic space with $\dim(q) = \dim(M)$. As $M^0 \neq 0$, we have $F_M(1)F_M(-1) = 0$, therefore the condition $\det(q)F_M(1)F_M(-1) \in k^2$ holds independently of the value of $\det(q)$. Hence by Th. 7.1 the quadratic form $q$ has an isometry with module $M$. □

**Corollary 7.4.** *Suppose that $F_M(1)F_M(-1) \neq 0$. Then the quadratic space $q$ has an isometry with module $M$ if and only if*

$$\det(q) = F_M(1)F_M(-1) \quad \textit{in } k^*/k^2.$$

*Proof.* This is a consequence of Th. 7.1. □

## 8. The field of real numbers

In this section the ground field $k$ is the field of real numbers $\mathbb{R}$. Let $q$ be a quadratic space over $\mathbb{R}$. It is well-known that $q$ is isomorphic to

$$X_1^2 + \cdots + X_r^2 - X_{r+1}^2 - \cdots - X_{r+s}^2$$

for some natural numbers $r$ and $s$. These are uniquely determined by $q$, and we have $r + s = \dim(V)$. The couple $(r, s)$ is called the *signature* of $q$.

Let $M$ be a module. Recall that $M = M^0 \oplus M^1 \oplus M^2$ with $M^i$ of type $i$. Let $F_M$ be the characteristic polynomial of $M$, and let $2\sigma$ be the number of roots of $F_M$ off the unit circle. Note that $\dim(M^2) = 2\sigma$.

Let us introduce some notation. For any integers $n, m, n', m'$, we write $(n, m) \geq (n', m')$ if $n \geq n'$ and $m \geq m'$, and we write $(n, m) \equiv (n', m') \pmod 2$ whenever $n \equiv n' \pmod 2$ and $m \equiv m' \pmod 2$.

We have seen in §4 that it suffices to consider semisimple modules (cf. Prop. 4.1). We first give the criterion in the semisimple case (see Prop. 8.1 below), and then use Prop. 4.1 to treat the case of arbitrary modules.

**Proposition 8.1.** *Assume that $M$ is semisimple and* $\dim(q) = \dim(M)$.

(a) *Suppose that the quadratic space $q$ has an isometry with module $M$. Then*

$$(r, s) \geq (\sigma, \sigma).$$

*If moreover $M^0 = 0$, then*

$$(r, s) \equiv (\sigma, \sigma) \pmod 2.$$

(b) *Conversely, suppose that $(r, s) \geq (\sigma, \sigma)$, and if moreover $M^0 = 0$, then $(r, s) \equiv (\sigma, \sigma) \pmod 2$. Then $q$ has an isometry with module $M$.*

*Proof.* (a) Suppose that the quadratic space $q$ has an isometry with module $M$. By Prop. 3.3, we have an orthogonal decomposition

$$(V, q) \simeq \bigoplus (V_f, q_f) \oplus H$$

where the sum is taken over all distinct monic, symmetric and irreducible factors of $F_M$, and $H$ is a hyperbolic space. Note that $\dim(H) = \dim(M^2) = 2\sigma$. This implies that $(r, s) \geq (\sigma, \sigma)$. If $M^0 = 0$, then every irreducible and symmetric polynomial $f$ appearing in the above decomposition is of degree two. Let $K_f = k[X]/(f)$. Then $V_f$ has a structure of $K_f$-vector space, and by Prop. 3.6, there exists a hermitian form $h_f : V_f \times V_f \to K_f$ such that

$$q_i(x, y) = \mathrm{Tr}_{K/\mathbb{R}}(h_f(x, y))$$

for all $x, y \in V_{f,}$. Let $(u_f, v_f)$ be the signature of $h_f$. Then the signature of $q_f$ is $(2u_f, 2v_f)$, and this implies that $(r, s) \equiv (\sigma, \sigma) \pmod 2$.

(b) Conversely, suppose that $r+s = \dim(M)$ and $(r, s) \geq (\sigma, \sigma)$. Note that $\dim(M) - 2\sigma = \dim(M) - \dim(M_2) \geq \dim(M^0)$, therefore $r + s - 2\sigma \geq \dim(M^0)$. As $\dim(M^1)$ is even, we also have $\dim(M) = r + s \equiv \dim(M^0) \pmod 2$. Set $r' = r - \sigma$ and $s' = s - \sigma$. Then $r' + s' \equiv r + s \equiv \dim(M^0) \pmod 2$, and $r' + s' \geq \dim(M^0)$. Let us write

$$r' = 2u + u_+ \text{ and } s' = 2v + v_-$$

with $u, v, u_+, v_- \in \mathbb{N}$ such that $u_+ + v_- = \dim(M^0)$. This is clearly possible if $\dim(M^0) > 0$. On the other hand, if $\dim(M^0) = 0$ then $M^0 = 0$, hence by hypothesis $(r, s) \equiv (\sigma, \sigma) \pmod 2$. This implies that $r'$ and $s'$ are even. In this case, set $u = r'/2$ and $v = s'/2$.

Note that $\dim(M^1) = 2u + 2v$. Recall that $M^1$ is a direct sum of modules of the type $[k[X]/(f)]^{n_f}$ with $f \in \mathbb{R}[X]$ symmetric, irreducible and $\deg(f) = 2$. Note that $u + v = \sum n_f$, where the sum is taken over all $f$ as above. Let $u_f, v_f \in \mathbb{N}$ be such that $0 \le u_f, v_f \le n_f$, $u_f + v_f = n_f$, and

$$\sum u_f = u, \quad \sum v_f = v,$$

the sums being taken over all the $f$ as above.

Set $K_f = k[X]/(f)$ and $V_f = K_f^{n_f}$. Let $h_f : V_f \times V_f \to K_f$ be a hermitian form of signature $(u_f, v_f)$, and let $q_f : V_f \times V_f \to \mathbb{R}$ be the quadratic space defined by $q_f(x, y) = \mathrm{Tr}_{K_f/\mathbb{R}}(h_f(x, y))$ for all $x, y \in V_f$. Then the signature of $q_f$ is $(2u_f, 2v_f)$. Let $q_1$ be the orthogonal sum of the spaces $q_f$ for all $f$ as above. Then the signature of $q_1$ is $(2u, 2v)$.

Let $q_0$ be the quadratic space of signature $(u_+, v_-)$, and let $q_2$ be the hyperbolic space of dimension $2\sigma$. Let $q' = q_0 \oplus q_1 \oplus q_2$. Then $q'$ has an isometry with module $M = M^0 \oplus M^1 \oplus M^2$.

Note that $\mathrm{sign}(q') = (u_+ + 2u + \sigma, v_- + 2v + \sigma) = (r, s) = \mathrm{sign}(q)$, hence $q' \simeq q$. This implies that $q$ has an isometry with module $M$. □

**Corollary 8.2.** *Let $F \in \mathbb{R}[X]$ be a symmetric polynomial such that $F(1)F(-1) \neq 0$. Then the quadratic space $q$ has a semisimple isometry with characteristic polynomial $F$ if and only if $(r, s) \ge (\sigma, \sigma)$ and $(r, s) \equiv (\sigma, \sigma) \pmod 2$.*

*Proof.* Let $M$ be the semisimple module with characteristic polynomial $F = F_M$. As $F_M(1)F_M(-1) \neq 0$, we have $M^0 = 0$, and the corollary follows from Prop. 8.1. □

A special case of this corollary is proved by Gross and McMullen in [GM02, Cor. 2.3].

Props. 8.1 and 4.1 lead to a criterion for arbitrary modules:

**Corollary 8.3** *Suppose that* $\dim(q) = \dim(M)$, *and set* $2\tau = \dim(M) - \dim(\overline{M}^0_{\mathrm{odd}})$.

(a) *Suppose that the quadratic space $q$ has an isometry with module $M$. Then*

$$(r, s) \ge (\tau, \tau).$$

*If moreover $\overline{M}^0_{\mathrm{odd}} = 0$, then*

$$(r, s) \equiv (\tau, \tau) \pmod 2.$$

(b) *Conversely, suppose that $(r, s) \ge (\tau, \tau)$, and if moreover $\overline{M}^0_{\mathrm{odd}} = 0$, then $(r, s) \equiv (\tau, \tau) \pmod 2$. Then $q$ has an isometry with module $M$.*

*Proof.* (a) As $q$ has an isometry with module $M$, by Prop. 4.1 the quadratic space $q$ is isomorphic to the orthogonal sum of a quadratic space $q'$ having an isometry with module $\overline{M}_{\mathrm{odd}}$ and of a hyperbolic space $H$ of dimension $2\tau$. The signature of $H$ is $(\tau, \tau)$, hence $(r, s) \ge (\tau, \tau)$. Let $(r', s')$ be the signature of $q'$. Note that all the roots of the polynomial $F_{\overline{M}_{\mathrm{odd}}}$ are on the unit circle. Therefore if $\overline{M}^0_{\mathrm{odd}} = 0$, by Prop. 8.1 we have $(r', s') \equiv (0, 0) \pmod 2$, hence $(r, s) \equiv (\tau, \tau) \pmod 2$.

