Groups Geom. Dyn. 18 (2024), 1427–1467
DOI 10.4171/GGD/787

Higher-dimensional digraphs from cube complexes and their spectral theory

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Abstract. We define k-dimensional digraphs and initiate a study of their spectral theory. The k-dimensional digraphs can be viewed as generating graphs for small categories called k-graphs. Guided by geometric insight, we obtain several new series of k-graphs using cube complexes covered by Cartesian products of trees, for $k \geq 2$. These k-graphs can not be presented as virtual products and constitute novel models of such small categories. The constructions yield rank-k Cuntz–Krieger algebras for all $k \geq 2$. We introduce Ramanujan k-graphs satisfying optimal spectral gap property and show explicitly how to construct the underlying k-digraphs.

1. Introduction

The study of higher-rank graphs and their C^*-algebras originates in the work of Robertson and Steger in [27] and expanded into a very active direction of research in operator algebras following the work of Kumjian–Pask [13], where the term k-graph was formalised. A higher-rank graph, or k-graph, with $k \geq 1$, is a small category with a functor into the monoid $\mathbb{N}^k$ that enjoys a unique factorisation property. While many important structural results about higher-rank graph C^*-algebras have been obtained, the supply of examples of k-graphs for $k \geq 3$ is limited. The main contribution of the present paper is to provide new, infinite series of examples of higher-rank graphs. Our main technical innovation is the concept of a k-dimensional digraph, where $k \geq 2$. Using input from the geometric group theory, we obtain several infinite series of explicit constructions of k-dimensional digraphs for $k \geq 3$, and from these we obtain novel examples of higher-rank graphs with rank at least 3 which are not of product type, in particular are not skew-products, as will be explained.

A k-dimensional digraph is a generalisation of a directed graph, or digraph. It is known that 1-graphs are free categories defined by directed graphs, see, e.g., [13] or [15, Proposition 3.12]. Our definition of k-dimensional digraph is made so that the natural category associated with it will be a k-graph. This definition is as follows.

Definition 1.1. Let $k \geq 2$ be a positive integer. A k-dimensional digraph

$$\mathrm{DG} = (V, E, o, t, k, \phi)$$

Mathematics Subject Classification 2020: 47D03 (primary); 20J99 (secondary).
Keywords: higher-dimensional digraphs, cube complexes, C^*-algebras.

is a directed graph with V a finite set of vertices, E finite set of edges, maps $o, t : E \to V$ which determine the origin and terminus of each $x \in E$, and the property that the edge set decomposes as a disjoint union $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k$ with E_i for $i = 1, \ldots, k$ regarded as edges of colour i, such that there is a bijection $\phi : Y \to Y$ on the set $Y := Y_{\mathrm{DG}}$ of all directed paths of length two formed of edges of colours given by ordered pairs (i, j) with $i \neq j$ in $\{1, 2, \ldots, k\}$, satisfying the following properties:

(F1) If xy is a path of length two with x of colour i and y of colour j, then $\phi(xy) = y'x'$ for a unique pair (y', x') where y' has colour j, x' has colour i, and the origin and terminus vertices of the paths xy and $y'x'$ coincide. We write this as $xy \sim y'x'$ and note that $\phi^2 = 1$.

(F2) For all $x \in E_i$, $y \in E_j$, and $z \in E_l$ so that xyz is a path on E, where i, j, l are distinct colours, if $x_1, x_2, x^1, x^2 \in E_i$, $y_1, y_2, y^1, y^2 \in E_j$, and $z_1, z_2, z^1, z^2 \in E_l$ satisfy

$$xy \sim y^1x^1, \quad x^1z \sim z^1x^2, \quad y^1z^1 \sim z^2y^2$$

and

$$yz \sim z_1y_1, \quad xz_1 \sim z_2x_1, \quad x_1y_1 \sim y_2x_2,$$

it follows that $x_2 = x^2$, $y_2 = y^2$, and $z_2 = z^2$.

One of the main motivations for this definition is to have a purely combinatorial finite set of data to deal with as input to defining a C^*-algebra. The definition of a k-graph in [13] involves an infinite category. In [11], Hazlewood, Raeburn, Sims, and Webster present an explicit construction of the (unique in the appropriate sense) k-graph associated with the data of a combinatorial graph with a prescription of colouring of edges and a complete and associative set of squares. While the complete and associative collection of squares contains the same combinatorial information as our k-dimensional digraph, the advantage of our concept of k-dimensional digraph is two-fold, as it involves only a finite number of conditions to be checked without reference to graph morphisms in an infinite category, and because it allows for providing a rich supply of examples from k-cube complexes via the geometric group theory. More precisely, the latter appeals to the theory of finitely-present groups with finite-index subgroups. When there are infinite-index subgroups, our approach could be used to construct k-graphs on infinitely many vertices from finite data.

If we seek for an analogy from topology, we may talk about a CW-complex having a simply-connected universal cover, so our k-dimensional digraph is an analogue of the complex and the k-graph in the existing literature is an analogue of the universal cover.

The constructions in [27] provided a generalisation of the Cuntz–Krieger algebras from topological Markov shifts introduced in [5]. They were motivated by an observation of Spielberg [33] clarifying that a free group Γ on finitely many generators, viewed as the fundamental group of a finite connected graph, acts on the boundary Ω of its Cayley graph such that in the associated crossed product $C(\Omega) \rtimes \Gamma$, one finds generating partial isometries for an ordinary Cuntz–Krieger algebra $\mathcal{A}$ associated with a matrix M that records incidence in the universal covering graph. This provided the basis for defining a *higher-rank Cuntz–Krieger* algebra $\mathcal{A}$ in [27]: the input consists of a finite alphabet and a family of commuting $(0, 1)$-matrices $M_1, M_2, \ldots, M_k$, with $k \geq 1$, having entries in the alphabet, satisfying a number of conditions, and controlling the formation of words of $\mathbb{N}^k$-valued shape. Shortly after [27], Kumjian–Pask [13] defined a k-graph Λ as a small category with an $\mathbb{N}^k$-valued degree of morphisms modelling the formation of words from [27] and with an associated C^*-algebra $C^*(\Lambda)$ generated by partial isometries subject to Cuntz–Krieger type relations, recovering [27].

The explicit examples in [27] feature $k = 2$. There, the foundation for the construction of the C^*-algebra is a group action on the boundary of an affine $\widetilde{A}_2$ building, from which one extracts a suitable alphabet and defines two commuting matrices with the properties (H0)–(H3) specified in [27]; see also [12]. Now, even though k-graphs were defined some time ago, not many explicit examples are known for $k \geq 3$; see though [19].

Here, we construct infinite series of examples of k-graphs, with $k \geq 3$, from groups acting freely and transitively on products of k regular trees of constant valencies, as were explicitly found by Rungtanapirom, Stix, and Vdovina in [29]. One of our key tools is Theorem 3.1 establishing that there is a k-graph, unique in the appropriate sense, associated with every k-dimensional digraph.

Theorem 1.2 (Cf. Theorem 3.1). *Suppose that* $\mathrm{DG} = (V, E, o, t, k, \phi)$ *is a k-dimensional digraph, $k \geq 2$. Then, there exist a small category Λ_{DG} with V as a set of objects and a functor $d : \Lambda_{\mathrm{DG}} \to \mathbb{N}^k$ which assigns value (degree) e_i to morphisms determined by edges in E_i, for all $1 \leq i \leq k$. Moreover, the functor d has the unique factorisation property and in particular,* $(\Lambda_{\mathrm{DG}}, d)$ *is a higher-rank graph.*

The other main tool shows that every k-cube complex gives rise to a k-dimensional digraph, as follows.

Theorem 1.3 (Cf. Theorem 3.3). *For any complex $\mathcal{X}$ with universal cover equal to the product of k regular trees, where $k \geq 2$, there is a k-dimensional digraph* $\mathrm{DG}(\mathcal{X})$ *defined by sending a vertex of $\mathcal{X}$ to a vertex in* $\mathrm{DG}(\mathcal{X})$*, and by sending each geometric edge in $\mathcal{X}$ to two edges in the edge set of* $\mathrm{DG}(\mathcal{X})$*, of the same colour and opposite orientations.*

The k-graphs resulting from our constructions are not skew-products of a k-graph by some finite group because such a setup would involve a fundamental group with torsion elements, while all fundamental groups of complexes in our construction are infinite torsion-free groups since they are fundamental groups of CAT(0) spaces.

The examples of k-graphs we obtain by Corollary 3.4 and Theorem 4.7 differ from, for example, the k-graphs constructed in [12, 26–28] since these papers only considered the case $k = 2$. Moreover, our examples are new for $k = 2$; see Remark 3.9. Our construction also differs from the more recent [18] when $k = 2$ and [19] for $k \geq 3$, both of which emulate [12, 27]. The core idea in these references is as follows: given a cell complex $\mathcal{X}$, each k-dimensional cell in $\mathcal{X}$ becomes a vertex in a k-graph $\Lambda(\mathcal{X})$, and for two such cells, there is an edge in $\Lambda(\mathcal{X})$ if the given cells are adjacent via a $(k-1)$-cell, for $k \geq 2$. In this construction, one may take pointed cells as vertices, or unpointed. Another way to distinguish our higher-rank graphs from the ones in [27] is to compute the products of coordinate matrices. The products of coordinate matrices in [27] have to be $(0, 1)$-matrices, but this is not necessarily the case in our construction (see Example 5.10).

To place our definition in the context of similar developments, we recall that a recipe for constructing 2-graphs was proposed in [13, Section 6], starting from two distinct directed graphs on the same set of vertices with commuting vertex matrices. A step forward was taken in [8]; see there Remark 2.3, where a certain associativity type condition was identified as sufficient. In [11], the authors distilled these earlier attempts at constructing higher-rank graphs and landed on a prescription requiring a skeleton, or a k-coloured graph (where the colouring refers to edges and employs k distinct colours) and a collection of building blocks termed squares that satisfy compatibility requirements; see [11, Theorem 4.4]. The squares here are certain coloured-graph morphisms.

A further simplification of the prescription of a k-graph from its skeleton, seen as a k-coloured graph, has been employed in concrete examples such as [14, Example 7.7] and [15, Section 8.2]. In fact, this last example articulates the requirements on the coloured graph that inspired our conditions (F1) and (F2) in Definition 1.1. It is interesting to note that the validity of the associativity condition (F2), for $k \geq 3$, is a priori highly nontrivial. There are connections to finding solutions of the Yang–Baxter equation; see for example [35, 37].

There is a strong connection between the geometry of CW-complexes, group and semigroup actions, higher-rank graphs, and the theory of C^*-algebras. The difficulty is that there are many ways to associate C^*-algebras with groups, semigroups, and CW-complexes, and this can lead to both isomorphic and non-isomorphic C^*-algebras. For the higher-rank graphs, there is a canonical way to associate a C^*-algebra, cf. [13], but it happens that non-isomorphic k-rank graphs lead to the same C^*-algebra. This conclusion is often achieved through the computation of K-theory and applications of the powerful Kirchberg–Phillips classification machinery for purely infinite simple unital nuclear C^*-algebras.

One important question is what is a genuine higher-rank? This means that our k-rank graph can not be obtained by some standard procedure from graphs of smaller ranks. We address this question by introducing the spectral theory of combinatorial higher-rank graphs. So far the spectrum of strongly connected higher-rank graphs was considered in [1, 14], through Perron–Frobenius theory, which leads to new explicit constructions of von Neumann factors. We generalise the results of [14] by constructing infinite series of III_λ factors for any k, and infinitely many values of λ.

We want to stress the following simple but important point about our construction of k-graphs: recall that for an undirected graph with (vertex) adjacency matrix A, we have $A(v, w) = 1 = A(w, v)$ if vertices v, w are connected by an edge. Thus, the adjacency matrix is *symmetric* and the eigenvalues are real. In a k-graph, we have directed edges in the various colours in its 1-skeleton. There is no reason why the adjacency matrix should be symmetric. What can be said in general about a k-graph is that if it is strongly connected, then its associated coordinate matrices jointly admit a unimodular Perron–

Frobenius eigenvector [1]. Now, in our construction of the k-graph $\Lambda(P)$ from a k-cube complex P, the procedure is such that it assigns to each undirected edge in the 1-skeleton of P a pair of morphisms (arrows) with opposite orientation in $\Lambda(P)$. As a consequence, the adjacency matrix for the complex in direction i is the same as the coordinate matrix M_i of $\Lambda(P)$ in colour i, for all $i = 1, \dots, k$. Thus, all our constructions of k-graphs have symmetric matrices.

With this in mind, we suggest a new class of higher-rank graphs, which we call Ramanujan k-rank graphs; see Section 5.2. Their coordinate matrices are symmetric, so all eigenvalues are real and it makes sense to consider the spectral gap. We show that our k-graphs satisfy the optimal spectral gap condition, which distinguishes them from the examples that have appeared in the literature so far.

The structure of the paper is the following: in a preliminary Section 2, we collect conventions and results about categories, groups acting on products of trees, k-cube complexes, in particular one-vertex k-cube complexes from k-cube groups, k-graphs and their C^*-algebras. Section 3 starts with one of our main results, Theorem 3.1, which prescribes the construction of a k-graph from a given k-dimensional digraph. We then associate a k-dimensional digraph with any complex covered by a product of k trees; see Theorem 3.3. The ensuing Corollary 3.4 describes the new family of k-graphs from k-cube complexes. In the case of a one-vertex complex P, or equivalently a k-cube group with complex P, we show that the resulting k-graphs are rigid in the sense of [16]; in particular, they are aperiodic and yield classifiable C^*-algebras in the sense of the Kirchberg–Phillips classification [22]. In Section 4, we construct k-cube complexes on N vertices from N-covers of one-vertex complexes, with $N \geq 2$, and prove that the C^*-algebras of the associated k-graphs with N vertices are covered by the Kirchberg–Phillips classification theory. In Corollary 5.3, we expand the scope of the constructions of 2-graphs in [14, Example 7.7] leading to factors of type $\mathrm{III}_{1/2}$ and give an explicit infinite family giving type $\mathrm{III}_{1/(2L)^2}$ factors, with L, an arbitrary integer. In Section 5.2, we introduce the notion of Ramanujan k-graphs and show that there are infinite families of such k-graphs; see Theorem 5.9. Example 5.10 details an explicit Ramanujan 3-graph on 25 vertices with optimal spectral gap. We compute the associated 25×25 incidence matrices and estimate the joint spectral gap of the 3-dimensional digraph with the help of MAGMA. The 3-graph moreover features the interesting property that while it arises from an infinite 3-cube group Γ_1 which is also an irreducible lattice, in the cover with 25 sheets, each of the three distinct alphabets in Γ_1 generates a finite group of order 25.

2. Preliminaries

2.1. Categories

We follow the principles laid out in [17] but also keep in mind the interpretation in [6, Chapter II, Sections 1.1 and 1.2]. A category $\mathcal{C}$ consists of objects $\mathrm{Obj}(\mathcal{C})$ and morphisms $\mathrm{Hom}(\mathcal{C})$. We often blur the distinction between $\mathcal{C}$ and $\mathrm{Hom}(\mathcal{C})$ and refer to the latter as the *elements* or *arrows* of $\mathcal{C}$. To each $f \in \mathcal{C}$, there are two objects associated, its *origin* and *terminus*, and the category is seen as a collection of elements endowed with a partial product governed by compatibility of objects. Of interest to us are categories associated with directed graphs, where the concatenation of edges on the graph will determine the composition of arrows in the category, upon natural reversal of origin and terminus.

2.2. Directed graphs

By a directed graph D we mean a set D^0 of vertices, a set D^1 of edges, and maps $o, t : D^1 \to D^0$ determining the origin and terminus of edges. Edges whose origin and terminus coincide, also called loops, will be allowed. We shall assume that D^0, D^1 are finite. We form a path ef when $t(e) = o(f)$ for $e, f \in D^1$ and extend this to finite directed paths $f_1 f_2 \cdots f_m$ on D, and likewise in the case of a finite number of graphs on the same vertex set; see Section 3.

2.3. Complexes covered by products of trees

We start by introducing our definition of k-cube complex. Then, we expand on the case of one-vertex k-cube complexes, for which we follow the notation and approach of [19, 35]. We refer to [4, 36], and especially [25, Section 1.2], for an introduction to square complexes and $(2m, 2l)$ groups, $m, l \geq 1$. We refer to [30] for details on CAT(0) complexes and to [10] for the basic theory of CW complexes. We use the letter T for an arbitrary regular tree, and $\mathcal{T}_l$ for the regular l-valent tree, where $l \geq 1$.

Definition 2.1. Let $k \geq 1$ be a positive integer. A CW complex $\mathcal{X}$ is a k-dimensional cube complex, or k-cube complex, if its universal cover is a Cartesian product of k trees $\mathrm{T}_1 \times \mathrm{T}_2 \times \cdots \times \mathrm{T}_k$, each of which has finite constant valency.

In the case of a one-vertex k-cube complex, for which we use the letter P, an equivalent definition is as the quotient space $P = Z \backslash G$ of a group G with a free and transitive action on a product $Z = \mathrm{T}_1 \times \mathrm{T}_2 \times \cdots \times \mathrm{T}_k$ of k trees. Such G are called k-cube groups; see Definition 2.3. For general k-cube complexes with more than one vertex, the similar definition as a quotient space $\mathcal{X} = Z \backslash G$ can be enforced upon replacing transitive action with cocompact action.

We leave the case of trees with possibly non-constant and/or infinite valency for future discussion.

