Essential Crossed Products for
Inverse Semigroup Actions:
Simplicity and Pure Infiniteness

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Abstract. We study simplicity and pure infiniteness criteria for C*-algebras associated to inverse semigroup actions by Hilbert bimodules and to Fell bundles over étale not necessarily Hausdorff groupoids. Inspired by recent work of Exel and Pitts, we introduce essential crossed products for which there are such criteria. In our approach the major role is played by a generalised expectation with values in the local multiplier algebra. We give a long list of equivalent conditions characterising when the essential and reduced C*-algebras coincide. Our most general simplicity and pure infiniteness criteria apply to aperiodic C*-inclusions equipped with supportive generalised expectations. We thoroughly discuss the relationship between aperiodicity, detection of ideals, purely outer inverse semigroup actions, and non-triviality conditions for dual groupoids.

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1 Introduction

Much is known about the ideal structure of reduced crossed products for group actions and of reduced groupoid C*-algebras of étale, Hausdorff, locally compact groupoids. More precisely, for an action α of a discrete group G on a separable C*-algebra A, there is a long list of equivalent conditions due to

Kishimoto and Olesen–Pedersen, which imply that the coefficient algebra A *detects ideals* in the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$ in the sense that $J \cap A = 0$ for an ideal J in $A \rtimes_{\alpha,\mathrm{r}} G$ implies $J = 0$. This theory has recently been generalised in [43] to actions by Hilbert bimodules or, equivalently, Fell bundles over groups. The goal of this article is to generalise the results in [43] to Fell bundles over groupoids and inverse semigroups. It turns out that many of these results extend, without major problems, to actions of inverse semigroups by Hilbert bimodules provided that the crossed product carries a conditional expectation. Such crossed products model, as a special case, section C^*-algebras associated to Fell bundles over Hausdorff, étale, locally compact groupoids. The existence of a canonical conditional expectation is closely related to the Hausdorffness of the underlying groupoid. This is already quite satisfactory, as it unifies the existing results for crossed products for group actions and for (twisted) groupoid C^*-algebras of Hausdorff étale groupoids and improves the criteria for detection of ideals for groupoid crossed products by Renault [55]. Another intriguing source of potential applications come from regular inclusions and the noncommutative Cartan inclusions introduced by Exel [20]. Namely, Exel [20] has found sufficient conditions for a regular inclusion with a conditional expectation to be of the form $A \subseteq A \rtimes_{\mathrm{r}} S$ for some inverse semigroup action by Hilbert bimodules. However, a real challenge and an important problem that has been open for decades now, is how to deal with non-Hausdorff groupoids. Eventually, the main point of this article has become how to attack this problem and get results about detection of ideals without a (genuine) conditional expectation.

Difficulties for non-Hausdorff groupoids were already noticed in [55], which includes an unexpected example by Skandalis of a minimal foliation with a non-simple C^*-algebra. Reduced crossed products for non-Hausdorff groupoids were studied further in [33, 34], but without progress on their ideal structure. Until recently, all general results about the ideal structure of groupoid C^*-algebras were limited to the Hausdorff case. There are several constructions of étale groupoids from other data for which Hausdorffness is unclear and not a natural assumption to make. Important classes of such groupoids are foliation C^*-algebras – restricted to complete transversals to make them étale – and the C^*-algebras of self-similar graphs defined by Exel and Pardo [24]. Very recently, there has been progress in the non-Hausdorff case in two directions. First, the simplicity of $C_{\mathrm{r}}^*(H)$ for minimal, topologically principal, second countable groupoids H has been studied in [13]. Secondly, Exel and Pitts [25] have defined "essential" (twisted) groupoid C^*-algebras for topologically principal groupoids in which $C_0(X)$ detects ideals. The construction in [25] appears to be rather *ad hoc*, however. It only works well for topologically principal groupoids.

Here we take the idea of Exel and Pitts much further. We define essential crossed products for all inverse semigroup actions on C^*-algebras by Hilbert bimodules. Our definition is conceptual and analogous to the definition of

reduced crossed products in [10]. Secondly, we derive a very powerful abstract criterion for a C*-subalgebra A in a C*-algebra B to detect ideals. Even more, our theory may show that A supports B in the sense that any non-zero positive element in B dominates some non-zero positive element in A with respect to the Cuntz preorder. This allows to prove that B is purely infinite and simple under suitable assumptions. Previous criteria for pure infiniteness of crossed products were restricted to groupoid C*-algebras of Hausdorff groupoids (see also [27, 30, 37, 41, 43, 46, 50, 57] for more work on pure infiniteness of crossed products).