(b) Since $(r, s) \equiv (\tau, \tau) \pmod 2$, we have $q \simeq q' \oplus H$, where $H$ is a hyperbolic space of dimension $2\tau$ and $q'$ is a quadratic space of dimension equal to $\dim(\overline{M}_{\mathrm{odd}})$. Let $(r', s')$ be the signature of $q'$. If $\overline{M}^0_{\mathrm{odd}} = 0$, then by hypothesis we have

$$(r, s) \equiv (\tau, \tau) \pmod 2.$$

Since the signature of $H$ is $(\tau, \tau)$, this implies that $(r', s') \equiv (0, 0) \pmod 2$. Since all the roots of the polynomial $F_{\overline{M}_{\mathrm{odd}}}$ are on the unit circle, Prop. 8.1 implies that the quadratic space $q'$ has an isometry with module $\overline{M}_{\mathrm{odd}}$, and by Prop. 4.1 this implies that $q$ has an isometry with module $M$. □

## 9. Global fields—the case of an irreducible minimal polynomial

The aim of this section is to give an answer to Milnor's question stated in the introduction in the case of global fields. Suppose that $k$ is a global field, let $q$ be a quadratic space, and let $f \in k[X]$ be an irreducible and symmetric polynomial. We have the following Hasse principle:

**Theorem 9.1.** *The quadratic space $q$ has an isometry with minimal polynomial $f$ if and only if such an isometry exists over every completion of $k$.*

The case $f(X) = X + 1$ is trivial, hence we may assume that $f(1)f(-1) \neq 0$. Before proving Th. 9.1, let us use the results of the previous two sections to obtain necessary and sufficient conditions for an isometry to exist. Let $F$ be a power of $f$ such that $\deg(F) = \dim(V) = 2n$.

For every real place $v$ of $k$, let $(r_v, s_v)$ denote the signature of $q$ over $k_v$, and let $\sigma_v$ be the number of roots of $F \in k_v[X]$ that are not on the unit circle.

We say that the *signature condition* is satisfied for $q$ and $F$ if for every real place $v$ of $k$, we have $(r_v, s_v) \geq (\sigma_v, \sigma_v)$ and $(r_v, s_v) \equiv (\sigma_v, \sigma_v) \pmod 2$.

We say that the *hyperbolicity condition* is satisfied for $q$ and $F$ if for all places $v$ of $k$ such that $F \in k_v[X]$ is a hyperbolic polynomial, the quadratic form $q_v$ over $k_v$ is hyperbolic.

**Corollary 9.2.** *The quadratic space $q$ has an isometry with minimal polynomial $f$ if and only if the signature condition and the hyperbolicity condition are satisfied, and*

$$\det(q) = F(1)F(-1) \quad \text{in } k^*/k^{*2}.$$

*Proof.* The necessity of the conditions follows from Corollaries 8.2, 3.5 and 5.2. Conversely, suppose that the signature condition is satisfied and $\det(q) = F(1)F(-1)$ in $k^*/k^{*2}$. Then by Cor. 8.2 and Th. 7.4, the quadratic space $q$ has an isometry with minimal polynomial $f$ over $k_v$ for every place $v$ of $k$. By Th. 9.1, this implies that $q$ has an isometry with minimal polynomial $f$. □

The following reformulation of Cor. 9.2 shows that it suffices to check a finite number of conditions. Let $q$ and $F$ be as above, with $\dim(q) = \deg(F) = 2n$. Let $S$ be the set of places of $k$ at which the Hasse invariant of $q$ is not equal to the Hasse invariant of the $2n$-dimensional hyperbolic space. Note that $S$ is a finite set.

**Corollary 9.3.** *The quadratic space $q$ has an isometry with minimal polynomial $f$ if and only if the following conditions are satisfied:*

(i) $F(1)F(-1) = \det(q)$ *in* $k^*/k^{*2}$.
(ii) *The signature condition holds.*
(iii) *If $v \in S$, then $F \in k_v[X]$ is not hyperbolic.*

*Proof.* It suffices to prove that (i)–(iii) imply the hyperbolicity condition. Let $v$ be a place of $k$ such that $F \in k_v[X]$ is hyperbolic. Then there exists $G \in k_v[X]$ such that $F = GG^*$. Note that $\deg(G) = n$. We have $F(1) = G(1)^2$ and $F(-1) = (-1)^n G(-1)^2$. By (i), we have $F(1)F(-1) = \det(q)$, hence $(-1)^n \det(q) = \operatorname{disc}(q) \in k_v^2$. On the other hand, (iii) implies that $v \notin S$, hence $q$ has the same Hasse invariant at $v$ as the $2n$-dimensional hyperbolic space. Therefore over $k_v$, the quadratic space $q$ has the same dimension, discriminant and Hasse invariant as the $2n$-dimensional hyperbolic space. If $v$ is an infinite place, then by (ii) the signature of $q$ at $v$ coincides with the signature of the $2n$-dimensional hyperbolic space. Hence $q$ is hyperbolic over $k_v$, in other words the hyperbolicity condition is satisfied. □

The following lemmas will be used in the proof of Th. 9.1, and also in §10.

Let $K = k[X]/(f)$, and let $^- : K \to K$ be the involution induced by $X \mapsto X^{-1}$. Let $E$ be the fixed field of the involution.

**Lemma 9.4.** *Let $v$ be a place of $k$. The following properties are equivalent:*

(i) *Every place of $E$ above $v$ splits in $K$.*
(ii) *The polynomial $f \in k_v[X]$ is hyperbolic.*
(iii) *For any $m \in \mathbb{N}$, the module $[k_v[X]/(f)]^m$ is hyperbolic.*

*Proof.* Let $w_1, \dots, w_r$ be the places of $E$ above $v$, and set $E_i = E_{w_i}$ and $K_i = K \otimes_E E_i$. Then $K_i$ is a field if $w_i$ is inert or ramified in $K$, a product of two fields if $w_i$ is split in $K$, and $k_v[X]/(f) \simeq K_1 \times \cdots \times K_r$.

(i)⇒(ii). Since every $w_i$ splits in $K$, all the $K_i$'s are products of two fields. This implies that $f = f_1 f_1^* \dots f_r f_r^*$ with $f_i \in k_v[X]$ monic and irreducible and $f_i \neq f_i^*$ for all $i = 1, \dots, r$. Therefore $f \in k_v[X]$ is hyperbolic.

(ii)⇔(iii) is clear.

(ii)⇒(i). Since $f \in k[X]$ is irreducible and $f \in k_v[X]$ is hyperbolic, we have $f = f_1 f_1^* \dots f_r f_r^*$ with $f_i \in k_v[X]$ monic and irreducible and $f_i \neq f_i^*$ for all $i = 1, \dots, r$. Therefore all the $K_i$'s are products of two fields, hence (i) holds.

**Lemma 9.5.** *Let $v$ be a place of $k$ satisfying the equivalent conditions of Lemma* 9.4. *Then:*

(i) *Any quadratic space over $k_v$ having an isometry with minimal polynomial $f$ is hyperbolic.*
(ii) *For any $m \in \mathbb{N}$, every quadratic space over $k_v$ having an isometry with module $[k_v[X]/(f)]^m$ is hyperbolic.*

*Proof.* Both assertions follow from Lemma 9.4 and Cor. 3.5. □

Recall that for any quadratic space $Q$, we denote by $w(Q)$ its Hasse invariant.

**Lemma 9.6.** *Let $m \in \mathbb{N}$, let $v$ be a finite place of $k$, and let $M = [k_v[X]/(f)]^m$. Suppose that $M$ is not hyperbolic. Let $\epsilon \in \{0, 1\}$. Then there exists a quadratic space $Q$ over $k_v$ such that $Q$ has an isometry with module $M$ and $w(Q) = \epsilon$.*

*Proof.* Note that $M$ is hyperbolic if and only if $f \in k_v[X]$ is hyperbolic, that is, if it is a product of polynomials of type 2 over $k_v$. As we are assuming that $M$ is not hyperbolic, the polynomial $f \in k_v[X]$ has at least one irreducible factor of type 1. Hence we have $f = f_1 f_2$ with $f_1, f_2 \in k_v[X]$ and $f_1$ irreducible, symmetric of even degree.

Recall that $K = k[X]/(f)$, $^- : K \to K$ is the involution induced by $X \mapsto X^{-1}$, and $E$ is the fixed field of this involution. Set $K_v = K \otimes_k k_v$. Then $K_v \simeq K_1 \times K_2$ with $K_i = k_v[X]/(f_i)$, the involution preserves $K_1$ and $K_2$, and we have $E = E_1 \times E_2$. Note that $K_1$ is a field, and $E_1$ is the fixed field of the restriction of the involution to $K_1$, hence $K_1/E_1$ is a quadratic extension.