To describe a k-cube complex for $k \geq 2$, it is useful to recall the formalism of 2-complexes (or square complexes) covered by products of two trees; see, e.g., [35]. We use the letter S to denote a generic 2-complex. A *square complex* S is a 2-dimensional combinatorial cell complex with 1-skeleton consisting of a graph $\mathcal{G}(S) = (V(S), E(S))$ with a set of vertices $V(S)$, and a set of oriented edges $E(S)$, and with 2-cells arising from a set of squares that are combinatorially glued to the graph $\mathcal{G}(S)$. More precisely, let $e \mapsto e^{-1}$ denote orientation reversal of an edge $e \in E(S)$, and suppose that (e_1, e_2, e_3, e_4) is a 4-tuple of oriented edges in $E(S)$ with the origin of e_{i+1} equal to the terminus of e_i (for i modulo 4). A square $\square = (e_1, e_2, e_3, e_4)$ is the orbit of (e_1, e_2, e_3, e_4) under the dihedral action generated by cyclically permuting the edges e_i and by the reverse orientation map

$$(e_1, e_2, e_3, e_4) \mapsto (e_4^{-1}, e_3^{-1}, e_2^{-1}, e_1^{-1}). \tag{2.1}$$

As customary, we think of a square-shaped 2-cell glued to the (topological realisation of the) respective edges of the graph $\mathcal{G}(S)$.

A vertical/horizontal structure (in short, a VH-structure) on a square complex is given by a bipartite structure of the set of unoriented edges $\overline{E(S)} = E_V \sqcup E_H$ such that for every vertex v in $V(S)$ the link at v is the complete bipartite graph on the resulting partition $E(v) = E(v)_V \sqcup E(v)_H$, with $E(v)$ denoting the set of oriented edges with origin v. Torsion-free cocompact lattices Γ in $\mathbb{A}\mathrm{ut}(\mathcal{T}_m) \times \mathbb{A}\mathrm{ut}(\mathcal{T}_l)$ with $m, l \geq 1$, not interchanging the factors and considered up to conjugation, correspond uniquely to finite square complexes S with a VH-structure of partition size $(2m, 2l)$ up to isomorphism. Simply transitive torsion-free lattices not interchanging the factors correspond to finite square complexes with one vertex and a VH-structure, necessarily of constant partition size.

2.4. One-vertex k-cube complexes

We first look at the case when $k = 2$. Let S be a square complex with one vertex $v \in S$ and a VH-structure $\overline{E(S)} = E_V \sqcup E_H$. Pictorially, this consists of a collection of squares, each of which has four vertices labelled v. Passing from the origin to the terminus of an oriented edge e in a square corresponds to a fixed-point free involution $e \to e^{-1}$ on $E(v)_V$ and on $E(v)_H$. Thus, the partition size is necessarily a tuple $(2m, 2l)$ of even integers, $m, l \geq 1$. The lattice identified with $\pi_1(S, v)$ admits a description in terms of two generating subsets A, B; see [35, Definition 5].

Definition 2.2. A *vertical/horizontal structure*, or *VH-structure*, in a group G is an ordered pair (A, B) of finite subsets $A, B \subseteq G$ such that the following hold.

(1) Taking inverses induces fixed-point free involutions on A and B.

(2) The union $A \cup B$ generates G.

(3) The product sets AB and BA have size $\#A \cdot \#B$ and satisfy $AB = BA$.

(4) The sets AB and BA do not contain 2-torsion.

The tuple $(\#A, \#B)$ is called the *valency vector* of the VH-structure in G.

If a group G admits a VH-structure (A, B) of valency vector $(\#A, \#B)$, then by [4, Section 6.1], there is a square complex $S_{A,B}$ with one vertex and a VH-structure in the sense of Section 2.3. The set of oriented edges of $S_{A,B}$ is the disjoint union $E(S_{A,B}) = A \sqcup B$, with the orientation reversion map given by $e \mapsto e^{-1}$, and with A, B labelling the edges in vertical and horizontal direction, respectively. The link of $S_{A,B}$ in v is the complete bipartite graph with vertices labelled by A and B, e.g. [35, Lemma 1], and [2, Theorem C] implies that the universal cover of $S_{A,B}$ is a product of trees. Conversely, given a square complex S with a VH-structure (A, B) and a single vertex, its fundamental group (i.e., the fundamental group of its topological realisation) admits a

VH-structure of valency $(\#A, \#B)$; see [29, Proposition 5.7]. We refer to it as a $(\#A, \#B)$-group. Example 3.8 shows a $(4, 4)$-complex with associated $(4, 4)$-group.

To describe the geometric squares of $S_{A,B}$, note that a relation $ab = b'a'$ in G with $a, a' \in A$ and $b, b' \in B$ (not necessarily pairwise distinct), as prescribed by Definition 2.2, leads to four algebraic relations obtained upon cyclic permutation and inversion; namely,

$$ab = b'a', \quad a^{-1}b' = ba'^{-1}, \quad a'^{-1}b'^{-1} = b^{-1}a^{-1}, \quad \text{and} \quad a'b^{-1} = b'^{-1}a. \tag{2.2}$$

This leads to the definition of a geometric square as a tuple of four Euclidean squares. All four vertices in each square coalesce into the single vertex v of $S_{A,B}$ when we glue the edges according to labels and orientation. Before we introduce our convention, we recall briefly two other (equivalent) conventions for describing geometric squares.

2.5. One convention – see, e.g., Rattaggi

The formalism of a geometric square seen as a 4-tuple of squares in Euclidean space is well known. In [25, Figure 4.1, p. 182], for example, the group relation $ab = b'a'$ is reflected by the 4-tuple of squares having edges labelled according to a one-way cyclic permutation in counterclockwise direction; see below:

$$\begin{tikzcd}
\bullet \arrow[d, "b'"'] \arrow[dr, phantom, "S_O"] & \bullet \arrow[l, "a'"'] & \bullet \arrow[d, "b"'] \arrow[dr, phantom, "S_R"] & \bullet \arrow[l, "a"'] & \bullet \arrow[r, "a'"] \arrow[dr, phantom, "S_H"] & \bullet \arrow[d, "b'"] & \bullet \arrow[r, "a"] \arrow[dr, phantom, "S_V"] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "a"'] & \bullet \arrow[u, "b"'] & \bullet \arrow[r, "a'"'] & \bullet \arrow[u, "b'"'] & \bullet \arrow[u, "b"] & \bullet \arrow[l, "a"] & \bullet \arrow[u, "b'"] & \bullet \arrow[l, "a'"]
\end{tikzcd} \tag{2.3}$$

The notation means that if S_O is regarded as a reference square, then S_H is obtained by reflection in the horizontal direction (about b), S_V by reflection in the vertical direction (about a), and, finally, S_R arises from rotation counterclockwise by π. Our use of S_O, S_R, S_H, S_V as notation for the squares is inspired by [19, Section 2].

The geometric square associated with $ab = b'a'$ in [25] and visualised in (2.3) is given by

$$\{(a, b, a', b'), (a', b', a, b), (a^{-1}, b'^{-1}, a'^{-1}, b^{-1}), (a'^{-1}, b^{-1}, a^{-1}, b'^{-1})\}.$$

2.6. A second convention – see Kimberley–Robertson

In [12], Kimberley–Robertson adopted a two-direction labelling of their squares which to a group relation $ab = b'a'$ assigns a 4-tuple of squares according to the convention below:

$$\begin{tikzcd}
\bullet \arrow[r, "a'"] \arrow[dr, phantom, "S_O"] & \bullet & \bullet \arrow[d, "b"'] \arrow[dr, phantom, "S_R"] & \bullet \arrow[l, "a"'] \arrow[d, "b'"] & \bullet \arrow[dr, phantom, "S_H"] & \bullet \arrow[l, "a'"'] & \bullet \arrow[r, "a"] \arrow[d, "b'"'] \arrow[dr, phantom, "S_V"] & \bullet \arrow[d, "b"] \\
\bullet \arrow[u, "b'"] \arrow[r, "a"'] & \bullet \arrow[u, "b"'] & \bullet & \bullet \arrow[l, "a'"] & \bullet \arrow[u, "b"] & \bullet \arrow[l, "a"] \arrow[u, "b'"'] & \bullet \arrow[r, "a'"'] & \bullet
\end{tikzcd} \tag{2.4}$$

The geometric square associated with $ab = b'a'$ and visualised in (2.4) is given by

$$\{(a, b, b', a'), (a'^{-1}, b'^{-1}, b^{-1}, a^{-1}), (a^{-1}, b', b, a'^{-1}), (a', b^{-1}, b'^{-1}, a)\}.$$

2.7. Our convention

We choose a convention that will facilitate our constructions of k-graphs later on, and in particular we swap the letters for vertical and horizontal directions, as follows: we keep the cyclic permutation in counterclockwise direction from [25] but choose labelling of edges as “starting” at one vertex by going out in both horizontal and vertical direction, similar to [12].

$$\begin{tikzcd} \bullet \arrow[r, "b"] & \bullet \\ \bullet \arrow[u, "a"] \arrow[r, "b'"'] \arrow[ur, phantom, "S_O"] & \bullet \arrow[u, "a'"'] \end{tikzcd} \quad \begin{tikzcd} \bullet \arrow[d, "a'"'] \arrow[dr, phantom, "S_R"] & \bullet \arrow[l, "b'"'] \arrow[d, "a"] \\ \bullet & \bullet \arrow[l, "b"] \end{tikzcd} \quad \begin{tikzcd} \bullet & \bullet \arrow[l, "b"'] \\ \bullet \arrow[u, "a'"] \arrow[ur, phantom, "S_H"] & \bullet \arrow[l, "b'"] \arrow[u, "a"'] \end{tikzcd} \quad \begin{tikzcd} \bullet \arrow[r, "b'"] \arrow[d, "a"'] \arrow[dr, phantom, "S_V"] & \bullet \arrow[d, "a'"] \\ \bullet \arrow[r, "b"'] & \bullet \end{tikzcd}. \tag{2.5}$$

Explicitly, we define a *geometric square* as visualised in (2.5) to be a tuple

$$\{(a, b, a'^{-1}, b'^{-1}), (a'^{-1}, b'^{-1}, a, b), (a', b^{-1}, a^{-1}, b'), (a^{-1}, b', a', b^{-1})\}, \tag{2.6}$$

where any two squares are seen as equivalent.

Since for our purposes it will be important to keep track of how such squares arise, we introduce the following more precise notation: for $a \in A$ and $b \in B$, we let

$$S_O^{a,b} := (a, b, a'^{-1}, b'^{-1}), \tag{2.7}$$

where a', b' are the unique elements in A and B, respectively, such that $ab = b'a'$. We refer to ab as the *vertical-horizontal pair of edges in* $S_O^{a,b}$ and to $b'a'$ as the *horizontal-vertical pair of edges in* $S_O^{a,b}$.

In [35], the last named author generalised VH-structure to the k-dimensional case, as follows.

Definition 2.3 (See [35, Definition 7]). A *k-cube structure* in a group G is an ordered k-tuple $(A_1, \dots, A_k)$ of finite subsets $A_i \subseteq G$ such that the following hold for all $i, j = 1, \dots, k$, $i \neq j$:

(1) Taking inverses induces fixed-point free involutions on A_i.

(2) The union $\cup A_i$ generates G.

(3) The product sets $A_i A_j$ and $A_j A_i$ have size $\#A_i \cdot \#A_j$ and $A_i A_j = A_j A_i$.

(4) The sets $A_i A_j$ and $A_j A_i$ do not contain 2-torsion.

(5) The group G acts simply transitively on a Cartesian product of k trees.

The tuple $(\#A_1, \dots, \#A_k)$ is the *valency vector* of the k-cube structure in G, and $A_1, \dots, A_k$ are generating sets of G.

We note that each pair $(A_i, A_j) \subseteq G$ for $i, j = 1, \dots, k$ with $i \neq j$ forms a subgroup $G_{i,j}$ of G equipped with a VH-structure. This observation can be used to show that to a given k-cube group G with generating family $(A_1, \dots, A_k)$, there is an associated k-cube complex $P_{(A_1,\dots,A_k)}$. Its 2-dimensional cells are prescribed by the square complexes S_{A_i,A_j} obtained from each $(\#A_i, \#A_j)$ group $G_{i,j}$ for $i \neq j$.

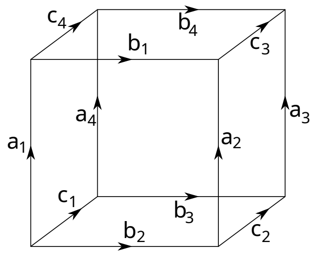

Figure 1. A generic 3-cube.

Remark 2.4. In studying k-cube groups and one-vertex k-cube complexes, we will move freely between two equivalent interpretations. Starting from a k-cube group G defined algebraically through properties (1)–(5) in Definition 2.3, the associated quotient space $Z\backslash G$ with Z a product of k trees is a k-cube complex with one vertex. Its construction as a CW complex from j-cells for $0 \leq j \leq k$ is detailed in [19, Definition 2.3]. Conversely, one may define a k-cube group G from combinatorial data by starting with k finite sets of even cardinalities (encoding edges), and building up by induction (on dimension of cells) a k-dimensional complex with the necessary compatibility to yield generating sets $A_1, A_2, \ldots, A_k$ for G; see [19, Definition 2.4].

We next introduce some notation for cubes in a one-vertex k-cube complex P for $k \geq 2$, with motivation and inspiration drawn from [19, Section 2]. We let E be the set of edges. Because P is the complex associated with a k-cube group, E partitions into k subsets as $E = E_1 \sqcup \cdots \sqcup E_k$ in such a way that if an edge e is in E_j, then $e^{-1} \in E_j$ for $j = 1, \ldots, k$. We refer to E_i as the subset of edges of *colour* i, for $i = 1, \ldots, k$, where the k colours are assumed distinct. The 2-cells of P are geometric squares of the form $S_O = (a, b, a'^{-1}, b'^{-1})$ regarded as the equivalence class $\{S_O, S_R, S_H, S_V\}$ described in (2.6), where $a, a' \in E_i$ and $b, b' \in E_j$ for $i \neq j$. By a *geometric square* we mean any square in $\{S_O, S_R, S_H, S_V\}$. Similar to [19], for distinct colours $i \neq j$ we let

$$F(i, j) = \{S = (a, b, a'^{-1}, b'^{-1}) \mid S \text{ is a geometric square with } a, a' \in E_i,\ b, b' \in E_j\},$$

and we denote by S^{ij} a generic square in $F(i, j)$ for all $i \neq j$ in $\{1, \ldots, k\}$. The 3-cells are determined by *geometric cubes*, all whose 6 faces are geometric squares; see Figure 1 for a (generic) cube.

More precisely, the 6 faces of the cube are geometric squares $(S_1^{ij}, S_2^{il}, S_3^{jl}, S_4^{jl}, S_5^{il}, S_6^{ij})$, with

$$\begin{aligned}
S_1^{ij} &= (a_1, b_1, a_2^{-1}, b_2^{-1}) && \text{(front face)},\\
S_2^{il} &= (a_2, c_3, a_3^{-1}, c_2^{-1}) && \text{(right face)},\\
S_3^{jl} &= (b_2, c_2, b_3^{-1}, c_1^{-1}) && \text{(bottom face)},\\
S_4^{jl} &= (b_1, c_3, b_4^{-1}, c_4^{-1}) && \text{(top face)},\\
S_5^{il} &= (a_1, c_4, a_4^{-1}, c_1^{-1}) && \text{(left face)},\\
S_6^{ij} &= (a_4, b_4, a_3^{-1}, b_3^{-1}) && \text{(back face)}.
\end{aligned}$$

In particular, any one of the 6 geometric squares in this list is given subject to the equivalence relation (2.6), and the geometric cube can be equivalently presented with any one of the 8 vertices in the bottom-left position. We stress that the labels $a_1, \dots, a_4, b_1, \dots, b_4, c_1, \dots, c_4$ here are formal symbols that keep track of how the cubes are glued in the complex. As already mentioned, if e is a label for an edge, then e^{-1} is the label recording orientation reversal.

For $4 \le l \le k$, the l-dimensional cells are l-cubes; see [19].

A given k-cell in a k-cube complex P has a topological realisation as the product of intervals $[0,1]^k$. Denoting by ε_i the standard basis elements in $\mathbb{R}^k$, for $i = 1, \dots, k$, we view a geometric edge in P as having *degree* ε_i if it lies in the span of the generator ε_i in its topological realisation. This agrees with the degree of paths in higher-rank graphs in Section 3.

2.8. Examples of cube groups

The cube groups in this section were introduced in [35]. They contain as a particular case arithmetic lattices and non-residually finite CAT(0) groups constructed in [29]. We refer to them as *RSV-groups*. They are *the first explicit examples* of arithmetic groups acting freely and transitively on products of k trees of constant valencies, for $k \ge 3$, as well as non-residually finite CAT(0) groups of dimensions $k \ge 3$. RSV groups are irreducible in the sense that they can not be presented as virtual products of group actions on products of smaller number of trees.

We recall here a construction of an explicit series of RSV lattices, which is infinite in several parameters, k, q, and δ. The significance of the irreducibility of these groups is that the associated k-graphs can not be presented as virtual products, so are entirely new.

For q, an odd prime, let $\delta \in \mathbb{F}_{q^2}^\times$ be a generator of the multiplicative group of the field with q^2 elements. If $i, j \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ satisfy $i \not\equiv j \pmod{q-1}$, then $1 + \delta^{j-i} \neq 0$, and it follows that there is a unique $x_{i,j} \in \mathbb{Z}/(q^2-1)\mathbb{Z}$ with $\delta^{x_{i,j}} = 1 + \delta^{j-i}$.

Set $y_{i,j} := x_{i,j} + i - j$, and note that

$$\delta^{y_{i,j}} = \delta^{x_{i,j}+i-j} = (1+\delta^{j-i}) \cdot \delta^{i-j} = 1 + \delta^{i-j}.$$

Define

$$\begin{aligned} l(i,j) &:= i - x_{i,j}(q-1), \\ k(i,j) &:= j - y_{i,j}(q-1), \end{aligned}$$

and further let $M \subseteq \mathbb{Z}/(q^2-1)\mathbb{Z}$ be a union of cosets under $(q-1)\mathbb{Z}/(q^2-1)\mathbb{Z}$ with $\#M = k$.