The reduced crossed product $A \rtimes_r S$ for an inverse semigroup action on A is defined in [10] using a weak conditional expectation $E \colon A \rtimes S \to A''$, where A'' is the bidual of A. We call the action *closed* if this expectation takes values in $A \subseteq A''$. This happens for inverse semigroup actions that are derived from actions of Hausdorff étale groupoids. In general, however, we must enlarge A to accommodate a conditional expectation. Let $\mathcal{M}_{\mathrm{loc}}(A)$ be the local multiplier algebra of A, that is, the inductive limit of the multiplier algebras of the essential ideals in A (see [3]). If $A = C_0(X)$, then $\mathcal{M}_{\mathrm{loc}}(A)$ is the inductive limit of $C_b(U)$ for dense open subsets $U \subseteq X$. So it is spanned by functions that are "densely defined" on X. Our main idea is to replace $E \colon A \rtimes S \to A''$ by a generalised conditional expectation $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$, which we briefly call an $\mathcal{M}_{\mathrm{loc}}$-*expectation*. Then we define the essential crossed product as the quotient of $A \rtimes S$ by the largest two-sided ideal on which EL vanishes. The main troublemakers in $C_r^*(H)$ for a non-Hausdorff groupoid H are elements $x \in C_r^*(H)$ for which $E(x)$ is supported on a nowhere dense subset of X. The expectation EL kills such elements. Hence they get killed in the essential crossed product.

The main achievement in this article is a conceptual understanding of the techniques used to prove that crossed products of various kinds are simple or purely infinite. We take this occasion to honour Emmy Noether, who pioneered the conceptual approach to mathematics despite strong resistance, and whose Habilitation in Göttingen was finally granted only in 1919. The basic concepts that make this paper work are aperiodicity and $\mathcal{M}_{\mathrm{loc}}$-expectations.

The concept of aperiodicity goes back to Kishimoto's proof that reduced crossed products for outer group actions on simple C*-algebras are again simple (see [38]). The crucial condition in Kishimoto's proof was rewritten in [43] in terms of normed A-bimodules in order to treat actions by Hilbert bimodules instead of by automorphisms. The right generality to study aperiodicity is a C*-inclusion $A \subseteq B$. We call the inclusion *aperiodic* if B/A equipped with the quotient norm and the induced A-bimodule structure satisfies Kishimoto's condition. Given an aperiodic inclusion, we prove that there is a maximal ideal $\mathcal{N}$ in B which is aperiodic as an A-bimodule and that $B/\mathcal{N}$ is the unique quotient of B for which the map $A \to B/\mathcal{N}$ is injective and detects ideals.

For inverse semigroup actions by Hilbert bimodules, the aperiodicity of $(A \rtimes S)/A$ is equivalent to the aperiodicity of certain Hilbert bimodules. Using the results of [43] we show the following. Let A be separable or of Type I. The inclusion $A \subseteq A \rtimes S$ is aperiodic if and only if the dual groupoid $\widehat{A} \rtimes S$ is topologically free (see Theorem 6.13). Here $\widehat{A}$ is the space of isomorphism classes of irreducible representations of A, and the action of S on A by Hilbert bimodules induces an action on $\widehat{A}$ by the Rieffel correspondence. We carefully discuss the concept of a "topologically free" étale groupoid in Section 2 because several slightly different definitions are used in the literature, and the dual groupoid $\widehat{A} \rtimes S$ is rather badly non-Hausdorff.

Aperiodicity becomes powerful when combined with a conditional expectation. The following theorem is a special case of Theorem 5.28.

Theorem 1. *Let $A \subseteq B$ be an aperiodic C*-inclusion with a conditional expectation $E \colon B \to A$. Let $\mathcal{N}_E$ be the largest two-sided ideal contained in $\ker E$. Let $A^+ := \{a \in A : a \geq 0\}$ and let $\precsim$ denote the Cuntz preorder on B^+. Then*

(1) *for every $b \in B^+$ with $b \notin \mathcal{N}_E$, there is $a \in A^+ \backslash \{0\}$ with $a \precsim b$;*

(2) *A supports $B/\mathcal{N}_E$;*

(3) *A detects ideals in $B/\mathcal{N}_E$, and $B/\mathcal{N}_E$ is the only quotient of B with this property;*

(4) *B is simple if and only if $\mathcal{N}_E = 0$ and $\overline{BIB} = B$ for all $0 \neq I \in \mathbb{I}(A)$;*

(5) *if B is simple, then B is purely infinite if and only if every element in $A^+ \backslash \{0\}$ is infinite in B.*

The first statement (1) is the key step here. It easily implies all the others.