We have $M \simeq M_1 \oplus M_2$ with $M_1 \simeq K_1^m$ and $M_2 \simeq K_2^m$. Let $h : M \times M \to K_v$ be the unit hermitian form. Then $h \simeq h_1 \oplus h_2$, where $h_i : M_i \times M_i \to K_v$, with $i = 1, 2$, is the restriction of $h$ to $M_i$. Let $\ell : K_v \to k_v$ be a non-zero linear form such that $\ell(\overline{x}) = \ell(x)$ for all $x \in K_v$. For any hermitian form $H$, set $q_H(x, y) = \ell(H(x, y))$. The quadratic space $q_h$ has an isometry with module $M$ by construction. If $w(q_h) = \epsilon$, then we set $Q = q_h$ and the lemma is proved.

Suppose that $w(q_h) \neq \epsilon$, and let $\alpha = \det(h_1)$; then $\alpha \in E_1^*$. Since $K_1/E_1$ is a quadratic extension, there exists $\beta \in E_1^*$ such that $\beta \notin \mathrm{N}_{K_1/E_1}(K_1^*)$. Let $h_1' : M_1 \times M_1 \to K_1$ be a hermitian form of determinant $\alpha\beta$. Then $h_1$ and $h_1'$ have same dimension and different determinants, hence by Th. 7.2 the quadratic spaces $q_{h_1}$ and $q_{h_1'}$ have equal dimension, determinant and different Hasse invariants. Let $h' = h_1' \oplus h_2$, and set $Q = q_{h'}$. Then $Q \simeq q_{h_1'} \oplus q_{h_2}$, hence $w(Q) = \epsilon$. Since $Q$ has an isometry with module $M$, this concludes the proof of the lemma. □

*Proof of Theorem 9.1.* Let $F = f^m$ and $\deg(f) = 2d$, and recall that $n = md$. Let $K = k[X]/(f)$, and let $^- : K \to K$ be the involution induced by $X \mapsto X^{-1}$. Let $E$ be the fixed field of the involution. Let $\theta \in E^*$ be such that $K = E(\sqrt{\theta})$, and for any place $w$ of $E$, let $(\ ,\ )_w$ denote the Hilbert symbol at $E_w$.

Let $v$ be a real place of $k$. Then the signature $(r_v, s_v)$ of $q$ at $k_v$ satisfies $(r_v, s_v) \geq (\sigma_v, \sigma_v)$ and $(r_v, s_v) \equiv (\sigma_v, \sigma_v) \pmod 2$. In particular, $r_v - \sigma_v$ is even. Set $r_v - \sigma_v = 2u_v$. Then $s_v - \sigma_v = 2(n - \sigma_v - u_v)$, and $n - \sigma_v - u_v \geq 0$. Therefore $0 \leq u_v \leq n - \sigma_v$. Let us denote by $2\tau_v$ the number of roots of $f$ that are not on the unit circle. Then $\sigma_v = m\tau_v$. Let us write $u_v = u_v^1 + \cdots + u_v^m$ for some integers $u_v^i$ such that $0 \leq u_v^i \leq d - \tau_v$.

Let $w_1, \ldots, w_{d-\tau_v}$ be the real places of $E$ above $v$ that extend to complex places of $K$. Let $\alpha_i \in E^*$ be such that $(\alpha_i, \theta)_{w_j} = 1$ if $j = 1, \ldots, u_i$, and that $(\alpha_i, \theta)_{w_j} = -1$ if $j = u_i + 1, \ldots, d - \tau_i v$.

Let $\ell : K \to k$ be a non-zero linear form such that $\ell(\overline{x}) = \ell(x)$ for all $x \in K$; if $\mathrm{char}(K) = 0$, then we choose $\ell$ to be the trace map, $\ell = \mathrm{Tr}_{K/k} : K \to k$. Let

$h' : V \times V \to K$ be the hermitian form defined by $h' = \langle \alpha_1, \dots, \alpha_m \rangle$ and let $q_{h'} : V \times V \to k$ be the quadratic space defined by $q_{h'}(x, y) = \ell(h'(x, y))$ for all $x, y \in V$. By construction, the signature at $v$ of $q_{h'}$ is $(r_v, s_v)$.

Let $S$ be the set of finite places of $k$ at which the Hasse invariants of $q$ and $q_{h'}$ are not equal. This is a finite set of even cardinality: indeed, the Hasse invariants of two quadratic spaces over $k$ differ at an even number of places, and $q$ and $q_{h'}$ are isomorphic at all the infinite places.

Let $T$ be the set of finite places of $k$ such that every place of $E$ above $v$ splits in $K$. Note that both quadratic spaces $q$ and $q_{h'}$ have isometries with minimal polynomial $f$ over every completion of $k$ (by hypothesis for $q$, by construction for $q_{h'}$). Therefore if $v \in T$, then both $q$ and $q_{h'}$ are hyperbolic over $k_v$ (cf. Lemma 9.5). Hence $q$ and $q_{h'}$ are isomorphic over $k_v$, and this implies that $v$ does not belong to $S$. Therefore $S$ and $T$ are disjoint.

For each $v \in S$, let us choose a place $w$ of $E$ which does not split in $K$; this is possible because $S$ and $T$ are disjoint. Let us denote by $S_E$ the set of those places. Then $S_E$ is a finite set of even cardinality.

For all $w \in S_E$, let $\beta_w \in E_w^*$ be such that $(\beta_w, \theta)_w = -1$; note that such a $\beta_w$ exists as $w$ does not split in $K$. By Hilbert's reciprocity, there exists $\beta \in E^*$ such that $(\beta, \theta)_w = (\beta_w, \theta)_w = -1$ if $w \in S_E$, and $(\beta_w, \theta)_w = 1$ for all the other places $w$ of $E$ (see for instance [O'M73, 71:19], or [PR10, Lemma 6.5]). Let $h : V \times V \to K$ be the hermitian form given by $h = \langle \beta\alpha_1, \dots, \alpha_d \rangle$ and let $q_h : V \times V \to k$ be the quadratic space defined by $q_h(x, y) = \ell(h(x, y))$ for all $x, y \in V$. Then by Th. 7.2, the Hasse invariants of $q_h$ and $q$ are equal. This implies that $q$ and $q_h$ have equal dimension, determinant, signatures and Hasse invariants, therefore these quadratic spaces are isomorphic. Note that $q_h$ has an isometry with minimal polynomial $f$ by construction, hence $q$ also has such an isometry, and this concludes the proof of the theorem. $\square$

## 10. A necessary and sufficient condition

Suppose that $k$ is a global field, and denote by $\Sigma_k$ the set of all places of $k$. Let $q$ be a quadratic space over $k$, and let $M$ be a module. The aim of this section is to give some necessary and sufficient conditions for $q$ to have an isometry with module $M$ (see Th. 10.11(b)). This was already started in the previous section. One of the results of §9 can be reformulated as follows:

**Theorem 10.1.** *Let $f \in k[X]$ be a symmetric, irreducible polynomial of even degree* (in other words, an irreducible polynomial of type 1). *Let $m \in \mathbb{N}$, let $M = [k[X]/(f)]^m$, and let $q$ be a quadratic space over $k$. Then $q$ has an isometry with module $M$ if and only if such an isometry exists over all the completions of $k$.*

*Proof.* As $f$ is irreducible, a quadratic space $q$ of dimension $m \deg(f)$ has an isometry with minimal polynomial $f$ if and only if $q$ has an isometry with module $M$. Hence the result follows from Th. 9.1. $\square$

We have $M = M^0 \oplus M^1 \oplus M^2$ with $M_i$ of type $i$. Recall that a quadratic space has an isometry with a module of type 2 if and only if it is hyperbolic (cf. Cor. 3.5). Hence $q$ has an isometry with module $M$ if and only it is isomorphic to an orthogonal sum of a quadratic space having an isometry with module $M^0 \oplus M^1$ and of a hyperbolic space. Therefore it suffices to consider modules with $M^2 = 0$.

On the other hand, we have seen that a quadratic space has an isometry with module $M$ if and only if it is the orthogonal sum of a quadratic space having an isometry with module $\overline{M}_{\mathrm{odd}}$ and of a hyperbolic space. Since $\overline{M}_{\mathrm{odd}}$ is semisimple, it is sufficient to treat the case of semisimple modules.

Suppose that $M$ is semisimple and $M^2 = 0$. We have $M = M^0 \oplus M^1$ with $M^0 = [k[X]/(X+1)]^{n_+} \oplus [k[X]/(X-1)]^{n_-}$ for some $n_+, n_- \in \mathbb{N}$, and

$$M^1 \simeq [k[X]/(f_1)]^{n_1} \oplus \cdots \oplus [k[X]/(f_r)]^{n_r},$$

where $f_1, \ldots, f_r \in k[X]$ are distinct irreducible polynomials of type 1 and $n_i \in \mathbb{N}$. Recall from §1 that this implies that $\deg(f_i)$ is even, and $f_i(1) f_i(-1) \neq 0$ for all $i = 1, \ldots, r$. Set $M_i = [k[X]/(f_i)]^{n_i}$.