If q is odd, it was shown in [29] that the following groups act freely and transitively on a product of k trees:

$$\Gamma_{M,\delta} = \left\langle a_i,\ i \in M \;\middle|\; \begin{array}{l} a_{i+(q^2-1)/2} a_i = 1 \text{ for all } i \in M, \\ a_i a_j = a_{k(i,j)} a_{l(i,j)} \text{ for all } i, j \in M \text{ with } i \not\equiv j \pmod{q-1} \end{array} \right\rangle.$$

Example 2.5. The smallest example in dimension $k = 3$ arises with $q = 5$ and M equal to the collection of cosets $i \in \mathbb{Z}/24\mathbb{Z}$ with i not dividing 4. This group, denoted Γ_1, acts vertex transitively on the product of three regular trees $\mathcal{T}_6 \times \mathcal{T}_6 \times \mathcal{T}_6$ and has the presentation

$$\Gamma_1 = \left\langle \begin{matrix} a_1, a_5, a_9, a_{13}, a_{17}, a_{21}, \\ b_2, b_6, b_{10}, b_{14}, b_{18}, b_{22}, \\ c_3, c_7, c_{11}, c_{15}, c_{19}, c_{23} \end{matrix} \;\middle|\; \begin{matrix} a_i a_{i+12} = b_i b_{i+12} = c_i c_{i+12} = 1 \text{ for all } i, \\ a_1 b_2 a_{17} b_{22}, a_1 b_6 a_9 b_{10}, a_1 b_{10} a_9 b_6, a_1 b_{14} a_{21} b_{14}, \\ a_1 b_{18} a_5 b_{18}, a_1 b_{22} a_{17} b_2, a_5 b_2 a_{21} b_6, a_5 b_6 a_{21} b_2, \\ a_5 b_{22} a_9 b_{22}, a_1 c_3 a_{17} c_3, a_1 c_7 a_{13} c_{19}, a_1 c_{11} a_9 c_{11}, \\ a_1 c_{15} a_1 c_{23}, a_5 c_3 a_5 c_{19}, a_5 c_7 a_{21} c_7, a_5 c_{11} a_{17} c_{23}, \\ a_9 c_3 a_{21} c_{15}, a_9 c_7 a_9 c_{23}, b_2 c_3 b_{18} c_{23}, b_2 c_7 b_{10} c_{11}, \\ b_2 c_{11} b_{10} c_7, b_2 c_{15} b_{22} c_{15}, b_2 c_{19} b_6 c_{19}, b_2 c_{23} b_{18} c_3, \\ b_6 c_3 b_{22} c_7, b_6 c_7 b_{22} c_3, b_6 c_{23} b_{10} c_{23}. \end{matrix} \right\rangle$$

Thus, Γ_1 is a 3-cube group with $A_1 = \{a_1, a_5, a_9, a_{13}, a_{17}, a_{21}\}$ and similar descriptions for A_2 and A_3. It is an arithmetic group, so it is residually finite. Of interest to us is the fact that it admits quotients of order 5^l, $l \in \mathbb{N}$; see Example 5.10.

In [29], the authors also constructed k-cube groups acting on a product of trees of distinct constant valencies. Explicitly, for any set of size k of distinct odd primes $p_1, \dots, p_k$, there is a group acting simply transitively on a product of k trees of valencies $p_1 + 1, \dots, p_k + 1$, obtained using Hamiltonian quaternion algebras.

Example 2.6. For $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, there is an explicit presentation of a group acting simply transitively on a product of three trees $\mathcal{T}_4 \times \mathcal{T}_6 \times \mathcal{T}_8$; see [29]. Indeed, with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denoting the quaternions, let

$$\begin{aligned} &a_1 = 1 + \mathbf{j} + \mathbf{k},\ a_2 = 1 + \mathbf{j} - \mathbf{k},\ a_3 = 1 - \mathbf{j} - \mathbf{k},\ a_4 = 1 - \mathbf{j} + \mathbf{k}, \\ &b_1 = 1 + 2\mathbf{i},\ b_2 = 1 + 2\mathbf{j},\ b_3 = 1 + 2\mathbf{k},\ b_4 = 1 - 2\mathbf{i},\ b_5 = 1 - 2\mathbf{j},\ b_6 = 1 - 2\mathbf{k}, \\ &c_1 = 1 + 2\mathbf{i} + \mathbf{j} + \mathbf{k},\ c_2 = 1 - 2\mathbf{i} + \mathbf{j} + \mathbf{k},\ c_3 = 1 + 2\mathbf{i} - \mathbf{j} + \mathbf{k},\ c_4 = 1 + 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \\ &c_5 = 1 - 2\mathbf{i} - \mathbf{j} - \mathbf{k},\ c_6 = 1 + 2\mathbf{i} - \mathbf{j} - \mathbf{k},\ c_7 = 1 - 2\mathbf{i} + \mathbf{j} - \mathbf{k},\ c_8 = 1 - 2\mathbf{i} - \mathbf{j} + \mathbf{k}. \end{aligned}$$

Then, $a_l^{-1} = a_{l+2}$, $b_l^{-1} = b_{l+3}$, and $c_l^{-1} = c_{l+4}$. The required group is given by

$$\Gamma_2 = \left\langle \begin{matrix} a_1, \dots a_4 \\ b_1, \dots, b_6 \\ c_1, \dots, c_8 \end{matrix} \;\middle|\; \begin{matrix} a_1 b_1 a_4 b_2, a_1 b_2 a_4 b_4, a_1 b_3 a_2 b_1, a_1 b_4 a_2 b_3, a_1 b_5 a_1 b_6, \\ a_2 b_2 a_2 b_6, a_1 c_1 a_2 c_8, a_1 c_2 a_4 c_4, a_1 c_3 a_2 c_2, a_1 c_4 a_3 c_3, \\ a_1 c_5 a_1 c_6, a_1 c_7 a_4 c_1, a_2 c_1 a_4 c_6, a_2 c_4 a_2 c_7, \\ b_1 c_1 b_5 c_4, b_1 c_2 b_1 c_5, b_1 c_3 b_6 c_1, b_1 c_4 b_3 c_6, b_1 c_6 b_2 c_3, b_1 c_7 b_1 c_8, \\ b_2 c_1 b_3 c_2, b_2 c_2 b_5 c_5, b_2 c_4 b_5 c_3, b_2 c_7 b_6 c_4, b_3 c_1 b_6 c_6, b_3 c_4 b_6 c_3 \end{matrix} \right\rangle.$$

This is also denoted $\Gamma_{3,5,7}$; see [29].

2.9. Higher-rank graphs

We recall the definition of a k-graph due to Kumjian–Pask, cf. [13], see also [24]. For an integer $k \geq 1$, we view $\mathbb{N}^k$ as a monoid under pointwise addition. A k-graph is a countable small category Λ together with an assignment of a *degree* $d(\mu) \in \mathbb{N}^k$ to every morphism $\mu \in \Lambda$ such that for all $\mu, \nu, \pi \in \Lambda$ the following hold:

(1) $d(\mu\nu) = d(\mu) + d(\nu)$;

(2) whenever $d(\pi) = m + n$ for $m, n \in \mathbb{N}^k$, there is a unique factorisation $\pi = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$.

Condition (2) is known as the *factorisation property* in the k-graph. The composition in $\mu\nu$ is understood in the sense of morphisms; thus, the source $s(\mu)$ of μ equals the range $r(\nu)$ of ν. Note that the morphisms of degree 0 (in $\mathbb{N}^k$) are the identity morphisms in the category. Denote this set by Λ^0, and refer to its elements as *vertices* of Λ. With $e_1, \ldots, e_k$ denoting the generators of $\mathbb{N}^k$, the set $\Lambda^{e_i} = \{\lambda \in \Lambda \mid d(\lambda) = e_i\}$ consists of edges (or morphisms) of degree e_i, for $i = 1, \ldots, k$. We write $v\Lambda^n$ for the set of morphisms of degree $n \in \mathbb{N}^k$ with range v.

Throughout this paper, we are concerned with k-graphs where Λ^0 and all Λ^{e_i}, $i = 1, \ldots, k$, are finite. A k-graph Λ so that $0 < \#v\Lambda^n < \infty$ for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$ is source-free and row-finite. Following [1], a finite k-graph Λ is *strongly connected* if $v\Lambda w \neq \emptyset$ for all vertices $v, w \in \Lambda^0$.

The *coordinate matrices* $M_1, \ldots, M_k \in \mathrm{Mat}_{\Lambda^0}(\mathbb{N})$ of Λ are $\Lambda^0 \times \Lambda^0$ matrices with

$$M_i(v, w) = |v\Lambda^{e_i} w|.$$

By the factorisation property, the matrices M_i pairwise commute for $i = 1, \ldots, k$. For $n = (n_i)_{i=1,\ldots,k} \in \mathbb{N}^k$, we define

$$M^n := \prod_{i=1}^{k} M_i^{n_i}.$$

We denote the spectral radius of a square matrix B by $\rho(B)$, and we let

$$\rho(\Lambda) := \big(\rho(M_1), \rho(M_2), \ldots, \rho(M_k)\big) \in [0, \infty)^k.$$

For $m \in \mathbb{Z}^k$, we write $\rho(\Lambda)^m$ for the product $\prod_{i=1}^k \rho(M_i)^{m_i}$.

Given a row-finite, source-free k-graph Λ, its associated C^*-algebra $C^*(\Lambda)$ is the universal C^*-algebra generated by a family $\{\mathbf{s}_\mu \mid \mu \in \Lambda\}$ of partial isometries satisfying

(CK1) $\{\mathbf{s}_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(CK2) $\mathbf{s}_\mu \mathbf{s}_\nu = \mathbf{s}_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(CK3) $\mathbf{s}_\mu^* \mathbf{s}_\mu = \mathbf{s}_{s(\mu)}$ for all μ;

(CK4) $\mathbf{s}_v = \sum_{\mu \in v\Lambda^n} \mathbf{s}_\mu \mathbf{s}_\mu^*$ for all $v \in \Lambda^0, n \in \mathbb{N}^k$.

3. Construction of k-graphs from k-cube groups: The one-vertex case

In this section, we construct k-graphs with one vertex, for $k \geq 2$. There are two main steps. The first is a general procedure by which we associate a category with a k-dimensional digraph as introduced in Definition 1.1 and prove that the conditions (F1) and (F2) inherent to the k-dimensional digraph imply the existence of a degree functor from the category to $\mathbb{N}^k$ that satisfies the required factorisation property. This step will be reminiscent of the construction of Artin monoids as quotients of free monoids by an equivalence relation on the collection of positive words identifying braid strings; see [3]. The degree functor will be similar to the degree map on an Artin monoid, cf. [31]. The second step is to provide k-dimensional digraphs, and here we shall use one-vertex cube complexes associated with k-cube groups as a source from which to construct such digraphs.

In Section 4, we use the results of this section combined with concrete constructions of covering maps in the context of complexes to produce higher-rank graphs with more than one vertex. We stress that our constructions are performed on the complexes, which depend on finite combinatorial data, and not on the k-graphs, which are categories with additional structure.

The next result is the abstract construction of the category associated with a k-dimensional digraph.

Theorem 3.1. *Suppose that* $\mathrm{DG} = (V, E, o, t, k, \phi)$ *is a k-dimensional digraph, $k \geq 2$. Then, there exist a small category Λ_{DG} with V as a set of objects and a functor $d : \Lambda_{\mathrm{DG}} \to \mathbb{N}^k$ which assigns value (degree) e_i to morphisms determined by edges in E_i, for all $1 \leq i \leq k$. Moreover, the functor d has the unique factorisation property, and in particular, $(\Lambda_{\mathrm{DG}}, d)$ is a higher-rank graph.*

Proof. First, let $\mathcal{C} := \mathcal{C}_{\mathrm{DG}}$ be the free category associated with the directed graph $\mathrm{DG} = (V, E, o, t)$, where we disregard the colouring of the edges; see, e.g., [17, Theorem 1, p. 49]. The object set of $\mathcal{C}$ is V and the arrows are given by finite strings, or paths, consisting of finite sequences $v_1, \ldots, v_m$ of objects connected by $m-1$ arrows $x_s : v_s \to v_{s+1}$ with the compatibility of objects $o(x_{s+1}) = t(x_s)$ for $1 \leq s \leq m-1$, where $m > 1$ is arbitrary.

To conform later with the conventions of higher-rank graphs, we assign length 0 to identity morphisms, or arrows $\langle v_1 \rangle$, as opposed to length 1 in [17]. An arrow $\langle v_1, x_1, v_2 \rangle$ of length 1 will be called an *elementary* arrow, where we again shift the value of the length by -1 compared to [17]. For $m > 1$, an arrow $A := \langle v_1, x_1, v_2, \ldots, v_{m-1}, x_{m-1}, v_m \rangle$ is determined by the finite path $x_1 x_2 \cdots x_{m-1}$ of length $m-1$ on the graph, and we shall often regard it as such, by disregarding the contribution of the objects in the notation. In the category, the same arrow is equal to a composition

$$A = A_{m-1} \circ A_{m-2} \circ \cdots \circ A_2 \circ A_1 \tag{3.1}$$

of $m-1$ elementary arrows $A_s = \langle v_s, x_s, v_{s+1} \rangle$, $1 \leq s \leq m$. For short, we write it as $x_{m-1} \circ \cdots \circ x_2 \circ x_1$. The category underlying the k-graph will be obtained as a quotient category of $\mathcal{C}$, as in [17, Proposition 1, p. 51].

We let $\mathcal{C}_Y$ be the collection of arrows in $\mathcal{C}$ arising from paths on the digraph DG belonging to Y,

$$\mathcal{C}_Y = \{\langle v, x, u, y, w\rangle \mid v, u, w \in V,\ x \in E_i,\ y \in E_j,\ 1 \le i \neq j \le k\}. \tag{3.2}$$

We refer to elements in $\mathcal{C}_Y$ as *bicoloured* arrows.

For objects v, w in $\mathcal{C}$, we let $\mathcal{C}(v, w)$ be the set of morphisms in $\mathcal{C}$ which are arrows from v to w. For each pair of objects $v, w \in \mathcal{C}$, we will define a binary relation $\mathcal{R}_{v,w}$ on the set of morphisms $\mathcal{C}(v, w)$. For this purpose, we introduce the following terminology: Given two arrows A, A' in $\mathcal{C}$, we say that A' is obtained from A by *diverting in $\mathcal{C}_Y$ over ϕ* if we have

$$A := \langle v_1, x_1, v_2, \dots, v_s, x_s, v_{s+1}, x_{s+1}, v_{s+2}, \dots, v_{m-1}, x_{m-1}, v_m\rangle, \tag{3.3}$$

$$A' := \langle v_1, x_1, v_2, \dots, v_s, x'_{s+1}, v'_{s+1}, x'_s, v_{s+2}, \dots, v_{m-1}, x_{m-1}, v_m\rangle \tag{3.4}$$

with $\langle v_s, x_s, v_{s+1}, x_{s+1}, v_{s+2}\rangle, \langle v_s, x'_{s+1}, v'_{s+1}, x'_s, v_{s+2}\rangle \in \mathcal{C}_Y$, and $x'_{s+1}x'_s = \phi(x_s x_{s+1})$, for some $1 \le s \le m-1$ and $m > 1$. Note that this makes sense on arrows of length at least 2. Note also that diverting A to A' over ϕ does not change the number of edges of given colour in A. Further, since $\phi^2 = 1$, if A' is obtained from A by diverting over ϕ, then also A is obtained from A' by diverting in $\mathcal{C}_Y$ over ϕ: simply use that $x_s x_{s+1} = \phi(x'_{s+1}x'_s)$.

Let $v, w \in \mathcal{C}$. For $A, A' \in \mathcal{C}(v, w)$, we let

$$A\mathcal{R}_{v,w}A'$$

if $A' = A$ or there is a finite sequence $A_0 = A, A_1, \dots, A_n = A'$ of elements in $\mathcal{C}$ so that A_{p+1} for $1 \le p \le n-1$ is obtained from A_p by diverting over ϕ. The number of edges of the same colour stays the same in each A_p.

We claim that $R_{v,w}$ is an equivalence relation. The reflexivity $A\mathcal{R}_{v,w}A$ is clear for each A.

To see that the relation is symmetric, let $A_0 = A, A_1, \dots, A_n = A'$ be a sequence of morphisms in C implementing the relation $A\mathcal{R}_{v,w}A'$. Then, the sequence $A'_0 = A'$, $A'_1 = A_{n-1}, \dots, A'_{n-1} = A_1$, $A'_n = A_0$ will implement $A'\mathcal{R}_{v,w}A$.

For transitivity, suppose that $A\mathcal{R}_{v,w}A'$ and $A'\mathcal{R}_{v,w}A''$ are represented by the sequences

$$A_0 = A,\ A_1, \dots, A_n = A', \quad B_0 = A',\ B_1, \dots, B_p = A''.$$

Then, it is clear that $A_0 = A, A_1, \dots, A_n = A', A_{n+1} = B_1, A_{n+2} = B_2, \dots, A_{n+p} = B_p$ implements $A\mathcal{R}_{v,w}A''$.