In order to treat general inverse semigroup crossed products or C*-algebras of non-Hausdorff étale groupoids, it is necessary to replace the conditional expectation in Theorem 1 by something weaker. At first, we tried a weak conditional expectation $E \colon B \to A''$ as in the definition of the reduced crossed product for inverse semigroup actions in [10]. An inspection of the proof of Theorem 1 led to the concept of a supportive weak conditional expectation, which suffices to make the proof of the theorem work. Besides looking at weak conditional expectations, we also looked at pseudo-expectations, which take values in the injective hull of A. Here we were motivated by the theorem of Zarikian that a crossed product for a group action has a unique pseudo-expectation if and only if the action is aperiodic (see [61, Theorem 3.5]). The injective hull of a commutative C*-algebra is equal to its local multiplier algebra (see [26]). And it turns out that a generalised conditional expectation with values in $\mathcal{M}_{\mathrm{loc}}(A)$ is always supportive. Thus Theorem 1 still holds with a generalised conditional expectation E that takes values in $\mathcal{M}_{\mathrm{loc}}(A)$. And this then suggests our definition of the essential crossed product.

As another test of our definition of the essential crossed product, we carry over the main results of Archbold–Spielberg [4] and Kawamura–Tomiyama [31]. Namely, if S is an inverse semigroup acting on a C*-algebra A by Hilbert bimodules and $\widehat{A} \rtimes S$ is topologically free, then A detects ideals in $A \rtimes_{\mathrm{ess}} S$ (Theorem 6.14). And for an étale locally compact groupoid H, $\mathrm{C}_0(X)$ detects ideals in $\mathrm{C}^*_{\mathrm{ess}}(H)$ if *and only if* H is topologically free (Theorem 7.29). Another very promising observation is that all derivations on A become inner in $\mathcal{M}_{\mathrm{loc}}(A)$ (see [3]). The corresponding result for A'' instead of $\mathcal{M}_{\mathrm{loc}}(A)$ plays a key role in the work of Olesen–Pedersen. The local multiplier algebra always embeds into Hamana's injective hull (see [26]). Hence every $\mathcal{M}_{\mathrm{loc}}$-expectation is a pseudo-expectation as well. These have been studied, for instance, in [52, 53, 61].

It is, however, often necessary to work with the reduced crossed product. One reason is that the essential crossed product is not functorial. Thus we want to know when the essential and the reduced crossed products are equal, meaning that they are both quotients of $A \rtimes S$ by the same ideal. We use that $E(A \rtimes S)$ embeds into the product $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$, consisting of uniformly bounded families of operators on the Hilbert spaces on which the irreducible representations of A act. The local multiplier algebra $\mathcal{M}_{\mathrm{loc}}(A)$ embeds into the quotient of $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ that is defined by the *essential* supremum of the pointwise norms – we disregard subsets that are meagre. As a result, the following are equivalent (see Corollary 4.17):

- $A \rtimes_{\mathrm{r}} S = A \rtimes_{\mathrm{ess}} S$;

- if $x \in (A \rtimes S)^+$ and $E(x) \neq 0$, then the set of $\pi \in \widehat{A}$ with $\|\pi''(E(x))\| \neq 0$ is not meagre;

- if $\varepsilon > 0$ and $x \in (A \rtimes S)^+$ satisfies $E(x) \neq 0$, then the set of $\pi \in \widehat{A}$ with $\|\pi''(E(x))\| > \varepsilon$ has non-empty interior.

We carry the theory above over to section C*-algebras $\mathrm{C}^*(H, \mathcal{A})$ for Fell bundles $\mathcal{A}$ over a locally compact, étale groupoid H with Hausdorff unit space X. This includes twisted groupoid C*-algebras as a special case. We define an essential section C*-algebra $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ for all Fell bundles $\mathcal{A}$. For a Fell line bundle $\mathcal{A}$, it coincides with the essential twisted groupoid C*-algebra defined by Pitts and Exel [25] if and only if H is topologically principal. We give many equivalent characterisations for $\mathrm{C}^*_{\mathrm{r}}(H, \mathcal{A}) = \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$. One of them is related to singular elements as defined in [13] for groupoid C*-algebras. We also discuss simplicity and pure infiniteness criteria for $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ that generalise and improve various results of this sort.