Set $M_0 = M^0$. Then $M = M_0 \oplus M_1 \oplus \cdots \oplus M_r$. Set $I = \{1, \ldots, r\}$, and $I_0 = I \cup \{0\} = \{0, \ldots, r\}$.

**Proposition 10.2.** *The quadratic space $q$ has an isometry with module $M$ over $k$ if and only if there exist quadratic spaces $q_0, \ldots, q_r$ defined over $k$ such that*

$$q \simeq q_0 \oplus \cdots \oplus q_r,$$

*and for all $i \in I_0$, the quadratic space $q_i$ has an isometry with module $M_i$ over all the completions of $k$.*

*Proof.* Suppose that $q$ has an isometry with module $M$ over $k$. Then by Prop. 3.3 there exist quadratic spaces $q_0, \ldots, q_r$ defined over $k$ such that $q \simeq q_0 \oplus \cdots \oplus q_r$ and that the quadratic space $q_i$ has an isometry with module $M_i$ over $k$ for all $i \in I_0$.

Let us prove the converse. By hypothesis there exist quadratic spaces $q_0, \ldots, q_r$ defined over $k$ such that $q \simeq q_0 \oplus \cdots \oplus q_r$, and for all $i \in I_0$, the quadratic space $q_i$ has an isometry with module $M_i$ over all the completions of $k$. By Th. 10.1 this implies that the quadratic space $q_i$ has an isometry with module $M_i$ over $k$ for all $i \in I$. As $M_0$ is of type 0 and $\dim(q_0) = \dim(M_0)$, the quadratic space $q_0$ has an isometry with module $M_0$. Since $q$ is the orthogonal sum of the $q_i$'s, this proves the proposition. □

Suppose that $q$ has an isometry with module $M$ over all the completions of $k$. Then by Prop. 3.3 there exist quadratic spaces $\tilde{q}_i^v$ having an isometry with module $M_i$ for all $i \in I_0$ and for all places $v$ of $k$ such that we have an isomorphism over $k_v$

$$q \simeq \tilde{q}_0^v \oplus \cdots \oplus \tilde{q}_r^v.$$

The quadratic spaces $\tilde{q}_i^v$ are not uniquely determined by $q$ and $M$. The strategy used in this section is to investigate under what condition one can modify them and obtain quadratic spaces $q_0^v, \ldots, q_r^v$ defined over $k_v$ such that that there exist quadratic spaces

$q_0, \dots, q_r$ defined over $k$ which are isomorphic to $q_0^v, \dots, q_r^v$ over $k_v$ for all places $v$ of $k$. By Prop. 10.2, the Hasse principle holds precisely when this is possible. We start with some definitions and lemmas.

Let us consider *collections* $C = \{q_i^v\}$, for $i \in I_0$ and $v \in \Sigma_k$, of quadratic spaces defined over $k_v$, and let us denote by $\mathcal{C}_M$ the set of $C = \{q_i^v\}$ of collections satisfying the condition

(i) For all $v \in \Sigma_k$ and all $i \in I_0$, the quadratic space $q_i^v$ has an isometry with module $M_i$ over $k_v$.

Further, let us denote by $\mathcal{C}_{M,q}$ the set of $C = \{q_i^v\} \in \mathcal{C}_{\mathcal{M}}$ of collections satisfying the additional condition

(ii) For all $v \in \Sigma_k$, we have $q \simeq q_0^v \oplus \cdots \oplus q_r^v$ over $k_v$.

The above considerations show that if $q$ has an isometry with module $M$ over $k_v$ for all the places $v$ of $k$, then there exist quadratic spaces $q_i^v$ satisfying (i) and (ii), in other words such that $C = \{q_i^v\} \in \mathcal{C}_{M,q}$. Hence we have

**Lemma 10.3.** *Suppose that the quadratic space $q$ has an isometry with module $M$ over $k_v$ for all $v \in \Sigma_k$. Then $\mathcal{C}_{M,q}$ is not empty.*

Note that if $C = \{q_i^v\} \in \mathcal{C}_{M,q}$, then $\dim(q_i^v) = \dim(M_i)$ and $\det(q_i^v) = [f_i(1)f_i(-1)]^{n_i}$ in $k_v^*/k_v^{*2}$ for all places $v$ of $k$, and for all $i \in I$. Therefore the collections in $\mathcal{C}_{M,q}$ can only differ by the Hasse invariants and the signatures of the quadratic spaces.

For any $C = \{q_i^v\} \in \mathcal{C}$ and any $v \in \Sigma_k$, set

$$S_v(C) = \{i \in I \mid w(q_i^v) = 1\}.$$

For all $i \in I$, set $d_i = [f_i(1)f_i(-1)]^{n_i}$, and let $d_0 = \det(q)d_1 \dots d_r$. Set $D = \sum_{i<j}(d_i, d_j) \in \mathrm{Br}_2(k)$. Recall that $w(q) \in \mathrm{Br}_2(k)$ is the Hasse invariant of $q$. For any $x \in \mathrm{Br}_2(k)$ and any $v \in \Sigma_k$, let us denote by $x_v \in \{0, 1\}$ the image of $x$ in $\mathrm{Br}_2(k_v)$.

**Proposition 10.4.** *Let $C = \{q_i^v\} \in \mathcal{C}_M$. Then $C \in \mathcal{C}_{M,q}$ if and only if* $\det(q_0^v) = d_0$,

$$|S_v(C)| \equiv w(q)_v + D_v \pmod 2$$

*for all $v \in \Sigma_k$, and* $\mathrm{sign}(q) = \mathrm{sign}(q_0^v \oplus \cdots \oplus q_r^v)$ *for all real places $v$ of $k$.*

*Proof.* Suppose that $C \in \mathcal{C}_{M,q}$. Then $q \simeq q_0^v \oplus \cdots \oplus q_r^v$ for all $v \in \Sigma_k$. In particular, if $v$ is a real place, then $\mathrm{sign}(q) = \mathrm{sign}(q_0^v \oplus \cdots \oplus q_r^v)$. For all $v \in \Sigma_k$ and all $i \in I$, we have $\det(q_i^v) = d_i$, hence $\det(q_0^v) = \det(q)d_1 \dots d_r = d_0$. Moreover, for all $v \in \Sigma_k$,

$$w(q)_v = w(q_0^v \oplus \cdots \oplus q_r^v) = w(q_0^v) + \cdots + w(q_r^v) + \sum_{i<j}(d_i, d_j) = |S_v(C)| + D_v,$$

as claimed.

Let us prove the converse. Let $v \in \Sigma_k$ be a finite place, and let us check that $q \simeq q_0^v \oplus \cdots \oplus q_r^v$ over $k_v$. As $C \in \mathcal{C}_M$, the quadratic space $q_i^v$ has an isometry with module $M_i$

over $k_v$ for all $i \in I_0$. Therefore $\det(q_i^v) = d_i$ for all $i \in I$. By hypothesis, $\det(q_0^v) = d_0 = \det(q)d_1 \dots d_r$. Thus

$$w(q_0^v \oplus \cdots \oplus q_r^v) = w(q_0^v) + \cdots + w(q_r^v) + \sum_{i<j}(d_i, d_j) = |S_v(C)| + D = w(q)_v.$$

Therefore $q_v$ and $q_0^v \oplus \cdots \oplus q_r^v$ have equal dimension, determinant and Hasse–Witt invariant, hence these quadratic spaces are isomorphic over $k_v$. If $v$ is a real place, then we are assuming that $\mathrm{sign}(q) = \mathrm{sign}(q_0^v \oplus \cdots \oplus q_r^v)$, hence $q \simeq q_0^v \oplus \cdots \oplus q_r^v$ over $k_v$. Thus condition (ii) holds. Since $C \in \mathcal{C}_M$, condition (i) holds as well, therefore we have $C \in \mathcal{C}_{M,q}$. □

**Corollary 10.5.** *Let $\tilde{C} = \{\tilde{q}_i^v\} \in \mathcal{C}_{M,q}$, and let $C = \{q_i^v\} \in \mathcal{C}_M$. Let $u \in \Sigma_k$ be a finite place, and let $\alpha, \beta \in I_0$ with $\alpha \neq \beta$ be such that*

(a) $q_i^v \simeq \tilde{q}_i^v$ *for all $v \neq u$ and for all $i \in I_0$;*
(b) $q_i^u \simeq \tilde{q}_i^u$ *for all $i \neq \alpha, \beta$;*
(c) $w(q_\alpha^u) \neq w(\tilde{q}_\alpha^u)$ *and* $w(q_\beta^u) \neq w(\tilde{q}_\beta^u)$*;*
(d) $\det(q_0) = \det(q)d_1 \dots d_r$.