We next claim that the equivalence relation $\mathcal{R}$ is preserved by the composition of morphisms. Suppose that $B \in \mathcal{C}(v', v)$ and $A\mathcal{R}_{v,w}A'$. Let $A_0 = A, A_1, \dots, A_n = A'$ implement the equivalence between A, A'. Then $C_0 = A_0 \circ B, C_1 = A_1 \circ B, \dots, C_n = A_n \circ B$ implements the relation

$$(A \circ B)\mathcal{R}_{v',w}(A' \circ B).$$

If now $D \in \mathcal{C}(w, w')$, then similarly we have $(D \circ A)\mathcal{R}_{v,w'}(D \circ A')$. In all, $\mathcal{R}$ is a congruence in the sense of [17, p. 52]. Hence, there is a category $\mathcal{C}/\mathcal{R}$ with object set V, the object set of $\mathcal{C}$, and the set of morphisms $(\mathcal{C}/\mathcal{R})(v, w)$ equal to the quotient $\mathcal{C}(v, w)/\mathcal{R}_{v,w}$, for $v, w \in V$. We denote the class in $\mathcal{C}(v, w)/\mathcal{R}_{v,w}$ of a morphism $A \in \mathcal{C}(v, w)$ by $\dot{A}$.

Next, we show the existence of the functor $d : \mathcal{C}/\mathcal{R} \to \mathbb{N}^k$. For an object v in $\mathcal{C}/\mathcal{R}$, we let $d(v) = 0$. For an elementary arrow A in $\mathcal{C}$ with $A \in \mathcal{C}(v, w)$ for $v, w \in V$, there is a unique colour $i \in \{1, \dots, k\}$ of the edge underlying the arrow. We set $d(A) := e_i$.

We extend this to an arbitrary arrow A in $\mathcal{C}$ by

$$d(A) := e_{i_1} + e_{i_2} + \cdots + e_{i_n} \text{ if } A = \langle v_1, x_1, v_2, \dots, x_n, v_{n+1}\rangle,\ x_s \in E_{i_s},\ 1 \le s \le n,$$

where i_s are not necessarily distinct. If two arrows A, A' are given as in (3.3) and (3.4) with $A\mathcal{R}_{v,w}A'$, we have that $d(A) = d(A')$ because at the vertex v_s where the path in A is diverted, we have $e_{i_s} + e_{i_{s+1}} = e_{i_{s+1}} + e_{i_s}$. Let

$$n_K = \begin{cases} \#\{s \mid 1 \le s \le n,\ i_s = K\} & \text{if } K \in \{i_1, i_2, \dots, i_n\} \\ 0 & \text{if } K \notin \{i_1, i_2, \dots, i_n\} \end{cases} \tag{3.5}$$

for $K = 1, \dots, k$. Then,

$$d(\dot{A}) := (n_1, n_2, \dots, n_k) \in \mathbb{N}^k$$

is well defined. Moreover, it satisfies $d(\dot{A}\dot{B}) = d(\dot{A}) + d(\dot{B})$ for $\dot{A} \in \mathcal{C}(v, w)/\mathcal{R}_{v,w}$ and $\dot{B} \in \mathcal{C}(v', v)/\mathcal{R}_{v',v}$. This is similar for composition on the right. This defines the functor $d : \mathcal{C}/\mathcal{R} \to \mathbb{N}^k$. We will show that d enjoys the unique factorisation property required of a higher-rank graph.

For this we fix an ordering of the colours and show, as an intermediate step, that every morphism $\dot{A}$ has a representative A in $\mathcal{C}$ with all edges of the same colour grouped together. For simplicity of notation, we assume that the colours appear in the order

$$\{1, 2, \dots, k\}.$$

We claim that each class $\dot{A}$ contains a representative

$$A = \langle v_1, x_1, \dots, x_{n_1}, v_{n_1+1}, x_{n_1+1}, \dots, x_{n_1+n_2}, \dots, v_m\rangle, \tag{3.6}$$

where $x_1, \dots, x_{n_1}$ are edges of colour i_1, followed by edges $x_{n_1+1}, \dots, x_{n_1+n_2}$ of colour i_2, and so on, with n_K edges of colour i_K at the end, where $i_1 > i_2 > \cdots > i_K$ are distinct colours in $\{1, \dots, k\}$. Thus, the largest colour appears nearest to the origin of the path on the digraph that determines the arrow, and the colours appear in decreasing order along the path towards it terminus.

Note that (3.6) is trivially satisfied if $\dot{A}$ is the class of an arrow A on the digraph that only follows one colour. Assume next that there are only two colours $i > j$ in a representative A for $\dot{A}$. Let v_s be the first vertex at which we have a bicoloured path

$\langle v_s, x_s, v_{s+1}, x_{s+1}, v_{s+2}\rangle$ with $x_s \in E_j, x_{s+1} \in E_i$. Then, we divert this into the equivalent path $\langle v_s, x'_{s+1}, v'_{s+1}, x'_s, v_{s+2}\rangle$ for unique $x'_s \in E_j$ and $x'_{s+1} \in E_i$. If there are no more edges from E_i past the vertex v_{s+2}, we are done. If not, continue the process until we are left with a representative for $\dot{A}$ of the form required in (3.6).

Assume now that there are three distinct colours $l > j > i$ in a representative A for $\dot{A}$. If all three colours appear in decreasing order l, j, i in the path supporting A, there is nothing to prove. If two colours appear in reverse order at a time, we reduce to the previous case. Assume now that a tricoloured path appears in increasing order of colours. For simplicity of notation, we may assume that this is an arrow in $\mathcal{C}$ of the form

$$\langle v_1, x, v_2, y, v_3, z, v_4\rangle, \quad v_1, \dots, v_4 \in V,\ x \in E_i,\ y \in E_j,\ z \in E_l.$$

This is the composition $\langle v_3, z, v_4\rangle \circ \langle v_2, y, v_3\rangle \circ \langle v_1, x, v_2\rangle$ (or $z \circ y \circ x$) of elementary arrows in colours $l > j > i$, and we claim that it is the same in $\mathcal{C}/\mathcal{R}$ as a (unique) arrow of the form

$$\langle v_1, \bar{z}, v'_2, \bar{y}, v'_3, \bar{x}, v_4\rangle, \quad \bar{z} \in E_l, \bar{y} \in E_j, \bar{x} \in E_i, \tag{3.7}$$

where $v'_2, v'_3 \in V$. By successive application of this reversing of order, it will follow that $\dot{A}$ admits a representative with the colours appearing as in (3.6).

Since $\langle v_1, x, v_2, y, v_3\rangle \in \mathcal{C}(v_1, v_3) \cap \mathcal{C}_Y$, there are unique $x^1 \in E_i, y^1 \in E_j$ such that

$$\langle v_1, x, v_2, y, v_3\rangle \mathcal{R}_{v_1,v_3} \langle v_1, y^1, u_1, x^1, v_3\rangle,$$

with $u_1 = t(y^1) = o(x^1)$. For simplicity, write this as $(y \circ x)\mathcal{R}_{v_1,v_3}(x^1 \circ y^1)$. Continuing this way, the bijection ϕ prescribes edges $x_1, x_2, x^2 \in E_i$, $y_1, y_2, y^2 \in E_j$, and $z_1, z^1, z_2, z^2 \in E_l$ such that

$$\begin{aligned}
(z \circ x^1)\mathcal{R}_{u_1,v_4}\,(x^2 \circ z^1), &\quad \text{with } o(x^2) = t(z^1) = u_2,\\
(z^1 \circ y^1)\mathcal{R}_{v_1,u_2}\,(y^2 \circ z^2), &\quad \text{with } o(y^2) = t(z^2) = u_3,\\
(z \circ y)\mathcal{R}_{v_2,v_4}\,(y_1 \circ z_1), &\quad \text{with } o(y_1) = t(z_1) = u_4,\\
(z_1 \circ x)\mathcal{R}_{v_1,u_4}\,(x_1 \circ z_2), &\quad \text{with } o(x_1) = t(z_2) = u_5,\\
(y_1 \circ x_1)\mathcal{R}_{u_3,v_4}\,(x_2 \circ y_2), &\quad \text{with } o(x_2) = t(y_2) = u_6,
\end{aligned}$$

where $u_1 = o(z^1)$ and $u_4 = t(x_1)$. By our assumption (F2) on DG, we have $u_2 = u_6$ and $u_3 = u_5$ and

$$\begin{aligned}
\bar{x} &:= x_2 = x^2,\\
\bar{y} &:= y_2 = y^2,\\
\bar{z} &:= z_2 = z^2.
\end{aligned}$$

With $v'_2 := u_3$ and $v'_3 := u_2$, this gives the claimed representative in (3.7), where we have used that the relation $\mathcal{R}$ preserves the composition of morphisms. Successive applications of this reversing of order in a tricoloured path show that $\dot{A}$ admits a representative as in

(3.6). Note at the same time that the class in $\mathcal{C}/\mathcal{R}$ of the morphism $\bar{x} \circ \bar{y} \circ \bar{z}$ decomposes uniquely as products of two morphisms along any choice in $\mathcal{R}_{v_1,v_4}$ which involves tricoloured paths. More precisely, the decompositions are

$$z \circ (y \circ x) \text{ corresponding to } e_l + (e_j + e_i), \tag{3.8}$$

$$z \circ (x^1 \circ y^1) \text{ corresponding to } e_l + (e_i + e_j), \tag{3.9}$$

$$\bar{x} \circ (z^1 \circ y^1) \text{ corresponding to } e_i + (e_l + e_j), \tag{3.10}$$

$$\bar{x} \circ (\bar{y} \circ \bar{z}) \text{ corresponding to } e_i + (e_j + e_l), \tag{3.11}$$

$$y_1 \circ (x_1 \circ \bar{z}) \text{ corresponding to } e_j + (e_i + e_l), \tag{3.12}$$

$$y_1 \circ (z_1 \circ x) \text{ corresponding to } e_j + (e_l + e_i). \tag{3.13}$$

If more than three colours appear in a representative A for $\dot{A}$, say $i_1 < i_2 < \cdots < i_K$ with $K \geq 4$, and if a path supporting A has colours in increasing order, then we resort to the previous cases. Thus, if $K = 4$ and a path appears with colours in the order $i_1 < i_2 < i_3 < i_4$, we first reverse the path onto colours i_1, i_4, i_3, i_2, working from the terminus of the path (source of its arrow in the category) to the left towards its origin. Then, we move the edges of colour i_1 past the ones of colours i_4, i_3, i_2, using the previous cases. This is the same for $K > 4$. In all, (3.6) follows.

Now, we are ready to prove the factorisation property of d. Suppose that $\dot{A}$ is in $\mathcal{C}/\mathcal{R}$ with $d(\dot{A}) = (n_1, n_2, \ldots, n_k) \in N^k$. We must show that whenever $(n_1, n_2, \ldots, n_k) = (m_1, m_2, \ldots, m_k) + (p_1, p_2, \ldots, p_k)$ in $\mathbb{N}^k$, there are unique morphisms $\dot{B}, \dot{C}$ in $\mathcal{C}/\mathcal{R}$ so that

$$\dot{A} = \dot{B}\dot{C}, \quad d(\dot{B}) = (m_1, m_2, \ldots, m_k), \quad \text{and} \quad d(\dot{C}) = (p_1, p_2, \ldots, p_k).$$

The proof is structured into cases determined by the number of non-zero entries n_s, that is, by the number of colours that appear in a morphism in the class $\dot{A}$.

Case 1: single colour. Thus, $n_i > 0$ for a unique $1 \leq i \leq k$. If $n_i = 1$, then by our construction of $\mathcal{C}/\mathcal{R}$ and d we know that $\dot{A}$ is the class of an elementary arrow $A = \langle v_1, x_1, v_2 \rangle$ with $x_1 \in E_i$ and only the trivial decomposition involving identities at v_1, v_2 is possible. If $n_i > 1$, then a representative for A consists of a path of length n_i along the edges in E_i, and so we can decompose $\dot{A} = \dot{B}\dot{C}$ with $d(\dot{B}) = m_i$, $d(\dot{C}) = p_i$ for any choice $n_i = m_i + p_i$ in $\mathbb{N}$.

Case 2: two colours. By our earlier claim (3.6), we may assume that

$$d(\dot{A}) = (n_1, n_2, \ldots, n_k)$$

with $n_i \geq 1, n_j \geq 1$ at $i > j$ in $\{1, \ldots, k\}$, and $n_l = 0$ at all other entries.

Case 2.1: $n_i = n_j = 1$. By the definition of $\mathcal{C}/\mathcal{R}$, the morphism $\dot{A}$ is the class of a bicoloured arrow $\langle v_1, x_1, v_2, x_2, v_3 \rangle$, with $x_1 \in E_i$ and $x_2 \in E_j$. Let $x_2' \in E_j$ and $x_1' \in E_i$ so that $\phi(x_1 x_2) = x_2' x_1'$, and put $v_2' = t(x_2') = o(x_1')$. Then,

$$A = \langle v_2', x_1', v_3 \rangle \langle v_1, x_2', v_2' \rangle = \langle v_2, x_2, v_3 \rangle \langle v_1, x_1, v_2 \rangle$$

give the unique decompositions $\dot{A} = \dot{B}\dot{C}$ according to the decompositions $e_i + e_j$ and $e_j + e_i$ of $d(\dot{A})$.

Case 2.2: $n_i > 1, n_j \geq 1$. Without loss of generality, we may assume that $n_i \geq n_j$.

The given morphism $\dot{A}$ is the class of an arrow following a path with n_i edges of colour i and n_j edges of colour n_j. Let $n_i = m_i + p_i$ and $n_j = m_j + p_j$, and consider first the case that $m_i, p_j \geq 1$. We then divert the path in A after $p_i + m_i - 1$ edges of colour i by applying Case 2.1 to obtain a representative for $\dot{A}$ in the form of a path with $p_i + m_i - 1$ edges of colour i, then one edge of colour j followed by an edge of colour i, and finally with $p_j + m_j - 1$ edges of colour j at the end. If $p_i + m_i - 2 = 0$ and $p_j + m_j - 2 = 0$, we are done, having recovered Case 2.1. Otherwise, if for example $p_i + m_i - 2 \geq 1$, then apply Case 2.1 to divert an edge of colour j onto one of colour i, thus obtaining a representative with $p_i + m_i - 2$ edges of colour i, then one edge of colour j followed by two of colour i, and finally $m_j + p_j - 1$ edges of colour j. If also $m_j + p_j - 2 \geq 1$, we divert another edge of colour j successively past the two of colour i to get $p_i + m_i - 2$ of colour i, two of colour j, two of colour i, and finally $p_j + m_j - 2$ of colour j. Continuing this process, we divert all m_i edges of colour i followed by the p_j edges of colour j into a path where there are p_j edges of colour j first, followed by m_i edges of colour i. This determines the required decomposition of $\dot{A} = \dot{B}\dot{C}$, where $d(\dot{B}) = m_i e_i + m_j e_j$ and $d(\dot{C}) = p_i e_i + p_j e_j$.

The remaining cases where only two colours are present are treated similarly.

Case 3: three colours. By our earlier claim, we may assume that

$$d(\dot{A}) = (n_1, n_2, \ldots, n_k)$$

with $n_i \geq 1, n_j \geq 1, n_l \geq 1$ at $i > j > l$ in $\{1, \ldots, k\}$, and $n_h = 0$ at all other entries $h \in \{1, \ldots, k\}$. Assuming $n_i = m_i + p_i$, $n_j = m_j + p_j$, and $n_l = m_l + p_l$, then considering cases as before and applying the factorisations (3.8)–(3.13) accordingly results in a factorisation $\dot{A} = \dot{B}\dot{C}$ with $d(\dot{B}) = m_i e_i + m_j e_j + m_l e_l$ and $d(\dot{C}) = p_i e_i + p_j e_j + p_l e_l$. We omit the details.

In the general case where $d(\dot{A})$ has nonzero entries in more than three distinct colours $i_1 > i_2 > \cdots > i_K$, $K \geq 4$, the problem reduces to decomposing along any expression of $(n_1, n_2, \ldots, n_K)$ where at least one of the entries n_{i_s}, $s = 1, \ldots, K$, is expressed as a sum $m_{i_s} + p_{i_s}$ for $m_{i_s}, p_{i_s} \geq 1$. This reduces to Cases 2 and 3. ∎

We remark that an alternative argument to construct a higher-rank graph from the input of a k-dimensional digraph could be obtained by appealing to [11]. The main step is to identify a complete collection of squares that is associative based on DG and then an application of [11, Theorem 4.4] provides the desired k-graph.

Remark 3.2. It is possible to express the bijection ϕ using the notation from [13]. If C and D are directed 1-graphs with a common set of vertices $V = C^0 = D^0$, distinct sets of edges C^1, D^1, and commuting vertex matrices, let

$$C^1 * D^1 = \{(x, y) \in C^1 \times D^1 \mid t(x) = o(y)\}.$$

Then, the bijection ϕ in (F1) is given by its restrictions $\phi_{i,j} : E_i^1 * E_j^1 \to E_j^1 * E_i^1$, for all $i \neq j$ in $\{1, 2, \ldots, k\}$. This construction is reminiscent of an older idea of a product of two (possibly directed) graphs, as described in Ore's monograph [21].

We note that condition (F2) in Definition 1.1 is vacuous when $k = 2$. We shall refer to $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k$ as the 1-*skeleton* of Λ_{DG}.

Theorem 3.3. *For any complex $\mathcal{X}$ with universal cover equal to the product of k regular trees, where $k \geq 2$, there is a k-dimensional digraph* $\mathrm{DG}(\mathcal{X})$ *defined by sending a vertex of $\mathcal{X}$ to a vertex in* $\mathrm{DG}(\mathcal{X})$*, and by sending each geometric edge in $\mathcal{X}$ to two edges in the edge set of* $\mathrm{DG}(\mathcal{X})$*, of the same colour and opposite orientations.*

We prove this theorem in stages. First, we prove the one-vertex case for $k = 2$, where the statement is a consequence of the description of a one-vertex 2-complex as $S_{A,B}$ for a VH-structure A, B. Then, we prove the case $k = 3$ by employing geometric cubes. First, we record a key consequence whose proof is immediate from Theorems 3.1 and 3.3.