Our more recent article [45] contains two important advances. First, all topologically free inverse semigroup actions are aperiodic. Secondly, pseudo-expectations are always supportive. This generalises Theorem 1 to all aperiodic inclusions $A \subseteq B$.

Steinberg and Szakács in [59] prove a criterion when the Steinberg algebra of an étale groupoid with totally disconnected object space is simple. Our results

imply an analogous criterion for reduced groupoid C*-algebras.

The paper is organised as follows. Section 2 introduces inverse semigroup actions on topological spaces and C*-algebras and the dual groupoid for an action on a C*-algebra, and it compares several concepts of topological freeness for such actions and for non-Hausdorff étale groupoids. Section 3 discusses basic notation about generalised conditional expectations and full and reduced crossed products for inverse semigroup actions. We show that the canonical weak conditional expectation on the reduced crossed product is faithful. Section 4 introduces the essential crossed product and the $\mathcal{M}_{\mathrm{loc}}$-expectation that defines it. We prove that this generalised expectation is faithful, and we characterise when the reduced and essential crossed products coincide.

Section 5 contains our general results on aperiodic C*-inclusions $A \subseteq B$. We show that for any such inclusion there is a unique quotient of B in which A detects ideals. Then we say that $A \subseteq B$ has the generalised intersection property and call the unique ideal $\mathcal{N}$ such that A detects ideals in $B/\mathcal{N}$ the hidden ideal. We define supportive generalised expectations and prove the generalisations of Theorem 1 discussed above. In Section 6, we specialise aperiodicity to inverse semigroup actions. We show that the inclusion $A \subseteq B$ into an exotic crossed product is aperiodic if and only if the underlying action is aperiodic in a suitable sense, and we reformulate aperiodicity in several equivalent ways. We use these equivalent characterisations of aperiodicity to rewrite our main results with different assumptions. In Section 7, we describe section algebras of Fell bundles over étale, locally compact groupoids through crossed products for inverse semigroup actions. Using this we define essential section algebras for Fell bundles over étale groupoids. We prove results about their ideal structure and when they are simple and purely infinite. And we compare the essential and reduced section algebras.

2   Preliminaries on inverse semigroup actions and étale groupoids

First we define inverse semigroup actions on topological spaces and compare them to étale groupoids. We allow arbitrary topological spaces, requiring neither Hausdorffness nor local compactness. We define actions of inverse semigroups on C*-algebras by Hilbert bimodules and relate them to regular inclusions and gradings by inverse semigroups. We define the dual groupoid $\widehat{A} \rtimes S$ of such an action. Then we discuss several variants of the concept of topological freeness.

2.1   Inverse semigroup actions on spaces and étale groupoids

An *inverse semigroup* is a semigroup S with the property that for each $t \in S$ there is a unique element $t^* \in S$ such that $tt^*t = t$ and $t^*tt^* = t^*$. Let

$$E(S) := \{e \in S : e^2 = e\}.$$

If $e, f \in E(S)$, then $e = e^*$ and $ef = fe$. If $t \in S$, then $t^*t, tt^* \in E(S)$. We call $E(S)$ the *idempotent semilattice* of S. A partial order on S is defined by $t \leqslant u$ for $t, u \in S$ if and only if $t = ut^*t$, if and only if there is $e \in E(S)$ with $t = ue$. By definition, $E(S) = \{e \in S : e \leqslant 1\}$.

Let X be an arbitrary topological space. A partial homeomorphism of X is a homeomorphism between two open subsets of X. Partial homeomorphisms with the composition of partial maps form a unital inverse semigroup, which we denote by $\Pi(X)$. Let S be a unital inverse semigroup. An *action of S on X by partial homeomorphisms* is a unital semigroup homomorphism $h \colon S \to \Pi(X)$. So it consists of open subsets X_t of X and homeomorphisms $h_t \colon X_t \to X_{t*}$ for all $t \in S$, such that $h_t \circ h_u = h_{tu}$ for all $t, u \in S$ and $h_1 = \operatorname{Id}_X$; then $h_{t*} = h_t^* = h_t^{-1}$ for all $t \in S$. We denote the domain of h_t by X_t, rather than by X_{t*} (which is a convention adopted in most of sources). We do this to lighten the notation, as we will talk mostly about domains of h_t's.