*Then $C \in \mathcal{C}_{M,q}$.*

*Proof.* By (b) and (c), we have $|S_u(C)| = |S_u(\tilde{C})|$. By (a), we have $|S_v(C) = |S_v(\tilde{C})|$ for all $v \neq u$, and $\mathrm{sign}(q_0^v \oplus \cdots \oplus q_r^v) = \mathrm{sign}(\tilde{q}_0^v \oplus \cdots \oplus \tilde{q}_r^v)$ if $v$ is a real place. Since $\tilde{C} \in \mathcal{C}_{M,q}$, by Prop. 10.4 we have $|S_v(\tilde{C})| \equiv w(q)_v + D_v \pmod 2$ for all $v \in \Sigma_k$, and $\mathrm{sign}(q) = \mathrm{sign}(\tilde{q}_1^v \oplus \cdots \oplus \tilde{q}_r^v)$ if $v$ is a real place. Hence we also have $|S_v(C)| \equiv w(q)_v + D_v \pmod 2$ for all $v \in \Sigma_k$, and $\mathrm{sign}(q) = \mathrm{sign}(q_0^v \oplus \cdots \oplus q_r^v)$ for all real places $v$ of $k$. By Prop. 10.4, this implies that $C \in \mathcal{C}_{M,q}$. □

**Lemma 10.6.** *Let $v$ be a finite, non-dyadic place of $k$, let $i \in I_0$, and let $Q$ a quadratic space over $k_v$ with module $M_i$.*

(a) *Suppose that $i \neq 0$. Then there exists a quadratic space $Q'$ over $k_v$ having an isometry with module $M_i$ such that $w(Q') = 0$.*
(b) *Suppose that $i = 0$, and let $d \in k^*/k^{*2}$. Then there exists a quadratic space $Q'$ over $k_v$ having an isometry with module $M_0$ such that $w(Q') = 0$ and $\det(Q') = d$.*

*Proof.* (a) If $w(Q) = 0$, there is nothing to prove. Suppose that $w(Q) = 1$. Since $v$ is non-dyadic, this implies that the quadratic space $Q$ is not hyperbolic. Therefore by Cor. 3.5 the module $M_i$ is not hyperbolic over $k_v$. By Lemma 9.6, there exists a quadratic space $Q'$ over $k_v$ having an isometry with module $M_i$ such that $w(Q') = 0$.

(b) Set $n_0 = \dim(M_0)$, and let $Q'$ be the $n_0$-dimensional quadratic space

$$Q' = \langle 1, \dots, 1, d \rangle.$$

Then $\det(Q') = d$ and $w(Q') = 0$. As any quadratic space of dimension $n_0$ has an isometry with module $M_0$, this completes the proof of the lemma. □

In order to give a necessary and sufficient condition for the Hasse principle to hold, the first step is to show that $\mathcal{C}_{M,q}$ contains a collection $C = \{q_i^v\}$ in $\mathcal{C}_{M,q}$ such that $w(q_i^v) = 0$

for almost all places $v$ of $k$ and all $i \in I_0$. Recall that $D = \sum_{i<j}(d_i, d_j) \in \mathrm{Br}_2(k)$. Let $T$ be the set of places $v$ of $k$ such that $D_v \neq 0$, and let $S$ be the set of places of $k$ at which the Hasse invariant of $q$ is not equal to the Hasse invariant of the hyperbolic space of dimension equal to $\dim(q)$. Let $\Sigma_2$ be the set of dyadic places and $\Sigma_\infty$ the set of infinite places of $k$. Set $\Sigma = S \cup T \cup \Sigma_2 \cup \Sigma_\infty$. Note that $\Sigma$ is a finite subset of $\Sigma_k$.

**Proposition 10.7.** *The set $\mathcal{C}_{M,q}$ contains a collection $C = \{q_i^v\}$ of quadratic forms defined over $k_v$ such that $w(q_i^v) = 0$ for all $v \notin \Sigma$ and all $i \in I_0$.*

*Proof.* Let $\tilde{C} = \{\tilde{q}_i^v\} \in \mathcal{C}_{M,q}$. Let $v$ be a place of $k$ such that $v \notin \Sigma$ and suppose that $|S_v(\tilde{C})| \neq 0$. It suffices to show that there exists a collection $C \in \mathcal{C}_{M,q}$ with $|S_v(C)| < |S_v(\tilde{C})|$.

Set $w_i^v = w(\tilde{q}_i^v)$. We are supposing that $|S_v(\tilde{C})| \neq 0$, hence there exists an $i$ with $w_i^v = 1$. Since $v \notin S \cup \Sigma_2$, we have $w(q)_v = 0$. Moreover $v \notin T$, hence $w(q)_v = w_0^v + \cdots + w_r^v$. Thus there exists $j \neq i$ such that $w_j^v = 1$.

By Lemma 10.6 there exist quadratic spaces $q_i^v$ and $q_j^v$ over $k_v$ having isometries with module $M_i$ respectively $M_j$ such that $\det(q_i^v) = d_i$, $\det(q_j^v) = d_j$ and $w(q_i^v) = w(q_j^v) = 0$. Set $q_\alpha^v = \tilde{q}_\alpha^v$ if $\alpha \neq i, j$.

Set $C = \{q_i^v\}$. Then $C$ satisfies the conditions of Cor. 10.5, hence $C \in \mathcal{C}_{M,q}$. Note that $C = \{q_i^v\}$ satisfies $w(q_i^v) = w(q_j^v) = 0$, therefore $|S_v(C)| < |S_v(\tilde{C})|$. This completes the proof of the proposition. □

For any collection $C = \{q_i^v\} \in \mathcal{C}_{M,q}$ and all $i \in I_0$, set

$$T_i(C) = \{v \in \Sigma_k \mid w(q_i^v) = 1\}.$$

Let $\mathcal{F}_{M,q}$ be the subset of $\mathcal{C}_{M,q}$ consisting of the collections $C = \{q_i^v\}$ of quadratic spaces over $k_v$ such that for all $i \in I_0$, the set $T_i(C)$ is finite.

**Theorem 10.8.** *Suppose that $q$ has an isometry with module $M$ over $k_v$ for all places $v$ of $k$. Then $q$ has an isometry with module $M$ if and only if there exists a collection $C = \{q_i^v\} \in \mathcal{F}_{M,q}$ such that for all $i \in I_0$, the cardinality of $T_i(C)$ is even.*

*Proof.* Suppose that $q$ has an isometry with module $M$. Then by Prop. 10.2, there exist quadratic spaces $q_0, \dots, q_r$ defined over $k$ such that $q \simeq q_0 \oplus \cdots \oplus q_r$, and the quadratic space $q_i$ has an isometry with module $M_i$ over all the completions of $k$ for all $i \in I_0$. Let $q_i^v = q_i \otimes_k k_v$, and let $C = \{q_i^v\}$. Then $C \in \mathcal{C}_{q,M}$, and for all $i = 0, \dots, r$, the set $T_i(C)$ is finite of even cardinality.

Conversely, let $C = \{q_i^v\} \in \mathcal{F}_{M,q}$ be such that $T_i(C)$ has even cardinality for all $i \in I_0$. Recall that as the quadratic space $q_i^v$ has an isometry with module $M_i$, we have $\dim(q_i^v) = \dim(M_i)$ and $\det(q_i^v) = d_i \in k_v^*/k_v^{*2}$ for all places $v$ of $k$, and all $i \in I_0$. Therefore by [O'M73, Chapter VII, Th. 72.1], for all $i \in I_0$ there exists a quadratic space $q_i$ such that $q_i \otimes_k k_v \simeq q_i^v$ for all $v \in \Sigma_k$. We have $q \simeq q_0 \oplus \cdots \oplus q_r$ over $k_v$ for all $v$, hence by the Hasse–Minkowski theorem we have $q \simeq q_0 \oplus \cdots \oplus q_r$. Therefore by Th. 10.1, the quadratic space $q$ has an isometry with module $M$. □

For any module $N$ and any $d \in k^*$, let $\Omega(N, d)$ be the set of finite places $v$ of $k$ such that for any $\epsilon \in \{0, 1\}$, there exists a quadratic space $Q$ over $k_v$ with $\mathrm{disc}(Q) = d$ and $w(Q) = \epsilon$ having an isometry with module $N \otimes_k k_v$.

For all $i, j \in I_0$, let $\Omega_{i,j} = \Omega(M_i, d_i) \cap \Omega(M_j, d_j)$.

**Remark 10.9.** Note that if $i, j \in I$, then $\Omega_{i,j}$ does not depend on $q$. If $M_0 \neq 0$ and $i = 0$, then $\Omega_{i,j}$ depends on $d_0 = \det(q) d_1 \dots d_r$.