Corollary 3.4. *Given $\mathcal{X}$ a k-cube complex with $k \geq 2$, there is a k-graph $\Lambda_{\mathrm{DG}(\mathcal{X})}$ with a vertex set equal to the vertex set of $\mathcal{X}$ and whose* 1*-skeleton contains two edges for each geometric edge of $\mathcal{X}$, in a colour preserving and orientation reversing assignment. Further, $\Lambda_{\mathrm{DG}(\mathcal{X})}$ is row-finite.*

Definition 3.5. Given a k-cube complex $\mathcal{X}$, its associated C^*-algebra is the higher-rank graph C^*-algebra $C^*(\Lambda_{\mathrm{DG}(\mathcal{X})})$.

To simplify the notation, we write $\Lambda(\mathcal{X})$ in place of $\Lambda_{\mathrm{DG}(\mathcal{X})}$. Continuing our convention from the proof of Theorem 3.1, we use letters such as x, y to denote both generic elements in the edge set E of $\mathrm{DG}(\mathcal{X})$ and their corresponding morphisms in $\Lambda(\mathcal{X})$ associated with elementary arrows. In particular, a bicoloured path xy on the digraph with $x \in E_i$, $y \in E_j$ for distinct colours $i \neq j$ will be $y \circ x$ as composition as morphisms in the k-graph.

As another point of notation, in the proof of the next result and that of Theorem 3.3 in the one-vertex case, we shall distinguish between labels such as a in a generating subset A of a k-cube group G and the label of the edge it defines in the associated k-digraph, for which we reserve a notation of the form α. In the group we have $aa^{-1} = 1 = 1_G$, so in the one-vertex complex a^{-1} means orientation reversal of the edge labelled with a, while in the k-dimensional digraph the edge labelled a is sent into distinct directed edges α_1, α_2 (with no cancellation inherited from the group).

Lemma 3.6. *Assume that $S_{A,B}$ is a one-vertex square complex with associated group G given by a VH-structure (A, B) with $\#A$ and $\#B$ both even positive integers. Suppose that*

$$A = \{a_1, \ldots, a_L, a_{L+1}, \ldots, a_{2L}\} \quad \textit{and} \quad B = \{b_1, \ldots, b_K, b_{K+1}, \ldots, b_{2K}\}, \tag{3.14}$$

with $a_r a_{L+r} = 1$ in G for all $r = 1, \ldots, L$ and $b_s b_{K+s} = 1$ in G for all $s = 1, \ldots, K$, with $K, L \geq 1$. In particular, for each $r = 1, \ldots, L$, we have that a_r and a_{L+r} label the same geometric edge in $S_{A,B}$, but with opposite orientations. Similarly for b_s, b_{K+s} with $s = 1, \ldots, K$.

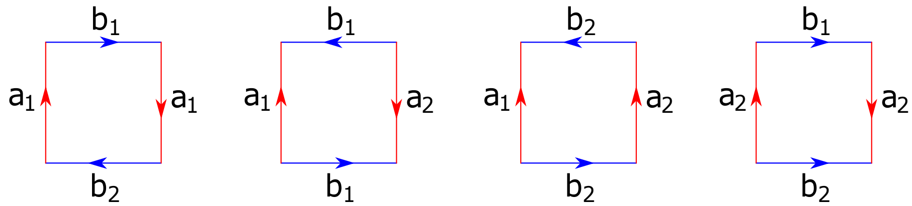

Figure 2. A concrete example of $(4,4)$-group from four squares.

Then, there is a 2-dimensional digraph in the sense of Definition 1.1 with the edge set $E(S_{A,B}) = E_1 \sqcup E_2$ obtained by associating with each a_r for $r = 1,\dots,2L$ a directed edge $\mathfrak{a}_r$ in E_1, and with each b_s for $s = 1,\dots,2K$ a directed edge $\mathfrak{b}_s$ in E_2.

Proof. We have that for each $a_r \in A$ and $b_s \in B$, with $r = 1,\dots,2L$, $s = 1,\dots,2K$, there are unique $a_{l(r,s)} \in A$ and $b_{m(r,s)} \in B$, with $l(r,s) \in \{1,\dots,2L\}$ and $m(r,s) \in \{1,\dots,2K\}$, such that

$$a_r b_s = b_{m(r,s)} a_{l(r,s)}.$$

In particular, $a_r b_s$ is contained as a vertical-horizontal pair of edges in a unique geometric square in the family

$$S_*^{a_r,b_s}, \quad r = 1,\dots,2L,\ s = 1,\dots,2K,\ * = O, H, V, R,$$

with $b_{m(r,s)} a_{l(r,s)}$ forming the horizontal-vertical pair of edges in $S_*^{a_r,b_s}$ starting and ending at the same vertices as $a_r b_s$ (which we recall coalesce to the single vertex v of $S_{A,B}$).

Let xy be a path of length two with $x \in E_1$ and $y \in E_2$. Then, xy is uniquely determined by $x = \mathfrak{a}_r$ for some $r = 1,\dots,2L$ and $y = \mathfrak{b}_s$ for some $s = 1,\dots,2K$. Let now $a_r, b_s, a_{l(r,s)}$, and $b_{m(r,s)}$ be as above. Then, $y' := \mathfrak{b}_{m(r,s)} \in E_2$ and $x' := \mathfrak{a}_{l(r,s)} \in E_1$ determine a unique path of length two $y'x'$ so that $xy \sim y'x'$.

This defines the required bijection $\phi : Y \to Y$ with $\phi(xy) = y'x'$ on the set Y of all paths of length two of distinct colours. ∎

Example 3.7. A simple construction of a 2-graph based on the procedure of Corollary 3.4 recovers a known example; see [23,37] and [16, Example 11.1 (1)], where $\theta(i,j) = (i,j)$ is the identity permutation of the set $\{1,2\} \times \{1,2\}$. Consider the $(2,2)$-group $G = \mathbb{Z} \times \mathbb{Z}$ with generating sets $A = \{a, a^{-1}\}$ corresponding to the first copy of $\mathbb{Z}$ and $B = \{b, b^{-1}\}$ for the second copy. We have the commutation relation $ab = ba$ as the basis for a geometric square $S_O^{a,b}$. The one-vertex complex has two loops. An application of Lemma 3.6 yields a 2-dimensional digraph with edge set a disjoint union of $E_1 = \{\mathfrak{a}_1, \mathfrak{a}_2\}$ and $E_2 = \{\mathfrak{b}_1, \mathfrak{b}_2\}$, thus four loops, with the bijection ϕ on the set of paths of length two of distinct colours read off from $S_O^{a,b}$, $S_H^{a,b}$, $S_V^{a,b}$, and $S_R^{a,b}$ as follows:

$$\mathfrak{a}_1\mathfrak{b}_1 \sim \mathfrak{b}_1\mathfrak{a}_1, \mathfrak{a}_1\mathfrak{b}_2 \sim \mathfrak{b}_2\mathfrak{a}_1, \mathfrak{a}_2\mathfrak{b}_1 \sim \mathfrak{b}_1\mathfrak{a}_2 \quad \text{and} \quad \mathfrak{a}_2\mathfrak{b}_2 \sim \mathfrak{b}_2\mathfrak{a}_2.$$

Example 3.8. We now present a 2-graph from this recipe where the group G is not of product type. As we will explain, Figure 2 shows an example of a $(4,4)$-group G, cf. [35].

The four squares are geometric squares representing the 2-cells of an associated complex $S_{A,B}$, where $A = \{a_1, a_2, a_3, a_4\}$ for $a_3 = a_1^{-1}$ and $a_4 = a_2^{-1}$, and $B = \{b_1, b_2, b_3, b_4\}$ for $b_3 = b_1^{-1}$ and $b_4 = b_2^{-1}$. Here, $L = K = 2$, cf. Lemma 3.6. With our convention in (2.7) we have, from left to right, $S_O^{a_1,b_1}$, $S_O^{a_1,b_3}$, $S_O^{a_1,b_4}$, and $S_O^{a_2,b_1}$.

The associated 2-graph $\Lambda(S_{A,B})$ from Corollary 3.4 has 1-skeleton determined by the 2-dimensional digraph whose edges are given by the disjoint union of

$$E_1 = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4\} \quad \text{and} \quad E_2 = \{\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3, \mathfrak{b}_4\};$$

see Lemma 3.6. Let us now describe explicitly the bijection $\phi : Y \to Y$. Note that the digraph has 4 loops of one colour (red) and 4 loops of the second colour (blue).

We have 16 paths of length two of the form xy, where $x \in E_1$ and $y \in E_2$, given by all the possible $\mathfrak{a}_i \mathfrak{b}_j$ with $i, j = 1, \ldots, 4$. Correspondingly, we have all possible vertical-horizontal pairs of edges $a_i b_j$ in the collection of geometric squares

$$S_*^{a_j,b_j}, \quad i, j = 1, \ldots, 4, \; * = O, V, R, H.$$

Pick for each $a_i b_j$ the unique $a_{l(i,j)} \in A$ and $b_{m(i,j)} \in B$ such that $b_{m(i,j)} a_{l(i,j)}$ is the corresponding horizontal-vertical pair of edges in the same square, and let $y' = \mathfrak{b}_{m(i,j)}$, $x' = \mathfrak{a}_{l(i,j)}$ as prescribed by the proof of Lemma 3.6.

Explicitly, corresponding to the horizontal-vertical pairs of edges in the geometric square

$$\{S_O^{a_1,b_1}, S_V^{a_1,b_1}, S_R^{a_1,b_1}, S_H^{a_1,b_1}\},$$

it is seen that $\phi(\mathfrak{a}_1\mathfrak{b}_1) = \mathfrak{b}_4\mathfrak{a}_3$, $\phi(\mathfrak{a}_3\mathfrak{b}_4) = \mathfrak{b}_1\mathfrak{a}_1$, $\phi(\mathfrak{a}_1\mathfrak{b}_2) = \mathfrak{b}_3\mathfrak{a}_3$, and $\phi(\mathfrak{a}_3\mathfrak{b}_3) = \mathfrak{b}_2\mathfrak{a}_1$.

Similarly, from the geometric square

$$\{S_O^{a_1,b_3}, S_V^{a_1,b_3}, S_R^{a_1,b_3}, S_H^{a_1,b_3}\}$$

we get $\phi(\mathfrak{a}_1\mathfrak{b}_3) = \mathfrak{b}_1\mathfrak{a}_4$, $\phi(\mathfrak{a}_3\mathfrak{b}_1) = \mathfrak{b}_3\mathfrak{a}_2$, $\phi(\mathfrak{a}_2\mathfrak{b}_3) = \mathfrak{b}_1\mathfrak{a}_3$, and $\phi(\mathfrak{a}_4\mathfrak{b}_1) = \mathfrak{b}_3\mathfrak{a}_1$; from

$$\{S_O^{a_1,b_4}, S_V^{a_1,b_4}, S_R^{a_1,b_4}, S_H^{a_1,b_4}\}$$

we get $\phi(\mathfrak{a}_1\mathfrak{b}_4) = \mathfrak{b}_2\mathfrak{a}_2$, $\phi(\mathfrak{a}_3\mathfrak{b}_2) = \mathfrak{b}_4\mathfrak{a}_4$, $\phi(\mathfrak{a}_4\mathfrak{b}_4) = \mathfrak{b}_2\mathfrak{a}_3$, and $\phi(\mathfrak{a}_2\mathfrak{b}_2) = \mathfrak{b}_4\mathfrak{a}_1$; finally, from the geometric square

$$\{S_O^{a_2,b_1}, S_V^{a_2,b_1}, S_R^{a_2,b_1}, S_H^{a_2,b_1}\}$$

we get $\phi(\mathfrak{a}_2\mathfrak{b}_1) = \mathfrak{b}_2\mathfrak{a}_4$, $\phi(\mathfrak{a}_4\mathfrak{b}_2) = \mathfrak{b}_1\mathfrak{a}_2$, $\phi(\mathfrak{a}_2\mathfrak{b}_4) = \mathfrak{b}_3\mathfrak{a}_4$, and $\phi(\mathfrak{a}_4\mathfrak{b}_3) = \mathfrak{b}_4\mathfrak{a}_2$; this describes the bijection ϕ completely.

The link of $S_{A,B}$ at its vertex v is the complete bipartite graph of type $(4, 4)$; see Figure 3.

It follows that $G := \pi_1(S_{A,B}, v)$ is a $(4, 4)$-group. In fact, G is the fundamental group of a CAT(0) complex with Gromov link condition; see [9]. We recall that every edge of the complex belongs to four squares; see Figure 4 for a fragment of the universal cover of the complex showing the edge a_1 belonging to four squares (in the universal cover).

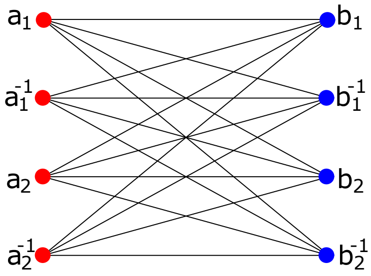

Figure 3. The link of the complex.

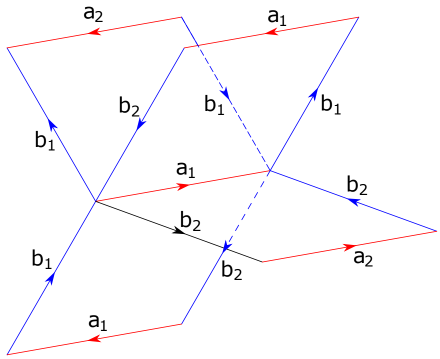

Figure 4. Fragment of the universal cover showing the edge a_1.

Remark 3.9. The group with the same VH-structure as in Figure 2 appears also in [12, Section 7], as the group 2×2.37 in their list. However, their 2-graph is different from the one in Example 3.8, since, if we translate the notions of [12] into higher-rank graphs, the 2-graph corresponding to the group 2×2.37 would have sixteen vertices. In general, the 2-graphs of [12] corresponding to $(2m, 2n)$ groups give 2-graphs with $4mn$ vertices, $4(m-1)mn$ edges of one colour, and $4mn(n-1)$ edges of another colour. The 2-graphs of [26, 28] have $3(q^2+q+1)$ vertices and $3(q^2+q+1)q$ edges of each colour for q being a prime power different from 3.

Proof of Theorem 3.3, the general case. Fix $k \geq 3$. Assume first that we have a one-vertex k-cube complex P; thus we may take it of form $P_{A_1,\dots,A_k}$ associated with a k-cube group G with underlying structure determined by the ordered tuple $(A_1, \dots, A_k)$. As explained, the edges of the complex are labelled by the generators of G. For each $i = 1, \dots, k$ we write

$$A_i = \{a_1^i, \dots, a_{L_i}^i, a_{L_i+1}^i, \dots, a_{2L_i}^i\},$$

with the convention that $a_r^i a_{L_i+r}^i = 1$ in G for $1 \leq r \leq L_i$. Define a digraph with the edge set $E = E_1 \sqcup E_2 \sqcup \cdots \sqcup E_k$ by the assignment that to each a_r^i corresponds a directed edge $\mathfrak{a}_r^i$ in E_i, with $i = 1, \dots, k$ and $r = 1, \dots, 2L_i$. We must identify a bijection ϕ on the set of bicoloured edges and establish conditions (F1) and (F2) of Definition 1.1.

Suppose that xy is a path of length two in E with $x \in E_i$ and $y \in E_j$ for distinct i, j in $\{1, \ldots, k\}$. Then, there are $a_r^i \in A_i$ and $b_s^j \in A_j$ for unique $r = 1, \ldots, 2L_i$ and $s = 1, \ldots, 2L_j$, such that $x = \alpha_r^i$ and $y = \alpha_s^j$. Since we have a cube complex, there is an associated square complex P_{A_i, A_j} with corresponding group G_{A_i, A_j}, where we choose the convention that A_i is vertical and A_j is horizontal direction. Lemma 3.6 implies that there is a unique path of length two $y'x'$ with $y' \in A_j$ and $x' \in A_i$, corresponding to a unique square with vertical-horizontal and horizontal-vertical pairs given by

$$a_r^i a_s^j = a_{m(r,s)}^j a_{l(r,s)}^i$$

such that $xy \sim y'x'$. Here, $a_{m(r,s)}^j \in A_j$ and $a_{l(r,s)}^i \in A_i$ are uniquely determined. This provides the desired bijection ϕ and settles requirement (F1).

Next, suppose that we are given a path xyz with $x \in E_i$, $y \in E_j$, $z \in E_l$ for distinct colours i, j, l in $\{1, \ldots, k\}$. The key ingredient is that by condition (5) in Definition 2.3, each directed cube, in the sense of (F2), arises as a directed copy of one of the 3-dimensional cubes of the complex. We now identify a directed cube satisfying the hypotheses of (F2) and an associated geometric 3-cube.