An inverse semigroup action h on X as above yields an étale topological groupoid with object space X, namely, the *transformation groupoid* $X \rtimes S$ (see [51, p. 140] or [19, Section 4]). We avoid the name "groupoid of germs" used by Exel because some authors use that name for another groupoid with a different germ relation. The arrows of $X \rtimes S$ are equivalence classes of pairs (t, x) for $x \in X_t \subseteq X$; two pairs (t, x) and (t', x') are equivalent if $x = x'$ and there is $v \in S$ with $v \leqslant t, t'$ and $x \in X_v$. The range and source maps $r, s \colon X \rtimes S \rightrightarrows X$ and the multiplication are defined by $r([t, x]) := h_t(x)$, $s([t, x]) := x$, and $[t, h_u(x)] \cdot [u, x] = [t \cdot u, x]$. We give $X \rtimes S$ the unique topology for which $[t, x] \mapsto x$ is a homeomorphism from an open subset of $X \rtimes S$ onto X_t for each $t \in S$. Then the range and source maps are local homeomorphisms $X \rtimes S \rightrightarrows X$. The multiplication is continuous. So $X \rtimes S$ is an étale topological groupoid. The subsets $U_t := \{[t, x] : x \in X_t\}$ are bisections of $X \rtimes S$, and they cover $X \rtimes S$. Now let H be an étale groupoid with object space X. We are going to write H as a transformation groupoid. A *bisection* of H is an open subset $U \subseteq H$ such that $r|_U$ and $s|_U$ are injective. If $U, V \subseteq H$ are bisections, then so are

$$U^{-1} := \{\gamma^{-1} : \gamma \in U\}, \qquad U \cdot V := \{\gamma \cdot \eta : \gamma \in U, \ \eta \in V\}.$$

The bisections of H with these operations form a unital inverse semigroup, which we denote by $\operatorname{Bis}(H)$. The unit bisection is the subset of all identity arrows in H. The inverse semigroup $\operatorname{Bis}(H)$ acts canonically on X by $h_U := r \circ (s|_U)^{-1}$ for $U \in \operatorname{Bis}(H)$.

DEFINITION 2.1. An inverse subsemigroup $S \subseteq \operatorname{Bis}(H)$ is called *wide* if $\bigcup S = H$ and $U \cap V$ is a union of bisections in S for all $U, V \in S$.

Proposition 2.2. *For any unital inverse subsemigroup $S \subseteq \mathrm{Bis}(H)$, the map*

$$\Phi \colon X \rtimes S \to H, \qquad [U, x] \mapsto (s|_U)^{-1}(x),$$

is a well defined, continuous, open groupoid homomorphism. It is an isomorphism if and only if $S \subseteq \mathrm{Bis}(H)$ is wide.

Proof. This is mostly proven in [19, Propositions 5.3 and 5.4]. Direct computations show that Φ is a well defined groupoid homomorphism. If $U \in S$, then $U' := \{[U, x] : x \in s(U)\}$ is a bisection of $X \rtimes S$ with $\Phi(U') = U$. Since both H and $X \rtimes S$ are étale and Φ is the identity map on objects, it follows that Φ restricts to a homeomorphism from U' onto U. The bisections U' for $U \in S$ cover $X \rtimes S$. Therefore, the map Φ is both continuous and open.
Clearly, the map Φ is surjective if and only if $\bigcup S = H$. And Φ is not injective if and only if there are $U, V \in S$ and $\gamma \in U \cap V$ with $[U, s(\gamma)] \neq [V, s(\gamma)]$ in $X \rtimes S$. By the definition of $X \rtimes S$, the latter holds if and only if $\gamma \notin W$ for all $W \in S$ with $W \subseteq U \cap V$. Hence there is such a $\gamma \in U \cap V$ if and only if $\bigcup_{W \in S, W \subseteq U \cap V} W \neq U \cap V$. So Φ is bijective if and only if S is wide. $\square$

Proposition 2.2 allows to translate properties of groupoids into the language of inverse semigroup actions, and vice versa.