Recall that for any collection $C = \{q_i^v\} \in \mathcal{F}_{M,q}$ and all $i \in I_0$, we have

$$T_i(C) = \{v \in \Sigma_k \mid w(q_i^v) = 1\}.$$

**Definition 10.10.** We say that $C = (q_i^v) \in \mathcal{F}_{M,q}$ is *connected* if for all $i \in I$ such that $|T_i(C)|$ is odd, there exist $j \in I$ with $j \neq i$ such that $|T_j(C)|$ is odd, and a chain $i = i_1, \dots, i_m = j$ of elements of $I$ with $\Omega_{i_t, i_{t+1}} \neq \emptyset$ for all $t = 1, \dots, m-1$. We say that $\mathcal{F}_{M,q}$ is *connected* if it contains a connected element.

**Theorem 10.11.** (a) *The quadratic space $q$ has an isometry with module $M$ over $k_v$ for all $v \in \Sigma_k$ if and only if $\mathcal{F}_{M,q}$ is not empty.*

(b) *The quadratic space $q$ has an isometry with module $M$ over $k$ if and only if $\mathcal{F}_{M,q}$ is connected.*

*Proof.* (a) It is clear that if $\mathcal{F}_{M,q} \neq \emptyset$, then the quadratic space $q$ has an isometry with module $M$ over $k_v$ all $v \in \Sigma_k$. The converse follows from Lemma 10.3 and Th. 10.7.

(b) If the quadratic space $q$ has an isometry with module $M$, then there exist quadratic spaces $q_0, \dots, q_r$ over $k$ such that $q \simeq q_0 \oplus \dots \oplus q_r$ and $q_i$ has an isometry with module $M_i$ for all $i \in I_0$. Set $q_i^v = q_i \otimes_k k_v$, and let $C = (q_i^v)$. Then $C \in \mathcal{F}_{M,q}$, and $|T_i(C)|$ is even for all $i \in I_0$. Therefore $C$ is a connected element of $\mathcal{F}_{M,q}$, hence $\mathcal{F}_{M,q}$ is connected.

Conversely, suppose that $\mathcal{F}_{M,q}$ is connected, and let $C = (q_i^v) \in \mathcal{F}_{M,q}$ be a connected element. Suppose that for some $i \in I_0$, the integer $|T_i(C)|$ is odd. Since $C$ is connected, there exist $j \in I_0$ with $j \neq i$ such that $|T_j(C)|$ is odd, and a chain $i = i_1, \dots, i_m = j$ of elements of $I$ with $\Omega_{i_t, i_{t+1}} \neq \emptyset$ for all $t = 1, \dots, m-1$. Let $v_t \in \Omega_{i_t, i_{t+1}}$. Then there exist quadratic spaces $\tilde{q}_t^{v_t}$ over $k_v$ with $w(\tilde{q}_t^{v_t}) \neq w(q_t^{v_t})$ and $\det(\tilde{q}_t^{v_t}) = d_t$ having an isometry with module $M_t$. Set $\tilde{q}_s^u = q_s^u$ if $(u, s) \neq (v_t, t)$. Set $\tilde{C} = (\tilde{q}_i^v)$; then $\tilde{C} \in \mathcal{F}_{M,q}$. We have $|T_i(\tilde{C})| \equiv 0 \pmod 2$, $|T_j(\tilde{C})| \equiv 0 \pmod 2$, and $|T_s(\tilde{C})| \equiv |T_s(C)| \pmod 2$ if $s \neq i, j$. Repeating this procedure we obtain a family of quadratic spaces $C' \in \mathcal{F}_{M,q}$ such that $|T_i(C)|$ is even for all $i \in I_0$. By Th. 10.8 this implies that $q$ has an isometry with module $M$. □

Note that condition (a) does not imply condition (b) in general (in other words, there are counter-examples to the Hasse principle): this follows from the examples of Prasad and Rapinchuk [PR10, Example 7.5].

## 11. The case of modules of mixed type

The aim of this section and the next is to give some applications of Th. 10.11. We keep the notation of the previous section; in particular, $k$ is a global field and $\Sigma_k$ is the set of places of $k$. Recall that $M$ is semisimple, and $M \simeq M^0 \oplus M^1$ with $M^0$ of type 0 and $M^1$

of type 1. If $M^1 = 0$, then we already have a complete criterion for the existence of an isometry with module $M$ (see Prop. 4.3). In this section, we consider the case where both $M^0$ and $M^1$ are non-zero. As we will see, the case where $\dim(M^0) \geq 3$ is especially simple, and will be considered first. Then we examine the case where $\dim(M^0) = 2$ or 1. Let $q$ be a quadratic space over $k$, and assume that $\dim(q) = \dim(M)$.

**Definition 11.1.** For every real place $v$ of $k$, let $(r_v, s_v)$ denote the signature of $q$ over $k_v$, and let $\sigma_v$ be the number of roots of $F_M \in k_v[X]$ that are not on the unit circle. We say that the *signature conditions* are satisfied if for every real place $v$ of $k$, we have $(r_v, s_v) \geq (\sigma_v, \sigma_v)$, and if moreover $M^0 = 0$, then $(r_v, s_v) \equiv (\sigma_v, \sigma_v) \pmod 2$.

**Proposition 11.2.** *Suppose that* $\dim(M^0) \geq 3$. *Then the quadratic space $q$ has an isometry with module $M$ if and only if the signature conditions are satisfied.*

The proof of Prop. 11.2, as well that of several other results of Sections 11 and 12, is based on Prop. 11.3 below. With the notation of §10, we have:

**Proposition 11.3.** *Suppose that there exists $i_0 \in I_0$ such that $\Omega_{i_0,i}(q) \neq \emptyset$ for all $i \in I_0$. Suppose that the quadratic space $q$ has an isometry with module $M$ over every completion of $k$. Then $q$ has an isometry with module $M$.*

For the proof of Prop. 11.3, we need the following lemmas. We use the notation of §10.

**Lemma 11.4.** *Let $C \in \mathcal{F}_{M,q}$. Then*

$$\sum_{v \in \Sigma_k} |S_v(C)| \equiv 0 \pmod 2.$$

*Proof.* By Prop. 10.4, we have

$$|S_v(C)| \equiv w(q)_v + D_v \pmod 2$$

for all $v \in \Sigma_k$. Hence

$$\sum_{v \in \Sigma_k} |S_v(C)| \equiv \sum_{v \in \Sigma_k} w(q)_v + \sum_{v \in \Sigma_k} D_v \pmod 2.$$

As $w(q)$ and $D$ are elements of $\mathrm{Br}_2(k)$, we have

$$\sum_{v \in \Sigma_k} w(q)_v \equiv 0 \pmod 2, \qquad \sum_{v \in \Sigma_k} D_v \equiv 0 \pmod 2.$$

This implies that $\sum_{v \in \Sigma_k} |S_v(C)| \equiv 0 \pmod 2$, as claimed. ◻

**Lemma 11.5.** *Let $C \in \mathcal{F}_{M,q}$. Then*

$$\sum_{i \in I_0} |T_i(C)| \equiv 0 \pmod 2.$$

*Proof.* Indeed, we have

$$\sum_{i\in I_0} |T_i(C)| = \sum_{v\in\Sigma_k} |S_v(C)|.$$

By Lemma 11.4, $\sum_{v\in\Sigma_k} |S_v(C)| \equiv 0 \pmod 2$, hence $\sum_{i\in I} |T_i(C)| \equiv 0 \pmod 2$.

*Proof of Proposition 11.3.* By Th. 10.11(a), the set $\mathcal{F}_{M,q}$ is not empty. Let $C = (q_i^v) \in \mathcal{F}_{M,q}$, and let $i \in I_0$ be such that $|T_i(C)|$ is odd. By Lemma 11.5, we have $\sum_{i\in I_0} |T_i(C)| \equiv 0 \pmod 2$, hence there exists $j \in I_0$ with $j \neq i$ such that $|T_i(C)|$ is odd. By hypothesis, we have $\Omega_{i,i_0} \neq \emptyset$ and $\Omega_{j,i_0} \neq \emptyset$, hence $C$ is connected. Therefore $\mathcal{F}_{M,q}$ is connected, hence by Th. 10.11(b) the quadratic space $q$ has an isometry with module $M$.