First, there is a unique square S_1^{ij} which contains a vertical-horizontal pair $a_r^i a_s^j$ with $a_r^i \in A_i, a_s^j \in A_j$ so that $x = \alpha_r^i$ and $y = \alpha_s^j$. Upon completing the square S_1^{ij} to $a_r^i a_s^j = a_{s^1}^j a_{r^1}^i$, as in the beginning of the proof, for unique $r^1 \in \{1, \ldots, 2L_i\}$ and $s^1 \in \{1, \ldots, 2L_j\}$, we have

$$xy \sim y^1 x^1 \quad \text{for } x^1 = \alpha_{r^1}^i \text{ and } y^1 = \alpha_{s^1}^j.$$

Next, we use x^1 and z to extract a geometric square S_2^{il}, determined by $a_{r^1}^i a_t^l = a_{t^1}^l a_{r^2}^i$, for unique $t^1 \in \{1, \ldots, 2L_l\}$ and $r^2 \in \{1, \ldots, 2L_i\}$, so that

$$x^1 z \sim z^1 x^2 \quad \text{for } z = \alpha_t^l,\ z^1 = \alpha_{t^1}^l \text{ and } x^2 = \alpha_{r^2}^i.$$

Finally, by using y^1, z^1 we extract a geometric square S_3^{jl}, determined by $a_{s^1}^j a_{t^1}^l = a_{t^2}^l a_{s^2}^j$, for unique $s^2 \in \{1, \ldots, 2L_j\}$ and $t^2 \in \{1, \ldots, 2L_l\}$, so that

$$y^1 z^1 \sim z^2 y^2 \quad \text{for } y^2 = \alpha_{s^2}^j \text{ and } z^2 = \alpha_{t^2}^l.$$

There is a unique geometric cube containing the squares S_1^{ij}, S_2^{il}, and S_3^{jl}, where the notation follows the convention after Figure 1, and $a_r^i a_s^j a_t^l$ is a path joining two vertices in the cube at longest possible distance due to condition (5) in Definition 2.3. Thus, in this geometric cube, we have also obtained the path $a_{t^2}^l a_{s^2}^j a_{r^2}^i$ opposite to $a_r^i a_s^j a_t^l$ and joining the same vertices in the cube.

If we perform the same argument starting with y, z to obtain $yz \sim z_1 y_1$, followed by x, z_1 to obtain $xz_1 \sim z_2 x_1$ and finally x_1, y_1 to get $x_1 y_1 \sim y_2 x_2$, we find unique squares $S_4^{jl}, S_5^{il}, S_6^{ij}$ determined by $a_s^j a_t^l = a_{t_1}^l a_{s_1}^j$, $a_r^i a_{t_1}^l = a_{t_2}^l a_{r_1}^i$, and $a_{r_1}^i a_{s_1}^j = a_{s_2}^j a_{r_2}^i$, respectively, for $r_2 \in \{1, \ldots, 2L_i\}$, $s_2 \in \{1, \ldots, 2L_j\}$, and $t_2 \in \{1, \ldots, 2L_l\}$. So

$$x_2 = \alpha_{r_2}^i, \quad y_2 = \alpha_{s_2}^j, \quad \text{and} \quad z_2 = \alpha_{t_2}^l.$$

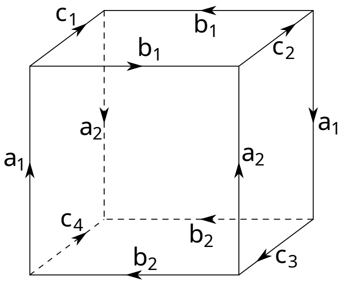

Figure 5. A geometric cube for the Γ_2 group.

Since $a^i_r a^j_s a^l_t$ is a common ijl-path, the squares $S^{ij}_1, S^{il}_2, S^{jl}_3, S^{jl}_4, S^{il}_5, S^{ij}_6$ determine the same geometric cube. We have that $a^l_{t_2} a^j_{s_2} a^i_{r_2}$ is another path in this 3-dimensional cube opposite to $a^i_r a^j_s a^l_t$ and joining vertices at the longest possible distance. Since there only is one path of the longest distance opposite to $a^i_r a^j_s a^l_t$ in a 3-dimensional cube, we must have

$$a^j_{s^2} = a^j_{s_2}, \quad a^l_{t^2} = a^l_{t_2}, \quad \text{and} \quad a^i_{r^2} = a^i_{r_2}.$$

Then, $x_2 = x^2$, $y_2 = y^2$, and $z_2 = z^2$, as required to fulfill condition (F2).

We now assume that $\mathcal{X}$ has N vertices, with $N \geq 2$, and we declare these to be the vertices of $\mathrm{DG}(\mathcal{X})$. As prescribed, each geometric edge in the complex is sent into two edges with opposite orientation in the edge set E of $\mathrm{DG}(\mathcal{X})$. For every path of length two xy with $x \in E_i$ and $y \in E_j$ for $i \neq j$ so that $o(y) = t(x)$, there is a unique geometric square S_1 in $\mathcal{X}$ in which xy corresponds to a vertical-horizontal pair of edges. Then, the corresponding horizontal-vertical pair of edges gives rise to $x' \in E_i$ and $y' \in E_j$ such that $xy \sim y'x'$, and this defines uniquely the bijection ϕ on the set Y of paths of length two of distinct colours required in (F1). The condition (F2) holds by the same argument as the one-vertex case because any ijl-coloured path xyz will be contained in a unique 3-cube in $\mathcal{X}$, determined through unique squares $S^{ij}_1, S^{il}_2, S^{jl}_3, S^{jl}_4, S^{il}_5, S^{ij}_6$. The difference is that since the complex has more than one vertex, we do not have a labelling of the edges by elements of the group G acting cocompactly on $\mathcal{X}$, but this does not affect the existence of the squares and of the 3-cube in which xyz determines a path with unique opposite path at the longest distance between the same vertices of the 3-dimensional cube. ∎

To visualise the argument in the proof of Theorem 3.3, we refer to Figure 1 and the generic geometric cube there. Let $x = a_1$ (or, for consistency, x is a directed edge in E_i labelled with $a_1 \in A_i$), $y = b_1$, and $z = c_3$. The argument produces the path $a^l_{t^2} a^j_{j^2} a^i_{r^2}$ given by $c_1 b_3 a_3$ following the squares

$$S^{ij}_1 = (a_1, b_1, a_2^{-1}, b_2^{-1}), \quad S^{il}_2 = (a_2, c_3, a_3^{-1}, c_2^{-1}), \quad \text{and} \quad S^{jl}_3 = (b_2, c_2, b_3^{-1}, c_1^{-1}).$$

Alternatively, it produces the path $a^l_{t_2} a^j_{j_2} a^i_{r_2}$ following the squares $(b_1, c_3, b_4^{-1}, c_4^{-1})$, $(a_1, c_4, a_4^{-1}, c_1^{-1})$, and $(a_4, b_4, a_3^{-1}, b_3^{-1})$.

Example 3.10. In Figure 5, we present a geometric cube which is part of the data of the 3-cube group Γ_2 from Section 2.8.

The generating sets of Γ_2 are $A_1=\{a_1,a_2,a_1^{-1},a_2^{-1}\}$, $A_2=\{b_1,b_2,b_3,b_1^{-1},b_2^{-1},b_3^{-1}\}$, and $A_3=\{c_1,c_2,c_3,c_4,c_1^{-1},c_2^{-1},c_3^{-1},c_4^{-1}\}$. There are $(|A_1|\cdot|A_2|\cdot|A_3|)/2^3=24$ cubes in total, where the factor 2^3 in the denominator corresponds to the fact that there are 8 vertices in the cube, and we can complete the cube starting with three edges of distinct colours from any one of them.

The cube in Figure 5 is obtained from the triple (a_1,b_1,c_2) of edges in the three alphabets by completing its faces with geometric squares. With the notation of Figure 1, the faces S_1^{12}, S_2^{13}, and S_3^{23} arise, respectively, from the group relations $a_1b_1a_4b_2$, $a_2c_2a_1c_3$, and $b_2c_4b_5c_3$ (identified with $b_2^{-1}c_3^{-1}b_2c_4^{-1}$). The remaining three faces correspond to the geometric squares $b_1c_2b_1c_5$, $a_1c_1a_2c_8$ (identified with $a_1c_1a_2c_4^{-1}$) and $a_4b_4a_1b_2$ (identified with $a_2^{-1}b_1^{-1}a_1b_2$).

3.1. Aperiodicity

In C^*-algebra theory, the classification of purely infinite, simple, unital, nuclear, C^*-algebras is a landmark result by Kirchberg–Phillips; see [22]. The aperiodicity of a higher-rank graph is an important property because together with cofinality it implies the simplicity of the associated C^*-algebra and further implies pure infiniteness if every vertex can be reached from a loop with an entrance. We next investigate the aperiodicity of $\Lambda(P)$ from Corollary 3.4.

We recall the necessary facts and notation from [13]. Let Λ be a k-graph. If $m=(m_i)_i$, $q=(q_i)_i\in\mathbb{N}^k$, we write $m\le q$ if $m_i\le q_i$ for all $i=1,\dots,k$. By Ω_k we denote the k-graph with vertex set $\Omega_k^0=\mathbb{N}^k$ and a set of elements (morphisms) consisting of pairs $(m,n)\in\mathbb{N}^k\times\mathbb{N}^k$ with $m\le n$ and $d(m,n)=n-m$. The set Λ^∞ of infinite paths consists of degree preserving functors $\omega:\Omega_k\to\Lambda$. An infinite path ω is *aperiodic* provided that for every $q\in\mathbb{N}^k$ and all $p\in\mathbb{Z}^k\setminus\{0\}$, there is $(m,n)\in\Omega_k$ such that $m+p\ge 0$ and $\omega(m+p+q,n+p+q)\ne\omega(m+p,n+p)$. The k-graph Λ satisfies the *aperiodicity condition (A)* provided that for every $v\in\Lambda^0$ there is an aperiodic path ω with $r(\omega)=v$.

In our case, the existence of an aperiodic infinite path will be provided by the theory of rigid k-monoids from [16]. Guided by the work of Lawson–Vdovina, we first extend the notion of left and right rigid to k-dimensional digraphs with one vertex. We note that the idea of rigidity below appeared in a first form in [34, Definition on p. 3, items (2), (3)].

Definition 3.11. Let DG be a k-dimensional digraph with one vertex and edge set $E=E_1\sqcup E_2\sqcup\cdots\sqcup E_k$ for $k\ge 2$. We say that

(1) DG is *right rigid* if for every $x'\in E_i$, $y'\in E_j$ with $i\ne j$ there are unique $x\in E_i$, $y\in E_j$ such that $xy'\sim yx'$.

(2) DG is *left rigid* if for every $x\in E_i$, $y\in E_j$, $i\ne j$, there are unique $x'\in E_i$, $y'\in E_j$ such that $xy'\sim yx'$.

Lemma 3.12. *Suppose that P is a one-vertex k-cube complex with underlying structure $(A_1,\dots,A_k)$ for $i=1,\dots,k$, where each A_i is of the form $\{a_1^i,\dots,a_{2L_i}^i\}$ and $a_r^ia_{L_i+r}^i=1$ in the associated group for all $1\le r\le L_i$. Let $\mathrm{DG}(P)$ be the associated k-dimensional digraph from Theorem 3.3. Then, $\mathrm{DG}(P)$ is left and right rigid.*

Proof. The two properties of being rigid above arise from the fact that the link of the vertex v in P has no multiple edges. Therefore, every top-left corner and every bottom-right corner appear exactly once in a geometric square. The formal proof is below.

Suppose that $x' \in E_i$ and $y' \in E_j$ for $i \neq j$ in $\{1, \ldots, k\}$. Since we have a cube complex, there is an associated square complex P_{A_i,A_j}. By our construction of $\mathrm{DG}(P)$, there are $a_r^i \in A_i$ and $b_s^j \in A_j$ for unique $r = 1, \ldots, 2L_i$ and $s = 1, \ldots, 2L_j$ such that $x' = \mathfrak{a}_r^i$ and $y' = \mathfrak{b}_s^j$. In the square complex P_{A_i,A_j}, there is a unique square of the form

$$S_O = \big(a_r^i, (b_s^j)^{-1}, (a_g^i)^{-1}, b_h^j\big)$$

for $g \in \{1, \ldots, 2L_i\}$ and $h \in \{1, \ldots, 2L_j\}$. The associated S_H is $(a_g^i, b_s^j, (a_r^i)^{-1}, (b_h^j)^{-1})$, and thus $a_g^i b_s^j = b_h^j a_r^i$ in G_{A_i,A_j}. Letting $x = \mathfrak{a}_g^i$ and $y = \mathfrak{b}_h^j$ gives $xy' \sim yx'$ in $\mathrm{DG}(P)$, as claimed.

Left rigidity is similar. Starting this time with $x \in E_i$ and $y \in E_j$ for distinct i, j in $\{1, \ldots, k\}$, we find $a_r^i \in A_i$ and $b_s^j \in A_j$ for unique $r \in \{1, \ldots, 2L_i\}$ and $s \in \{1, \ldots, 2L_j\}$ so that $x = \mathfrak{a}_r^i$ and $y = \mathfrak{b}_s^j$. Consider the unique square

$$S_O = \big((a_r^i)^{-1}, b_s^j, a_m^i, (b_n^j)^{-1}\big)$$

in P_{A_i,A_j}, and form its associated S_V, which is $(a_r^i, b_n^j, (a_m^i)^{-1}, (b_s^j)^{-1})$. Then, $a_r^i b_n^j = b_s^j a_m^i$ in G_{A_i,A_j}, so letting $x' = \mathfrak{a}_m^i$ and $y' = \mathfrak{b}_n^j$ leads to $xy' \sim yx'$ in the digraph, as claimed. ∎

Given a one-vertex k-cube complex P, the associated k-graph $\Lambda(P)$ is a monoid, being a category with a single object. It is a k-monoid in the sense of [16], with alphabets $E_i = \{\mathfrak{a}_1^i, \ldots, \mathfrak{a}_{2L_i}^i\}$ for $i = 1, \ldots, k$ where each $L_i \geq 1$.

Corollary 3.13. *Given a one-vertex k-cube complex P, the graph $\Lambda(P)$ is left and right rigid and satisfies the aperiodicity condition. In particular, $C^*(\Lambda(P))$ is simple.*

Proof. Let P be a one-vertex k-complex, which we may assume as in the hypothesis of Lemma 3.12. Let $\Lambda(P) = \Lambda(\mathrm{DG}(P))$ be the associated k-graph from Corollary 3.4. The right rigidity of the digraph implies that for any choice of elements y', x' with y' in the alphabet E_j and x' in E_i, where $i \neq j$, there are unique elements x and y in the alphabets E_i and E_j, respectively, so that $y' \circ x = x' \circ y$. This means that $\Lambda(P)$ is right rigid. Left rigid follows in a similar way. We conclude from [16, Corollary 11.10 and Lemma 4.15] that $\Lambda(P)$ is effective and hence admits an aperiodic infinite path. As there is only one vertex, the aperiodicity condition is satisfied. Since $\Lambda(P)$ is also cofinal, $C^*(\Lambda(P))$ is simple by [13, Proposition 4.8]. ∎

The constructions of [27] produce a purely infinite simple rank two Cuntz–Krieger algebra $\mathcal{A}$. This uses in a crucial way the fact that every word w of a given shape $m = (m_1, m_2) \in \mathbb{N}^2$ admits at least two distinct extensions w', w'', in the sense that the origin of w', w'' (with suitable interpretation) equals the terminus of w, and both have the same shape e_j for all $j = 1, 2$.

For a row-finite and source-free k-graph Λ, [13, Proposition 4.9] puts forward conditions that would imply $C^*(\Lambda)$ is purely infinite simple. A correct version of these conditions was identified in [32, Proposition 8.8], which we present here (writing *cycle* instead of *loop*): given a finitely aligned k-graph Λ, a morphism $\mu \in \Lambda \setminus \Lambda^0$ is a *cycle with an entrance* if $s(\mu) = r(\mu)$ and there exists $\alpha \in s(\mu)\Lambda$ having $d(\alpha) \leq d(\mu)$ and being distinct from the initial segment of μ of degree $d(\alpha)$. Thus, for some factorisation $\mu = \mu_1\mu_2$ where $d(\mu_1) = n \leq d(\mu)$, there exists $\alpha \neq \mu_1$ with $d(\alpha) = n$ and $r(\alpha) = r(\mu_1)$. Therefore, upon interpreting the concatenation of edges on the digraph $\mathrm{DG}(P)$ as the composition of morphisms in the associated $\Lambda(P)$, see (3.1), and by interpreting the constructions of [27] in terms of higher-rank graphs, the existence of a cycle with an entry requires that for a given μ_2 there are two distinct extensions, with the additional property that the origin of μ_2 is the terminus of one of the extensions. As we will show below, our k-graphs satisfy the stronger aperiodicity condition used in [27].

Proposition 3.14. *Let P be a one-vertex k-complex as in the hypothesis of Lemma 3.12 for $k \geq 2$ and let $\Lambda(P)$ be the associated one-vertex k-graph. If $L_i \geq 2$ for $i = 1, \ldots, k$, then the vertex in $\Lambda(P)$ supports at least two distinct cycles of length two in colour i; hence, $C^*(\Lambda(P))$ is purely infinite. Furthermore, $C^*(\Lambda(P))$ falls under the Kirchberg–Phillips classification theory and is thus determined by its K-theory.*

Proof. Fix $i \in \{1, \ldots, k\}$ with $L_i \geq 2$. Then, we can form the length-two cycles $\mu = \mathfrak{a}_1^i \mathfrak{a}_{L_i+1}^i$ and $\nu = \mathfrak{a}_2^i \mathfrak{a}_{L_i+2}^i$ based at v with $d(\mu)_i = 2 = d(\nu)_i$. Now, $\mathfrak{a}_2^i$ provides an edge α with nontrivial degree $d(\alpha) \leq d(\mu)$ which is an entry to μ not already contained in μ. We conclude that $C^*(\Lambda(P))$ is purely infinite. The last claim follows by [32, Corollary 8.15], see also [7, Remark 5.2], by appealing to the classification result in [22]. ∎

4. Construction of k-graphs with several vertices

In this section, we present our construction of k-graphs with several vertices, for $k \geq 2$, and provide examples and applications.