Definition 2.3. A subset $Y \subseteq X$ is *h-invariant* for an action $h \colon S \to \Pi(X)$ if $h_t(Y \cap X_t) \subseteq Y$ for all $t \in S$. If Y is invariant, then there is a restricted action $h|_Y \colon S \to \Pi(Y)$, which is defined by $(h|_Y)_t := h_t|_Y \colon Y \cap X_t \to Y \cap X_{t*}$ for all $t \in S$. A subset $Y \subseteq X$ is *H-invariant* for a groupoid H with object space X if and only if $s^{-1}(Y) = r^{-1}(Y)$ as subsets of H.

Remark 2.4. A subset $Y \subseteq X$ is $X \rtimes_h S$-invariant if and only if it is *h*-invariant (see [23, Proposition 5.4]).

Definition 2.5. Let X be a topological space and $h \colon S \to \Pi(X)$ an inverse semigroup action. For $t \in S$, define

$$X_{1,t} := \bigcup_{e \leqslant t, e \in E(S)} X_e.$$

The action is called *closed* if $X_{1,t}$ is relatively closed in X_t for all $t \in S$.

Lemma 2.6. *The action h is closed if and only if the space of units X is closed in $X \rtimes S$.*

Proof. This follows because the subset $X_{1,t}$ is equal to the intersection of X_t with the unit bisection in $X \rtimes S$ and $(X_t)_{t \in S}$ is an open cover of X. $\square$

Remark 2.7. Closed inverse semigroup actions are important because a topological groupoid is Hausdorff if and only if the object space is Hausdorff and the units form a closed subset of the arrows (see [10, Lemma 5.2]).

2.2 Inverse semigroup actions on C*-algebras

DEFINITION 2.8 ([11]). An *action* of a unital inverse semigroup S on a C*-algebra A (by Hilbert bimodules) consists of Hilbert A-bimodules $\mathcal{E}_t$ for $t \in S$ and Hilbert bimodule isomorphisms $\mu_{t,u} \colon \mathcal{E}_t \otimes_A \mathcal{E}_u \xrightarrow{\sim} \mathcal{E}_{tu}$ for $t, u \in S$, such that

(A1) for all $t, u, v \in S$, the following diagram commutes (associativity):

$$
\begin{array}{ccc}
(\mathcal{E}_t \otimes_A \mathcal{E}_u) \otimes_A \mathcal{E}_v & \xrightarrow{\;\mu_{t,u} \otimes_A \mathrm{Id}_{\mathcal{E}_v}\;} & \mathcal{E}_{tu} \otimes_A \mathcal{E}_v \searrow^{\mu_{tu,v}} \\
\mathrm{ass} \updownarrow & & \hspace{2cm} \mathcal{E}_{tuv} \\
\mathcal{E}_t \otimes_A (\mathcal{E}_u \otimes_A \mathcal{E}_v) & \xrightarrow[\;\mathrm{Id}_{\mathcal{E}_t} \otimes_A \mu_{u,v}\;]{} & \mathcal{E}_t \otimes_A \mathcal{E}_{uv} \nearrow_{\mu_{t,uv}}
\end{array}
$$

(A2) $\mathcal{E}_1$ is the identity Hilbert A, A-bimodule A;

(A3) $\mu_{t,1} \colon \mathcal{E}_t \otimes_A A \xrightarrow{\sim} \mathcal{E}_t$ and $\mu_{1,t} \colon A \otimes_A \mathcal{E}_t \xrightarrow{\sim} \mathcal{E}_t$ for $t \in S$ are the maps defined by $\mu_{1,t}(a \otimes \xi) = a \cdot \xi$ and $\mu_{t,1}(\xi \otimes a) = \xi \cdot a$ for $a \in A$, $\xi \in \mathcal{E}_t$.

We shall not use actions by partial automorphisms in this article. *So all actions of inverse semigroups on C*-algebras are understood to be by Hilbert bimodules.* Any S-action by Hilbert bimodules comes with canonical involutions $\mathcal{E}_t^* \to \mathcal{E}_{t*}$, $x \mapsto x^*$, and inclusion maps $j_{u,t} \colon \mathcal{E}_t \to \mathcal{E}_u$ for $t \leqslant u$ that satisfy the conditions required for a saturated Fell bundle in [20] (see [11, Theorem 4.8]). Thus S-actions by Hilbert bimodules are equivalent to saturated Fell bundles over S.