**Lemma 11.6.** *Let $N$ be a module of type* 0*, and let $d \in k^*$.*

(a) *If* $\dim(N) \geq 3$*, then every finite place of $k$ is in $\Omega(N, d)$.*
(b) *If* $\dim(N) = 2$ *and $d \neq -1$ in $k^*/k^{*2}$, then every finite place of $k$ is in $\Omega(N, d)$.*

*Proof.* Since $N$ is of type 0, every quadratic space of dimension equal to $\dim(N)$ has an isometry with module $N$. Therefore the result follows from [O'M73, 63:23]. □

**Proposition 11.7.** *Suppose that* $\dim(M^0) \geq 3$*, or* $\dim(M^0) = 2$ *and* $\det(q) \neq -d_1 \dots d_r$ *in $k^*/k^{*2}$. If the quadratic space $q$ has an isometry with module $M$ over every completion of $k$, then $q$ has an isometry with module $M$ over $k$.*

*Proof.* If $\dim(M^0) \geq 3$, then Lemma 11.6(a) implies that every finite place of $k$ is in $\Omega(M^0, d_0)$. Therefore $\Omega_{0,i} \neq \emptyset$ for all $i \in I_0$. Suppose that $\dim(M^0) = 2$ and $\det(q) \neq -d_1 \dots d_r$ in $k^*/k^{*2}$. Recall that $d_0 = \det(q)d_1 \dots d_r$ in $k^*/k^{*2}$. Hence $d_0 \neq -1$ in $k^*/k^{*2}$, and therefore by Lemma 11.6(b) every finite place of $k$ is in $\Omega(M^0, d_0)$. This implies that $\Omega_{0,i} \neq \emptyset$ for all $i \in I_0$ in this case as well, and hence the proposition follows from Prop. 11.3. □

*Proof of Proposition 11.2.* The necessity of the signature conditions follows from Prop. 8.1. Let us show that they are also sufficient. By Prop. 11.7, it suffices to show that $q$ has an isometry with module $M$ over $k_v$ for all $v \in \Sigma_k$. For real places, this is a consequence of Prop. 8.1. Let $v$ be a finite place. If $M^1 = 0$, then $M$ is of type 0, and every quadratic space of dimension equal to $\dim(M)$ has an isometry with module $M$. Suppose that $M^1 \neq 0$, and note that this implies that $\dim(M^1) \geq 2$. We have $M^1 \otimes_k k_v \simeq N_1^v \oplus N_2^v$ where $N_1^v$ is of type 1 and $N_2^v$ of type 2. If $N_1^v \neq 0$, then the result follows from Th. 7.1. Suppose that $N_1^v = 0$, and let $2m_2 = \dim(N_2^v)$. Since $\dim(M^1) \geq 2$, we have $\dim(M) \geq 5$, hence $q$ is isotropic over $k_v$, and its Witt index is $\geq m_2$. By Prop. 4.3, this implies that $q$ has an isometry with module $M$ over $k_v$. This concludes the proof of the proposition. □

**Proposition 11.9.** *Suppose that* $\dim(M^0) = 2$ *and* $\det(q) \neq -d_1 \dots d_r$ *in $k^*/k^{*2}$. Then $q$ has an isometry with module $M$ if and only the following two conditions hold:*

(a) *The signature conditions are satisfied.*
(b) *If $v$ is a finite place and if $M^1 \otimes_k k_v$ is hyperbolic, then the Witt index of $q$ over $k_v$ is* $\geq \frac{1}{2}\dim(M^1)$.

*Proof.* By Prop. 11.7, we have to show that the conditions hold if and only if $q$ has an isometry with module $M$ over $k_v$ for all $v$. For real places, this is a consequence of Prop. 8.1. Let $v$ be a finite place. Recall that $M^1 \neq 0$, and let $M^1 \otimes_k k_v \simeq N_1^v \oplus N_2^v$ where $N_1^v$ is of type 1 and $N_2^v$ of type 2. If $N_1^v \neq 0$, then the result follows from Th. 7.1. Suppose that $N_1^v = 0$, and note that this means that $M^1 \otimes_k k_v$ is hyperbolic. By Prop. 4.3, this implies that $q$ has an isometry with module $M$ over $k_v$ if and only if the Witt index of $q$ over $k_v$ is $\geq \frac{1}{2}\dim(M^1)$, and this is precisely condition (b). □

**Proposition 11.10.** *Suppose that* $\dim(M^0) = 2$ *and* $\det(q) = -d_1 \dots d_r$ *in* $k^*/k^{*2}$. *Then* $q$ *has an isometry with module* $M$ *if and only if* $q \simeq q_0 \oplus q'$ *where* $q_0$ *is a hyperbolic plane, and* $q'$ *is a quadratic space over* $k$ *having an isometry with module* $M^1$.

*Proof.* If $q \simeq q_0 \oplus q'$ with $q_0$ a hyperbolic plane and $q'$ a quadratic space having an isometry with module $M^1$, then $q$ has an isometry with module $M$.

Conversely, suppose that $q$ has an isometry with module $M$. Then $q \simeq q_0 \oplus q'$ with $q_0$ having an isometry with module $M_0$, and $q'$ having an isometry with module $M^1$. As $M^1$ is of type 1, we have $\det(q') = d_1 \dots d_r$ in $k^*/k^{*2}$. By hypothesis, $\det(q) = -d_1 \dots d_r$ in $k^*/k^{*2}$. Therefore $\det(q_0) = -1$ in $k^*/k^{*2}$. Since $\dim(q_0) = 2$, this implies that $q_0$ is isomorphic to a hyperbolic plane. □

Recall that $d_0 = \det(q)d_1 \dots d_r$ in $k^*/k^{*2}$.

**Proposition 11.11.** *Suppose that* $\dim(M^0) = 1$. *Then* $q$ *has an isometry with module* $M$ *if and only if* $q \simeq q_0 \oplus q'$ *where* $q_0 \simeq \langle d_0 \rangle$ *and* $q'$ *is a quadratic space having an isometry with module* $M^1$.

*Proof.* If $q \simeq q_0 \oplus q'$ with $q_0 \simeq \langle d_0 \rangle$ and $q'$ having an isometry with module $M^1$, then $q$ has an isometry with module $M$.

Conversely, suppose that $q$ has an isometry with module $M$. Then $q \simeq q_0 \oplus q'$ with $q_0$ having an isometry with module $M_0$, and $q'$ having an isometry with module $M^1$. As $M^1$ is of type 1, we have $\det(q') = d_1 \dots d_r$ in $k^*/k^{*2}$. Since $\dim(q_0) = \dim(M_0) = 1$ and $d_0 = \det(q)d_1 \dots d_r$ in $k^*/k^{*2}$, we have $q_0 \simeq \langle d_0 \rangle$. □

## 12. Modules of type 1

We keep the notation of Sections 10 and 11. In particular, $k$ is a global field and $\Sigma_k$ is the set of places of $k$. In this section, we assume that $M$ is a semisimple module of type 1. Recall that this means that $M \simeq M_1 \oplus \dots \oplus M_r$, where $M_i = [k[X]/(f_i)]^{n_i}$ for some symmetric, irreducible polynomials $f_i \in k[X]$ of even degree, and for some $n_i \in \mathbb{N}$. We use the notation $I = \{1, \dots, r\}$, and $K_i = k[X]/(f_i)$. Let $q$ be a quadratic space over $k$ such that $\dim(q) = \dim(M)$. Recall that we denote by $F_M \in k[X]$ the characteristic polynomial of $M$. If $v$ is a real place of $k$, then we denote by $(r_v, s_v)$ the signature of $q$ at $v$, and by $\sigma_v$ the number of roots of $F_M$ off the unit circle.

Recall that the *signature conditions* are satisfied for $q$ and $M$ if for all real places $v$ of $k$, we have $(r_v, s_v) \geq (\sigma_v, \sigma_v)$, and $(r_v, s_v) \equiv (\sigma_v, \sigma_v) \pmod 2$.

We say that the *hyperbolicity conditions* are satisfied for $q$ and $M$ if for all places $v$ of $k$ such that $M \otimes_k k_v$ is a hyperbolic module (that is, a module of type 2), the quadratic form $q_v$ over $k_v$ is hyperbolic.

We have the following

**Theorem 12.1.** *The quadratic space $q$ has an isometry with module $M$ over all the completions of $k$ if and only if the signature conditions and the hyperbolicity conditions are satisfied and* $\det(q) = F_M(1)F_M(-1)$ *in* $k^*/k^{*2}$.

*Proof.* This follows from Cor. 3.5, Cor. 7.4, and Prop. 8.1. □

We will see that the necessary and sufficient conditions of Th. 10.11 can be interpreted in terms of splitting properties of the fields $K_i$. We start with a few lemmas. Let us recall that for any module $N$ and any $d \in k^*$, we denote by $\Omega(N, d)$ the set of finite places $v$ of $k$ such that for any $\epsilon \in \{0, 1\}$, there exists a quadratic space $Q$ over $k_v$ with $\operatorname{disc}(Q) = d$ and $w(Q) = \epsilon$ having an isometry with module $N$.