Towards this aim, we need a procedure to get k-cube complexes with several vertices. It is known that in a complex with several vertices one cannot consistently identify the label of an edge with the label as a generator in the fundamental group. We come around this challenge by introducing additional layers of labels, corresponding to covers with N sheets, similar to what is done for $N = 2$ in [15, Section 8.1]. In general, this is a hard problem since there exist complexes without nontrivial finite covers, cf. [4]. Even when the complexes are known to admit N-covers, corresponding to subgroups of index N of the fundamental group, see, e.g., [10, Theorem 1.38], it is difficult to construct covers explicitly. One challenge is that the subgroups can be defined in many different ways. For us a cover will be defined by picture, meaning that it is explicitly defined by the images of vertices, edges, faces, and so on. In all our pictures, the covering map amounts to forgetting the upper indexes, and we can explicitly see that we have a local homeomorphism at each point.

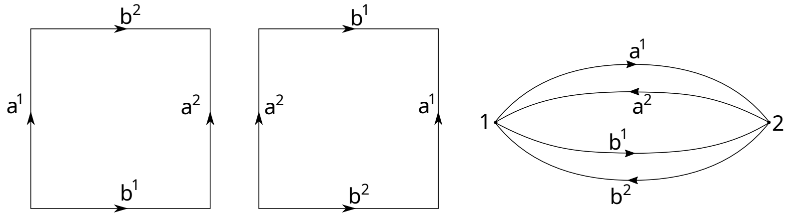

Figure 6. A double cover of the torus.

Recall that in a one-vertex k-cube complex with generating structure $A_1, \dots, A_k$, we view edges as being coloured, with each A_i for $i = 1, \dots, k$ endowed with a distinct colour. For a complex $\mathcal{X} = \widetilde{P}$ obtained as an N-cover of a one-vertex k-cube complex $P_{A_1,\dots,A_k}$, the coloured edges are given by $p^{-1}(A_i)$ for $i = 1, \dots, k$, where $p : \mathcal{X} = \widetilde{P} \to P_{A_1,\dots,A_k}$ is the covering map. A k-complex, when viewed as undirected, is always connected.

Proposition 4.1. *Suppose that G is a k-cube group with associated k-cube complex $P_{A_1,\dots,A_k}$. Then, $P_{A_1,\dots,A_k}$ admits a double cover $p : \widetilde{P} \to P_{A_1,\dots,A_k}$ with $\widetilde{P}$ a complex with 2 vertices.*

We establish this proposition by writing down an explicit double cover, which will be prescribed “by picture” on 2-cells and 3-cells of the complexes under consideration; see Lemmas 4.3 and 4.4. Before presenting the proof, we point out a consequence.

Corollary 4.2. *Each k-cube group G admits a subgroup of index 2.*

Proof. By, e.g., [10, Theorem 1.38], for a given path-connected, locally path-connected, and semilocally simply connected space X, there is a bijection between the set of basepoint preserving isomorphism classes of path-connected covering spaces $p : (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. The correspondence associates the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ with the covering $(\widetilde{X}, \widetilde{x_0})$, and the number of sheets of the covering equals the index of $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ in $\pi_1(X, x_0)$; see [10, Proposition 1.32]. Applying this to the 2-cover from Proposition 4.1 yields the existence of a subgroup of G of index 2. ■

In order to motivate our constructions of coverings, we review a construction of a 2-cover of the complex associated with the torus $\mathbb{T}^2$.

Recall from, for example, [10, p. 14] that $\mathbb{T}^2$ is obtained from a 2-cell given by a square with pairs of opposite edges having the same orientation and label a (vertically) or b (horizontally), by gluing it onto the wedge of two circles. A double cover with *two vertices* arises from two squares with oriented edges having distinct labels a^1, a^2, b^1, and b^2, with the upper index 1 or 2 indicating the source vertex, as shown in Figure 6. The complex is obtained by attaching the two squares (the 2-cells) to the graph with vertices 1 and 2 in Figure 6. If P is the complex associated with the (2, 2)-group in Example 3.7, then the cover $\widetilde{P}$ just described will lead to a 2-graph with two vertices; see Theorem 4.7 and [15, Figure 2, Section 8.1].

The proof of Proposition 4.1 relies on two lemmas, with the first one detailing an explicit double cover for the 2-cells of the given k-cube complex.

Lemma 4.3. *Suppose that we have a one-vertex square complex $S = S_{A,B}$ with VH-structure (A, B) and vertex v. Then, there is a 2-cover $p : \widetilde{S} \to S$ given by a square complex $\widetilde{S}$ with two vertices v_1 and v_2 whose squares are given by the prescription: the inverse image of a geometric square $S_O^{a,b} = (a, b, c^{-1}, d^{-1})$ in $S_{A,B}$ consists of two geometric squares, $S_1 = (a^1, b^2, (c^2)^{-1}, (d^1)^{-1})$ and $S_2 = (a^2, b^1, (c^1)^{-1}, (d^2)^{-1})$ in $\widetilde{S}$, and the covering map is determined by*

$$p(*^1) = p(*^2) = * \quad \textit{for } * = a, b, c, d.$$

Equivalently, if we denote $v_1 = 1$ and $v_2 = 2$, the covering p is depicted on the squares by

$$\begin{tikzcd} \color{red}{2} \arrow[r, "b^2"] & 1 & 1 \arrow[r, "b^1"] & \color{red}{2} & 1 \arrow[r, "b"] & 1 \\ 1 \arrow[u, "a^1"] \arrow[r, "d^1"'] \arrow[ur, phantom, "S_1"] & \color{red}{2} \arrow[u, "c^2"'] & \color{red}{2} \arrow[u, "a^2"] \arrow[r, "d^2"'] \arrow[ur, phantom, "S_2"] & 1 \arrow[u, "c^1"'] & 1 \arrow[u, "a"] \arrow[r, "d"'] \arrow[ur, phantom, "S_O^{a,b}"] & 1 . \end{tikzcd} \tag{4.1}$$

Note that in the 2-cover there are two geometric edges for each geometric edge in $S_{A,B}$, so for example to $a \in S_{A,B}$ having origin and terminus vertex 1 (identified with v), there will correspond a^1 and a^2 in $\widetilde{S}$, with a^1 having origin 1 and terminus 2, and a^2 with origin 2 and terminus 1.

Proof. We need only to observe that

$$p(S_*^{a^1,b^2}) = S_*^{a,b} = p(S_*^{a^2,b^1}) \quad \text{for } * = O, H, V, R.$$

Thus, the square complex $\widetilde{S}$ is well defined. The map p is a local homeomorphism because it is defined on the cells and sends edges to edges and vertices to vertices. ∎

Lemma 4.4. *Let $P_{A_1,\dots,A_k}$ be a k-cube complex associated with a k-cube group G with underlying structure determined by the ordered tuple $(A_1, \dots, A_k)$, with $\#A_i = 2L_i$ for every $i = 1, \dots, k$. There is a 2-cover $\widetilde{P}$ of $P_{A_1,\dots,A_k}$ determined as follows: On each 2-dimensional cell, the covering p is defined in Lemma 4.3. On a 3-dimensional geometric cube C, such as is described in Figure 1 where we assume $a_r \in A_i$, $b_s \in A_j$, and $c_t \in A_l$, for $r, s, t = 1, \dots, 4$ and $i, j, l \in \{1, \dots, k\}$, the cover $\widetilde{C}$ of C consists of two geometric cubes, see Figure 7, with labelling of edges $\{a_r^\varepsilon\}$, $\{b_s^\varepsilon\}$, and $\{c_t^\varepsilon\}$ for $\varepsilon = 1, 2$, and with the covering map given by*

$$p : \widetilde{C} \to C, \quad p(a_r^1) = p(a_r^2) = a_r, \quad r = 1, \dots, 4, \tag{4.2}$$

and similarly for $p(b_s^\varepsilon)$ and $p(c_t^\varepsilon)$. For $4 \le l \le k$, the map p is defined on an arbitrary l-cube by its prescription on the underlying 3-cubes.

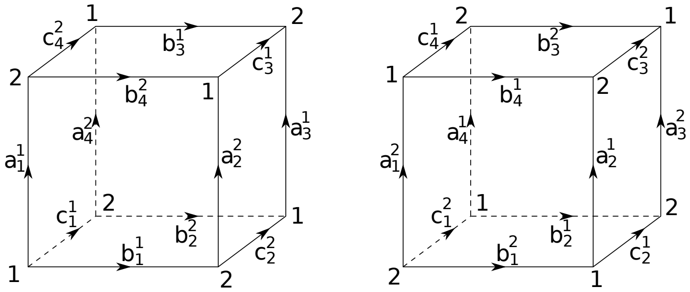

Figure 7. A two-cover of a generic geometric cube.

Proof. At $k = 3$, it suffices to verify that p on $\widetilde{C}$ is well defined. But this is clear from the construction of the map p in (4.2). The map p is constructed recursively on higher-dimensional cubes: if C is an l-cube for $4 \leq l \leq k$, then p is prescribed consistently on all $(l-1)$-dimensional faces of C, similarly to how (4.2) is obtained on 3-cubes from its prescription in (4.1) on squares. ∎

Proof of Proposition 4.1. This follows by applying Lemmas 4.3 and 4.4. ∎

To obtain k-complexes with N vertices for $N > 2$, there are several ways to use k-cube groups. Many of the k-cube groups are residually finite, so, because of the 1-to-1 correspondence between the subgroups of index N and N-covers of the corresponding k-cube complex, we can get infinitely many k-cube complexes with N vertices. In principle, different subgroups of the same index N can lead to different coverings. If $N \geq 3$, then the labelling of vertices is hard to sort out and we do not know of an explicit prescription similar to (4.2). We shall use the following general procedure.

Proposition 4.5. *Suppose that G is a residually finite k-cube group with $k \geq 2$. To each normal subgroup H of G of finite index N there is a k-complex $\mathcal{X}$ with N vertices obtained by the following prescription: let $Q : G \to S_N$ the homomorphism obtained by composing the embedding of G/H into the symmetric group on N letters S_N given by Cayley's theorem with the quotient map $q : G \to G/H$. For $a \in G$, the permutation $Q(a)$ in S_N encodes the edges in the complex, with a^n labelling an edge from the vertex n to the vertex $n' = Q(a)(n)$ for $n = 1, \dots, N$.*

Proof. Let P be the one-vertex complex determined by G. Suppose that H is a subgroup of G so that $(G : H) = N$. The complex $\mathcal{X}$ is constructed by associating with each square $S_O = (a, b, (a')^{-1}, (b')^{-1})$ in P a total of N squares

$$\big(Q(a), Q(b), Q\big((a')^{-1}\big), Q\big((b')^{-1}\big)\big)$$

in the new, N-vertex complex, with vertices labelled by elements in $\{1, 2, \dots, N\}$. Indeed, applying Q to the relation $ab = b'a'$ in G gives the identity $Q(a)Q(b) = Q(b')Q(a')$ in S_N, which in turn yields N squares in the complex $\mathcal{X}$ determined by

$$Q(a)Q(b)(n) = Q(b')Q(a')(n), \quad \text{for all } n = 1, \dots, N.$$

More precisely, for each n, let $m = Q(a)Q(b)(n) = Q(b')Q(a')(n)$ and consider the square with labelling $(a')^n(b')^s$ in vertical-horizontal direction from vertex n to vertex m via vertex s, and with labelling $b^n a^r$ in horizontal-vertical direction via vertex r, where $m = Q(a)(r)$, $r = Q(b)(n)$, $Q(a')(n) = s$, and $Q(b')(s) = m$; see the following diagram, where 1 in the right square denotes the vertex in the complex of G:

$$\begin{tikzcd} s \arrow[r, "(b')^s"] & m & 1 \arrow[r, "b"] & 1 \\ n \arrow[u, "(a')^n"] \arrow[r, "b^n"'] \arrow[ur, phantom, "S"] & r \arrow[u, "a^r"'] & 1 \arrow[u, "a"] \arrow[r, "b'"'] \arrow[ur, phantom, "S_O^{a,b}"] & 1 . \arrow[u, "a'"'] \end{tikzcd} \tag{4.3}$$

The covering map $p : \mathcal{X} \to P$ is defined by collapsing all squares of the form S onto the given square $S_O^{a,b}$ in P, for each square in P. ■

We extend the notions of left and right rigid to k-dimensional digraphs and k-graphs with more than one vertex in the natural way. The idea is that being rigid means that if two edges can form a corner (either bottom-left or top-right), then they do form a unique corner.

Definition 4.6. (a) Suppose that DG is a k-dimensional digraph. Then, DG is right rigid if for $x \in E_i$ and $y \in E_j$ edges of distinct colours $i \neq j$ so that $o(x) = o(y)$, there are unique $x' \in E_i$ and $y' \in E_j$ with $xy' \sim yx'$. Left rigid is defined in a similar way.

(b) Suppose that (Λ, d) is a k-graph, and let $E_i = d^{-1}(\{e_i\})$ be its alphabets, for $1 \leq i \leq k$. We say that Λ is right rigid if for $x \in E_i$ and $y \in E_j$ with the same origin, for $i \neq j$, there are unique $x' \in E_i$ and $y' \in E_j$ with the property that $y' \circ x = x' \circ y$. Left rigidity of Λ is defined in a similar manner.

Theorem 4.7. *Suppose that P is a one-vertex k-cube complex that admits an N-cover $p : \widetilde{P} \to P$ as in Proposition 4.5, with $\mathcal{X} = \widetilde{P}$ the associated k-cube complex with N vertices, for $k \geq 2$ and $N \geq 2$. The following are valid.*

(a) *The k-dimensional digraph* DG$(\mathcal{X})$ *determined in Theorem 3.3 is left and right rigid.*

(b) *The k-graph $\Lambda(\mathcal{X}) := \Lambda(\mathrm{DG}(\mathcal{X}))$ associated with* DG$(\mathcal{X})$ *by Corollary 3.4 is strongly connected, left rigid, and right rigid.*

Proof. Part (a) follows from Theorem 3.3. Turning to part (b), to show that $\Lambda(\mathcal{X})$ is strongly connected, let v, w be distinct vertices. By the construction of cover in Proposition 4.5, there is an element of the k-cube group whose action on v gives w. Associated with this element, there is a path $y_1 y_2 \cdots y_m$ in the 1-skeleton of the k-complex P, and this has a unique lift to a path in $\mathcal{X}$ from $v = o(y_1)$ to $w = t(y_m)$. In particular, $v_s = t(y_s)$, $v_{s+1} = o(y_s)$ are adjacent vertices in the complex for each $1 \leq s \leq m - 1$ (identifying $v = v_1$). Our definition of the k-dimensional digraph gives two directed edges, with opposite orientation, having source v_s and terminus v_{s+1}, respectively the opposite, for each $s = 1, \ldots, m-1$. This allows forming directed paths in $\mathrm{DG}(\mathcal{X})$, hence in $\Lambda(\mathcal{X})$, from v to w and from w to v, as needed.

To see that $\Lambda(\mathcal{X})$ is rigid, it suffices to note that every vertex in the cover $\widetilde{P}$ has the same link as the one vertex of P, and in particular its link contains no multiple edges. Therefore, the proof of Lemma 3.12 carries through. ∎

Corollary 4.8. *The k-graph $\Lambda(\mathcal{X})$ from Theorem 4.7 satisfies the aperiodicity condition.*

Proof. Since $\Lambda(\mathcal{X})$ is rigid, the existence of an aperiodic path in $\Lambda(\mathcal{X})$ based at a given vertex is guaranteed as in the one-vertex case; see [16, Lemma 4.15 and Corollary 11.10]. ∎

Proposition 4.9. *Assume the hypotheses of Theorem 4.7, where $P = P_{A_1,\ldots,A_k}$ with A_i given as in Lemma 3.12 for $i = 1, \ldots, k$. If $|A_i| \geq 2$ for all $i = 1, \ldots, k$, then every vertex $\Lambda(\mathcal{X})$ supports at least two cycles of each colour. In particular, $C^*(\Lambda(\mathcal{X}))$ is purely infinite and therefore classifiable by the Kirchberg–Phillips classification theory.*

Proof. Let $\Lambda(\mathcal{X})^0$ denote the vertices, or identities, in our k-graph. Since every geometric edge in P gives rise to N geometric edges in $\widetilde{P}$, we have that for each colour $i \in \{1, \ldots, k\}$, every vertex $v \in \Lambda(\mathcal{X})^0$ admits $N|A_i|$ incident edges, namely edges with origin or terminus v. Furthermore, by our construction of $\Lambda(\mathcal{X})$, we also know that each edge is contained in a length-two cycle. Thus, for a given $v \in \Lambda(\mathcal{X})^0$, there are at least two cycles $\mu = x_2 \circ x_1$ and $\nu = x_4 \circ x_3$ based at v and consisting of edges of colour i, with the terminus of x_1 possibly distinct from the terminus of x_3. Then, x_4 is an entry to μ of smaller degree and not already contained in μ. In this consideration, the vertex v already supports a cycle with an entrance, but since our k-graph is strongly connected, we could have chosen a cycle μ based at a different vertex w and apply the same consideration. Now, [32, Proposition 8.8 and Corollary 8.15] apply to give the claimed conclusion. ∎

We next illustrate our construction of k-graphs with more than one vertex with an explicit example of an infinite family of k-graphs with two vertices, for all $k \geq 2$. The construction was partly outlined in [15, Section 8.1], as corresponding to the uniform labelling l_u, and explicit factorisation rules of the 2-vertex graph were given in the case of the mixed labelling l_m. Here, we describe completely the case l_u as an application of Theorem 4.7.

Proposition 4.10. *For $k \geq 2$ and any k-tuple $(L_1, \ldots, L_k)$ of positive integers, there exists an aperiodic strongly connected 2-vertex k-rank graph Λ with $|v\Lambda^{e_i}w| = 2L_i$, where v and w are the vertices in Λ and $i = 1, \ldots, k$.*

Proof. Fix $k \geq 2$ and for each $i = 1, \ldots, k$ let $\mathcal{L}_i$ be an alphabet with L_i letters. Let $\mathbb{F}_i = \mathbb{F}_{L_i}$ be the free group generated by $\mathcal{L}_i$ for each $i = 1, \ldots, k$. The product group $\mathbb{F}_1 \times \cdots \times \mathbb{F}_k$ acts simply and transitively on the product of trees $\mathcal{T}_{2L_1} \times \cdots \times \mathcal{T}_{2L_k}$ and yields in the quotient a complex P with one vertex and skeleton a wedge of $\sum_{i=1}^{k} L_i$ circles. The 2-cells in P arise from pairs $a \in \mathcal{L}_i$, $b \in \mathcal{L}_j$ for $i \neq j$ with the commutation relation $ab = ba$ as in Example 3.7; for each such pair, there is a torus glued to the wedge of circles.