DEFINITION 2.9 ([44, Definition 6.15]). An *S-graded C*-algebra* is a C*-algebra B with closed subspaces $B_t \subseteq B$ for $t \in S$ such that $\sum_{t \in S} B_t$ is dense in B, $B_t B_u \subseteq B_{tu}$ and $B_t^* = B_{t*}$ for all $t, u \in S$, and $B_t \subseteq B_u$ if $t \leqslant u$ in S. The grading is *saturated* if $B_t \cdot B_u = B_{tu}$ for all $t, u \in S$. We call $A := B_1 \subseteq B$ the *unit fibre* of the grading.

Remark 2.10. For group gradings, it is customary to require the fibres to be linearly independent. We have no use for such a condition. We will, however, restrict to "topological gradings" for several important results. In the group case, our notion of a topological grading specialises to the usual one, and that implies immediately that the fibres are linearly independent.

A (saturated) S-grading $(B_t)_{t \in S}$ on B defines a (saturated) Fell bundle over S using the operations in B. Conversely, the crossed product construction allows to realise any (saturated) Fell bundle through a grading on a suitable C*-algebra.

One important source of inverse semigroup actions are Fell bundles over étale groupoids (see Section 7.1). Another important source are regular inclusions:

DEFINITION 2.11. Let $A \subseteq B$ be a C*-subalgebra. We call the elements of

$$N(A, B) := \{ b \in B : bAb^* \subseteq A, \; b^*Ab \subseteq A \}$$

normalisers of A in B (see [39]). We call the inclusion $A \subseteq B$ *regular* if it is non-degenerate and $N(A, B)$ generates B as a C*-algebra (see [56]).

PROPOSITION 2.12. *The following are equivalent for a C*-inclusion $A \subseteq B$:*

(1) *A is a regular subalgebra of B;*

(2) *A is the unit fibre for some inverse semigroup grading on B;*

(3) *A is the unit fibre for some saturated inverse semigroup grading on B.*

If these equivalent conditions hold, then

$$\mathcal{S}(A, B) := \{M \subseteq N(A, B) : M \text{ is a closed linear subspace}$$
$$\text{and } AM \subseteq M, \ MA \subseteq M\}$$

with the operations $M \cdot N := \overline{\text{span}} \{mn : m \in M, \ n \in N\}$ and $M^* := \{m^* : m \in M\}$ is an inverse semigroup. And the subspaces $M \in \mathcal{S}(A, B)$ form a saturated $\mathcal{S}(A, B)$-grading on B.

Proof. This goes back to [20]. See also [44, Lemma 6.25 and Proposition 6.26]. □

LEMMA 2.13. *Any non-degenerate C*-inclusion $A \subseteq B$ with commutative B is regular.*

Proof. Let B^+ be the minimal unitisation of B. If $u \in B^+$ is unitary, then $A \cdot u \subseteq N(A, B)$ because B is commutative. The unital C*-algebra B^+ is spanned by the unitaries it contains. Hence B is the linear span of $A \cdot u$ for unitaries $u \in B^+$. □

2.3 Dual groupoids

Let A be a C*-algebra with an action $\mathcal{E}$ of a unital inverse semigroup S. Let $\widehat{A}$ and $\check{A} = \text{Prim}(A)$ be the space of irreducible representations and the primitive ideal space of A, respectively. Open subsets in $\widehat{A}$ and in $\check{A}$ are in natural bijection with ideals in A. The action of S on A induces an action $(\check{\mathcal{E}}_t)_{t \in S}$ on $\check{A}$ by partial homeomorphisms by [11, Lemma 6.12]. We explain how this action lifts to an action $(\widehat{\mathcal{E}}_t)_{t \in S}$ of S on $\widehat{A}$.

Let $t \in S$. Then $\widehat{s(\mathcal{E}_t)}$ is an open subset of $\widehat{A}$, consisting of all $\pi \in \widehat{A}$ that are non-degenerate on $s(\mathcal{E}_t) = \langle \mathcal{E}_t \,|\, \mathcal{E}_t \rangle$. Let $\pi \in \widehat{A}$. The tensor product $\mathcal{E}_t \otimes_A \mathcal{H}_\pi$ is non-zero if and only if π belongs to $\widehat{s(\mathcal{E}_t)}$. And then the left multiplication action of A on $\mathcal{E}_t \otimes_A \mathcal{H}_\pi$ is another irreducible representation of A, which we call $\widehat{\mathcal{E}}_t(\pi) \in \widehat{A}$. This defines a homeomorphism $\widehat{\mathcal{E}}_t : \widehat{s(\mathcal{E}_t)} \xrightarrow{\sim} \widehat{r(\mathcal{E}_t)}$ with inverse $\widehat{\mathcal{E}_{t^*}}$. The family $\widehat{\mathcal{E}} := (\widehat{\mathcal{E}}_t)_{t \in S}$ forms an action of S on $\widehat{A}$ by partial homeomorphisms.