**Lemma 12.2.** *Let $f \in k[X]$ be a symmetric, irreducible polynomial of even degree, and $m \in \mathbb{N}$. Set $N = [k[X]/(f)]^m$ and let $d = [f(1)f(-1)]^m$. Let $v$ be a finite place of $k$. Then $v \in \Omega(N, d)$ if and only if $N \otimes_k k_v$ is not hyperbolic.*

*Proof.* If $N \otimes_k k_v$ is hyperbolic, then every quadratic space with module $N \otimes_k k_v$ is hyperbolic (cf. Cor. 3.5), therefore $v \notin \Omega(N, d)$. Conversely, suppose that $N \otimes_k k_v$ is not hyperbolic. Then by Lemma 9.6, for any $\epsilon \in \{0, 1\}$ there exists a quadratic space $Q$ having an isometry with module $N \otimes_k k_v$ such that $w(Q) = \epsilon$. By Cor. 5.2, we have $\det(Q) = d$, hence $v \in \Omega(N, d)$. □

**Notation 12.3.** Let $E$ be an extension of finite degree of $k$, let $K$ be a quadratic extension of $E$, and let $x \mapsto \overline{x}$ be the non-trivial automorphism of $K$ over $E$. Let us denote by $\Sigma^{\mathrm{s}}(K)$ the set of $v \in \Sigma_k$ such that every place of $E$ above $v$ splits in $K$. Let $\Sigma^{\mathrm{ns}}(K)$ be the complement of $\Sigma^{\mathrm{s}}(K)$ in $\Sigma_k$; in other words, the set of $v \in \Sigma_k$ such that there exists a place of $E$ above $v$ that is not split in $K$.

Let $\Sigma'_k$ be the set of finite places of $k$. Then we have

**Lemma 12.4.** *For all $i \in I$, we have $\Omega(M_i, d_i) = \Sigma^{\mathrm{ns}}(K_i) \cap \Sigma'_k$.*

*Proof.* Let $v \in \Sigma'_k$. By Lemma 9.4, we have $v \in \Sigma^{\mathrm{ns}}(K_i)$ if and only if $M_i \otimes_k k_v$ is not hyperbolic, and by Lemma 12.2 this is equivalent to $v \in \Omega(M_i, d_i)$. □

For all $i, j \in I$, set $\Sigma^{\mathrm{ns}}_{i,j} = \Sigma^{\mathrm{ns}}(K_i) \cap \Sigma^{\mathrm{ns}}(K_j)$.

**Theorem 12.5.** *Assume that there exists $i_0 \in I$ such that $\Sigma^{\mathrm{ns}}_{i_0,i} \neq \emptyset$ for all $i \in I$. Suppose that $q$ has an isometry with module $M$ over all the completions of $k$. Then $q$ has an isometry with module $M$.*

*Proof.* Let $i \in I$, and let us show that there exists a finite place $v$ of $k$ such that $v \in \Sigma^{\mathrm{ns}}_{i_0,i}$. Indeed, let $u$ be a real place of $k$ with $u \in \Sigma^{\mathrm{ns}}_{i_0,i}$. Let $L$ be a Galois extension of $k$ containing the fields $K_{i_0}$ and $K_i$, and let $G = \mathrm{Gal}(L/k)$. Let us denote by $c$ the conjugacy class of the complex conjugation in $G$ corresponding to an extension of the place $u$ to $L$.

By the Chebotarev density theorem, there exists a finite place $v$ of $k$ such that the conjugacy class of the Frobenius automorphism at $v$ is equal to $c$. Let $v$ be such a place. Then all the places of $E_{i_0}$, respectively $E_i$, above $v$ are inert in $K_{i_0}$, respectively $K_i$. Therefore, $v \in \Sigma^{\mathrm{ns}}_{i_0,i} = \Sigma^{\mathrm{ns}}(K_{i_0}) \cap \Sigma^{\mathrm{ns}}(K_i)$. Since $v \in \Sigma'_k$, by Lemma 12.4 this implies that $v \in \Omega(M_{i_0}, d_{i_0}) \cap \Omega(M_i, d_i) = \Omega_{i_0,i}$. Thus $\Omega_{i_0,i} \neq \emptyset$ for all $i \in I$. Since here $I = I_0$, Prop. 11.3 gives the desired result. □

**Corollary 12.6.** *Suppose that there exists $i_0 \in I$ such that $\Sigma^{\mathrm{ns}}_{i_0,i} \neq \emptyset$ for all $i \in I$. Then $q$ has an isometry with module $M$ if and only if the hyperbolicity and signature conditions are satisfied and* $\det(q) = F_M(1)F_M(-1)$ *in $k^*/k^{*2}$.*

*Proof.* This follows from Ths. 12.1 and 12.5. □

The hypotheses of Th. 12.5 and Cor. 12.6 are often satisfied; for instance, we have

**Corollary 12.7.** *Suppose that there exists a real place $v$ of $k$ such that all the roots of $F_M \in k_v[X]$ are on the unit circle. Then $q$ has an isometry with module $M$ if and only if the hyperbolicity and signature conditions are satisfied and* $\det(q) = F_M(1)F_M(-1)$ *in $k^*/k^{*2}$.*

*Proof.* Indeed, we have $v \in \Sigma^{\mathrm{ns}}(K_i)$ for all $i \in I$, hence $v \in \Sigma^{\mathrm{ns}}_{i,j}$ for all $i, j$. Therefore the result follows from Cor. 12.6. □

Recall that a number field is CM if it is a totally imaginary quadratic extension of a totally real number field. We say that the module $M$ is of *type CM* if $k = \mathbb{Q}$ and the fields $K_i$ are CM fields for all $i \in I$. □

**Corollary 12.8.** *Suppose that $M$ is a module of type CM. Then the quadratic space $q$ has an isometry with module $M$ if and only if the hyperbolicity conditions are satisfied,* $\det(q) = F_M(1)F_M(-1)$ *in $k^*/k^{*2}$, and the signature of $q$ is even.*

*Proof.* Indeed, the hypothesis of Cor. 12.7 is satisfied, and the signature condition amounts to saying that the signature of $q$ is even. □

## References

[B84] Bayer-Fluckiger, E.: Definite unimodular lattices having an automorphism of given characteristic polynomial. Comment. Math. Helv. **59**, 509–538 (1984) Zbl 0558.10029 MR 0780074

[B99] Bayer-Fluckiger, E.: Lattices and number fields. In: Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math. 241, Amer. Math. Soc., Providence, RI, 69–84 (1999) Zbl 0951.11016 MR 1718137

[B14] Bayer-Fluckiger, E.: Embeddings of tori in orthogonal groups. Ann. Inst. Fourier (Grenoble) **64**, 113–125 (2014) Zbl 06387267

[BMa13] Bayer-Fluckiger, E., Maciak, P.: Upper bounds for the Euclidean minima of abelian fields of odd prime power conductor. Math. Ann. **357**, 1071–1089 (2013) Zbl 1278.11098 MR 3118625

[BMar94] Bayer-Fluckiger, E., Martinet, J.: Formes quadratiques liées aux algèbres semi-simples. J. Reine Angew. Math. **451**, 51–69 (1994) Zbl 0801.11020 MR 1277294

[BCM03] Brusamarello, R., Chuard-Koulmann, P., Morales, J.: Orthogonal groups containing a given maximal torus. J. Algebra **266**, 87–101 (2003) Zbl 1079.11023 MR 1994530

[F12] Fiori, A.: Characterization of special points of orthogonal symmetric spaces. J. Algebra **372**, 397–419 (2012) Zbl 06180116 MR 2990017

[GR13] Garibaldi, S., Rapinchuk, A.: Weakly commensurable S-arithmetic subgroups in almost simple algebraic groups of types B and C. Algebra Number Theory **7**, 1147–1178 (2013) Zbl 1285.20045 MR 3101075

[G04] Gille, P.: Type des tores maximaux des groupes semi-simples. J. Ramanujan Math. Soc. **19**, 213–230 (2004) Zbl 1193.20057 MR 2139505

[GM02] Gross, B., McMullen, C. T.: Automorphisms of even unimodular lattices and unramified Salem numbers. J. Algebra **257**, 265–290 (2002) Zbl 1022.11016 MR 1947324

[JRV14] Jouve, F., Rodriguez-Villegas, F.: On the bilinear structure associated to Bezoutians. J. Algebra **400**, 161–184 (2014) Zbl 1296.11023 MR 3147370

[Lee14] Lee, T.-Y.: Embedding functors and their arithmetic properties. Comment. Math. Helv. **89**, 671–717 (2014) Zbl 06361417 MR 3260846

[Le69] Levine, J.: Invariants of knot cobordism. Invent. Math **8**, 98–110 (1969) Zbl 0179.52401 MR 0253348

[Le80] Levine, J.: Algebraic Structure of Knot Modules. Lecture Notes in Math. 772, Springer, Berlin (1980) Zbl 0422.57007 MR 0564114

[M69] Milnor, J.: Isometries of inner product spaces. Invent. Math. **8**, 83–97 (1969) Zbl 0177.05204 MR 0249519

[O'M73] O'Meara, O. T.: Introduction to Quadratic Forms. Reprint of the 1973 edition. Classics Math., Springer, Berlin (2000) Zbl 1034.11003 MR 1754311

[PR10] Prasad, G., Rapinchuk, A. S.: Local-global principles for embedding of fields with involution into simple algebras with involution. Comment. Math. Helv. **85**, 583–645 (2010) Zbl 1223.11047 MR 2653693

[Sch85] Scharlau, W.: Quadratic and Hermitian Forms. Grundlehren Math. Wiss. 270, Springer, Berlin (1985) Zbl 0584.10010 MR 0770063