Let $\widetilde{P}$ be the associated 2-cover from Proposition 4.1. By Theorem 4.7 and Corollary 4.8, there is a k-graph $\Lambda := \Lambda(\widetilde{P})$ with the desired property: for each geometric edge in P, say having label $a \in \mathcal{L}_i$, there are two geometric edges labelled a^1 and a^2 in $\widetilde{P}$, and each of these gives exactly one edge in the associated k-graph Λ between the two vertices. ∎

Example 4.11. To illustrate Proposition 4.9 and Proposition 4.10, suppose that $k = 2$ and $L_1 = L_2 = 1$. The associated Λ has as 1-skeleton the graph with two vertices in Figure 6, where we identify v as vertex 1 and w as vertex 2. If we view the coloured edges in direction $e_1 \in \mathbb{N}^2$ as labelled by $\mathcal{L}_1$ and in direction $e_2 \in \mathbb{N}^2$ to be labelled by $\mathcal{L}_2$, then by Proposition 4.10, we have

$$w\Lambda^{e_1}v = \{a^1, \overline{a^2}\}, \quad w\Lambda^{e_2}v = \{b^1, \overline{b^2}\}, \quad v\Lambda^{e_1}w = \{a^2, \overline{a^1}\}, \quad v\Lambda^{e_1}w = \{b^2, \overline{b^1}\}.$$

The 8 factorisation rules are as follows:

$$\begin{aligned} &a^1b^2 = b^1a^2, \quad a^2\overline{b^2} = \overline{b^1}a^1, \quad \overline{a^1}b^1 = b^2\overline{a^2}, \quad \overline{a^2}\overline{b^2} = \overline{b^1}\overline{a^1}, \\ &a^2b^1 = b^2a^1, \quad a^1\overline{b^1} = \overline{b^2}a^2, \quad \overline{a^2}b^2 = b^1\overline{a^1}, \quad \overline{a^1}\overline{b^1} = \overline{b^2}\overline{a^2}. \end{aligned}$$

If $k = 2$ and $L_1 = L_2 = 2$, then the corresponding 2-graph on two vertices has the same 1-skeleton, and for example $|v\Lambda^{e_1}w| = |w\Lambda^{e_1}v| = 4$, and similarly in colour e_2.

5. Applications

5.1. Von Neumann algebras from strongly connected k-graphs

We now present a large supply of von Neumann type III_λ factors from k-graphs as in [14], for infinitely many values of λ in $(0, 1]$. We start with some preparation.

We refer to [29, Section 6] for the notion of adjacency operator in i-direction for a k-cube complex, where $i \in \{1, \dots, k\}$. The basic ingredients are as follows: let X be a k-cube complex with a vertex set (of its 1-skeleton) denoted X_0 and with universal cover, a product $\mathrm{T}_1 \times \mathrm{T}_2 \times \cdots \times \mathrm{T}_k$ of regular trees. For each $i = 1, \dots, k$ and $V, W \in X_0$, we write $V \sim_i W$ if the two vertices in the complex are adjacent in the i-direction of X. The *adjacency operator* $\mathcal{A}_i$ *in* i*-direction* is defined on $L^2(X_0)$ by

$$\mathcal{A}_i(f)(V) = \sum_{W \sim_i V} f(W).$$

Since all complexes considered here are locally finite in a strong sense, meaning that at every vertex there are finitely many edges in each direction i, or of each colour i, for $i \in \{1, \dots, k\}$, the operators $\mathcal{A}_i$ become $|X_0| \times |X_0|$ matrices. It was further observed in [29, Remark 6.4] that whenever each pair of edges starting at a vertex of X in direction i, j, with $i \neq j$, belong to a unique square in X, then $\mathcal{A}_i$ and $\mathcal{A}_j$ commute.

Proposition 5.1. *Let $\mathcal{X}$ be a k-cube complex with N vertices covered by a cartesian product of k trees with valencies $n_1, n_2, \dots, n_k \in \mathbb{Z}^+$, respectively, where $k \geq 2$ and $N \geq 1$. Let $\Lambda(\mathcal{X})$ be the associated k-graph as in Theorem 4.7. Then, with the notation of subsection 2.9, we have*

$$\rho\big(\Lambda(\mathcal{X})\big) = (n_1, n_2, \dots, n_k). \tag{5.1}$$

Proof. The assumption on $\mathcal{X}$ says that there are n_i edges (disregarding orientation) of colour i for each $i = 1, \dots, k$. The graph $\Lambda(\mathcal{X})$ is constructed by assigning two edges in its skeleton, of opposite orientation, for each geometric edge in $\mathcal{X}$. Therefore, if M_i denotes the coordinate matrix of $\Lambda(\mathcal{X})$ in colour i, we have that M_i is the same as the adjacency operator in i-direction $\mathcal{A}_i$. Thus, it is a symmetric matrix with the largest positive eigenvalue equal to the valency of the tree in colour i. Hence, $\rho(M_i) = n_i$ for each $i = 1, \dots, k$, as claimed. ■

Remark 5.2. The graph of Proposition 4.10 satisfies $\rho(\Lambda) = (2L_1, \dots, 2L_k)$.

Given a strongly connected k-graph Λ, it was shown in [1, Corollary 4.6] that $C^*(\Lambda)$ admits KMS states at inverse temperature $\beta = 1$ for the one-parameter action $\alpha\colon \mathbb{R} \to \operatorname{Aut} C^*(\Lambda)$, the so-called preferred dynamics, characterised by $\alpha_t(\mathbf{s}_\mu) = e^{it \log \rho(\Lambda)\cdot d(\mu)}\mathbf{s}_\mu$, $t \in \mathbb{R}$, $\mu \in \Lambda$. Following [14], define $\mathcal{S} := \{\rho(\Lambda)^{d(\mu)-d(\nu)} \mid \mu, \nu \in \Lambda \text{ are cycles}\}$ and let $\lambda := \sup\{s \in \mathcal{S} \mid s < 1\}$. By the main result of [14], Theorem 3.1, we have $\lambda \in (0, 1]$ and the von Neumann algebra generated by the image of $C^*(\Lambda)$ in the GNS representation π_φ corresponding to an extremal KMS state φ is the injective type III_λ factor.

Our application here is motivated by [14, Example 7.7]; see also [20, 38]. It consists of producing an infinite family of von Neumann factors $(\pi_\varphi(C^*(\Lambda)))''$ of type III_λ associated with k-graphs in this fashion. Recall from [14, Section 6] that the group of periods of a strongly connected graph Λ is defined as $\mathcal{P}_\Lambda = \mathcal{P}_v^+ - \mathcal{P}_v^+$, where for an arbitrary vertex $v \in \Lambda^0$, $\mathcal{P}_v^+$ is the subsemigroup $d(v\Lambda v)$ of $\mathbb{N}^k$. Equivalently, $\mathcal{P}_\Lambda$ is the subgroup of $\mathbb{Z}^k$ determined as $\{d(\mu) - d(\nu) \mid \mu, \nu \text{ are cycles in } \Lambda\}$. As shown in [14, Theorem 7.3], the set $\mathcal{S}$ above is the closure inside the positive real half-line of the set $\{\rho(\Lambda)^g \mid g \in \mathcal{P}_\Lambda\}$.

Corollary 5.3. *For $k \geq 2$ and any k-tuple $(L_1, \dots, L_k)$ of positive integers, let Λ be the k-graph with two vertices from Proposition 4.10. There is a III_λ von Neumann factor $(\pi_\varphi(C^*(\Lambda)))''$, where π_φ is the GNS representation of $C^*(\Lambda)$ corresponding to an extremal KMS$_1$ state ϕ, and the type is determined as*

$$\lambda = \sup\big\{(2L_1)^{m_1}(2L_2)^{m_2}\cdots(2L_k)^{m_k} \mid (m_1, m_2, \dots, m_k) \in \mathcal{P}_\Lambda\big\} \cap (0, 1].$$

In particular, if $L_1 = \cdots = L_k = L$, then $\lambda = (2L)^{-2}$.

Proof. Fix $k \geq 2$ and positive integers $L_1, \dots, L_k$, and let Λ be as specified. Then, Λ is strongly connected, and we may apply [14, Theorem 3.1] to obtain the claimed von Neumann factors.

The remaining task is to compute $\mathcal{P}_\Lambda$. By our construction of Λ, it is not hard to see that $\mathcal{P}_\Lambda$ is generated by $m \in \mathbb{Z}^k$ where either $m_i = 2$ for a unique $i \in \{1, \dots, k\}$ while $m_l = 0$ at $l \neq i$ or $m_i = m_j = 1$ for some $i \neq j$ in $\{1, \dots, k\}$ and $m_l = 0$ for $l \notin \{i, j\}$.

If $L_1 = L_2 = \cdots = L_k = L$, then $\rho(\Lambda)^m = (2L)^{\sum_{i=1}^k m_i}$ with $m \in \mathcal{P}_\Lambda$, and because $\sum_{i=1}^k m_i \in 2\mathbb{Z}$, the required type is attained as $\lambda = (2L)^{-2}$. ∎

It was pointed out in [14, Remark 7.6] that the type of the von Neumann factors arising from extremal KMS_1 states depends only on the skeleton of the k-graph, and not on its factorisation rules. In our examples in Corollary 5.3, this means that the type of the von Neumann factor depends only on the complex $\widetilde{P}$ built up as a 2-cover of the one-vertex complex $(\mathcal{T}_{2L_1} \times \cdots \times \mathcal{T}_{2L_k})\backslash(\mathbb{F}_1 \times \cdots \times \mathbb{F}_k)$.

5.2. Spectral theory of k-graphs

Alon and Boppana prove that asymptotically in families of finite $(q+1)$-regular graphs X_n with diameter tending to ∞, the largest absolute value of a nontrivial eigenvalue $\lambda(X_n)$ of the adjacency operator A_{X_n} has limes inferior $\underline{\lim}_{n\to\infty} \lambda(X_n) \geq 2\sqrt{q}$.

Now, instead of graphs we may consider cube complexes covered by products of trees $\mathrm{T}_1 \times \cdots \times \mathrm{T}_k$, such that T_i has valency q_i, and look at adjacency operators $\mathcal{A}_i$ in direction i corresponding to an individual tree T_i.

Definition 5.4. Let X be a finite k-cube complex that has constant valency $q_i + 1$ in all directions $i = 1, \ldots, k$. Then, X is a *cubical Ramanujan complex* if for each $i \in \{1, \ldots, k\}$, the eigenvalues λ of $\mathcal{A}_i$ satisfy either the equality $\lambda = \pm(q_i + 1)$ or the bound

$$\lambda \leq 2\sqrt{q_i}.$$

Each such complex yields a k-graph Δ such that $\rho(\Delta) = (q_1 + 1, q_2 + 1, \ldots, q_k + 1)$.

There are explicit constructions of Ramanujan cube complexes for several infinite families in [29]. We consider next the complexes from [29], corresponding to congruence subgroups of arithmetic lattices. We reformulate some results of [29] in the light of the present paper.

Theorem 5.5 (Cf. [29, Section 6]). *For p, a prime, l, a positive integer, and $N \geq 2$, there are infinitely many k-cube complexes with N vertices covered by products of k trees, where $k \leq p - 1$ and each tree is of valency $p^l + 1$, satisfying optimal spectral properties, namely with a spectral gap, the interval $[2\sqrt{q}, q+1]$, for $q = p^l$.*

Proof. Such k-cube complexes were constructed in [29, Section 6]. They correspond to congruence quotients of arithmetic groups. The number of vertices of such complexes is given by the order of the group $\mathrm{PGL}(2, p^l)$. ∎

Remark 5.6. There are also non-residually finite complexes which have interesting k-graphs although they do not necessarily exhibit the optimal spectral gap. Such complexes with one vertex were constructed in [29, Section 5]. Applying Lemma 4.4, we get such complexes with 2 vertices for all values $k \geq 1$.

Now, we extend the notion of the Ramanujan cube complexes to higher-rank graphs.

Definition 5.7. We say that a coordinate matrix of a k-graph is L-regular for $L \in \mathbb{N}$ if the sum of all row entries is equal to L.

Definition 5.8. Let Λ be a k-graph with L_i-regular coordinate matrices M_i having positive second eigenvalue λ_i, for $i = 1, \dots, k$. We say that the k-graph Λ is Ramanujan if

$$\lambda_i \le 2\sqrt{L_i - 1} \quad \text{for all } i = 1, \dots, k.$$

Theorem 5.9. *For each $k \ge 2$, there is an infinite family of Ramanujan k-graphs with $N \ge 2$ vertices. More precisely, N is determined as the index of congruence subgroups of the RSV-groups $\Gamma_{M,\delta}$ from Section 2.8.*

Proof. This is a direct application of Theorem 5.5 in conjunction with Theorem 4.7. For a complex $\mathcal{X}$ with N vertices, there is a k-graph $\Lambda(\mathcal{X})$ with N vertices by an application of Theorem 4.7. ∎

We note that both one-vertex cube complexes covered by the product of k trees and one-vertex higher-rank graphs are trivially Ramanujan, so we will require in addition the number of vertices to be greater than, for example, the maximum of $(L_i - 1)^2$, $i = 1, \dots, k$.

Example 5.10. We now describe an explicit Ramanujan 3-graph with 25 vertices in the above infinite family. Let $p = 5$ and consider the group Γ_1 from Section 2.8 acting on a product of three trees with valencies $(6, 6, 6)$. Let P denote the one-vertex 3-complex associated with G. The existence of the claimed 3-graph is assured by Proposition 4.5 and Theorem 4.7 because Γ_1 has a quotient L of order 25 (indeed, it has quotients of order 5^l for all $l \ge 1$). Let $\mathcal{X}$ denote the resulting complex with 25 vertices, and let Λ be its associated 3-graph.

Certain subsets of generators of Γ_1 already generate a group of order 25 in the cover, as may be verified using MAGMA. For example, the image $Q(a_1)$ in S_{25} is the product of disjoint cycles

$$\begin{aligned} &Q(a_1)\\ &= (1, 15, 24, 8, 17)(2, 11, 25, 9, 18)(3, 12, 21, 10, 19)(4, 13, 22, 6, 20)(5, 14, 23, 7, 16). \end{aligned}$$

With the notation of Proposition 4.5, we have isomorphisms of groups

$$\begin{aligned} L &\cong \langle Q(a_1), Q(a_5), Q(a_9)\rangle,\\ L &\cong \langle Q(b_2), Q(b_6), Q(b_{10})\rangle,\\ L &\cong \langle Q(c_3), Q(c_7), Q(c_{11})\rangle, \end{aligned}$$

and thus all three groups in the right-hand side are abstractly isomorphic to a finite group of order 25. Let K_1, K_2, and K_3, respectively, denote the Cayley graphs of the finite group of order 25 coming from the three preceding isomorphisms.

This means that while the presentation of the infinite group Γ_1 requires generators of all three colours a_i, b_j, c_k (as Γ_1 is irreducible), in the presentation of the finite group of order 25, generators of only one colour suffice. The finite cover is the complex $\mathcal{X}$, and fixing each colour yields the Cayley graph of a finite group of order 25. In other words, each of the generating sets A_1, A_2, and A_3 gives Cayley graphs in three different sets of generators (colours) of the *same* finite group.

The adjacency matrices M_i of the Cayley graphs K_i, $i = 1, 2, 3$, may be computed using MAGMA, using that $Q(G)$ acts as permutations in S_{25}. It turns out that M_1, M_2, M_3 are equal. As noted in the proof of Proposition 5.1, the adjacency matrices M_1, M_2, M_3 of the complex are also the adjacency matrices of the 3-graph Λ. Each M_i is 6-regular in the sense of Definition 5.7, for $i = 1, 2, 3$, as may be seen from the concrete description of the matrices obtained with MAGMA. An application of Theorem 5.9 gives that Λ is a Ramanujan 3-graph, so the second-largest eigenvalue λ_i of M_i is dominated by $2\sqrt{5}$, for $i = 1, 2, 3$.

In this example, the spectral gap is strictly in the optimal bound; namely, the second eigenvalue of M_1 is dominated by 3.24, according to MAGMA computations. This bound is lower than the theoretically predicted $2\sqrt{5}$.

In addition, using MAGMA shows that the product $M_1 M_2 M_3$ is not a (0, 1)-matrix, which distinguishes this example from [27] and all papers inspired by it. For example, the diagonal entries in $M_1 M_2 M_3$ are all equal to 12. The remaining entries are 6, 7, or 15.

6. Matrices of the Cayley graph of an order 25-group

$$
\left[\begin{array}{ccccccccccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right].
$$

Acknowledgements. We thank Mark Lawson and Aidan Sims for useful comments. We are grateful to the anonymous referee for an extensive list of suggestions towards improving the presentation and several proofs.

Funding. This research was initiated during a visit by A. Vdovina to Oslo in connection to the Master Class on "Equilibrium states in semigroup theory, K-theory and number theory", 4–6 November, 2019, supported by the Trond Mohn Foundation through the project "Pure Mathematics in Norway". It was carried out while N. Larsen was a Research Fellow in the Cluster of Excellence Mathematics Münster at WWU, Germany. She thanks for warm hospitality and excellent working conditions in the Cluster.

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Received 20 June 2022.

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