DEFINITION 2.14. We call $\widehat{\mathcal{E}}$ the *dual action* to the action $\mathcal{E}$. The transformation groupoid $\widehat{A} \rtimes S$ is called the *dual groupoid* of $\mathcal{E}$.

Let $\mathbb{I}(A)$ for a C*-algebra A denote the lattice of (closed, two-sided) ideals in A.

DEFINITION 2.15. We call $I \in \mathbb{I}(A)$ *$\mathcal{F}$-invariant* for a Hilbert A-bimodule $\mathcal{F}$ if $I \cdot \mathcal{F} = \mathcal{F} \cdot I$. We call $I \in \mathbb{I}(A)$ *$\mathcal{E}$-invariant* for an action $\mathcal{E}$ of an inverse semigroup S if I is $\mathcal{E}_t$-invariant for all $t \in S$. Let $\mathbb{I}^{\mathcal{E}}(A)$ denote the set of all $\mathcal{E}$-invariant ideals in A. We call $\mathcal{E}$ *minimal* if $\mathbb{I}^{\mathcal{E}}(A) = \{0, A\}$, that is, the only $\mathcal{E}$-invariant ideals in A are 0 and A.

Remark 2.16. Let B be an S-graded C*-algebra with grading $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$. Then $\mathbb{I}^{\mathcal{E}}(A) = \{J \cap A : J \in \mathbb{I}(B)\}$ (see [44, Proposition 6.19]).

LEMMA 2.17. *An ideal I in A is $\mathcal{E}$-invariant if and only if the corresponding open subset $\widehat{I} \subseteq \widehat{A}$ is invariant for the dual groupoid $\widehat{A} \rtimes S$.*

Proof. By Remark 2.4, $\widehat{A} \rtimes S$-invariance is the same as invariance under the dual S-action $(\widehat{\mathcal{E}}_t)_{t \in S}$ on $\widehat{A}$. Therefore, we need to show that I is $\mathcal{E}$-invariant if and only if $\widehat{\mathcal{E}}_t(\widehat{I} \cap \widehat{D}_t) = \widehat{I} \cap \widehat{D}_{t*}$ for all $t \in S$. This follows from a known fact for any Hilbert A-bimodule $\mathcal{F}$: an ideal I in A is $\mathcal{F}$-invariant if and only if $\widehat{I}$ is invariant under the partial homeomorphism $\widehat{\mathcal{F}}$ (see [40] or the proof of [1, Proposition 3.10]). $\qquad\square$

DEFINITION 2.18. An inverse semigroup action $\mathcal{E}$ on a C*-algebra A is called *closed* if the dual S-action on $\widehat{A}$ is closed.

Remark 2.19. Let H be an étale locally compact groupoid with a Hausdorff unit space X and let S be a wide inverse semigroup of bisections of H. If we equip $A = C_0(X)$ with the canonical action of S, then the dual action on $\widehat{A} \cong X$ is the canonical S-action and $\widehat{A} \rtimes S \cong H$. Hence the S-action on A is closed if and only if H is Hausdorff (see Remark 2.7).

2.4 Non-Triviality conditions for étale groupoids

We carefully distinguish several versions of the concept of topological freeness. They are all equivalent for groupoids that are second countable, locally compact and Hausdorff. We will, however, meet groupoids where the object space is the spectrum of a C*-algebra, which is often badly non-Hausdorff, and the unit space is not closed in the groupoid.

DEFINITION 2.20. Let H be an étale groupoid and $X \subseteq H$ its unit space. The *isotropy group* of a point $x \in X$ is $H(x) := s^{-1}(x) \cap r^{-1}(x) \subseteq H$. We call H

(1) *effective* if any open subset $U \subseteq H$ with $r|_U = s|_U$ is contained in X;

(2) *topologically free* if, for every bisection $U \subseteq H \backslash X$, the set $\{x \in X : H(x) \cap U \neq \varnothing\}$ has empty interior;

(3) *AS topologically free* if, for finitely many bisections $U_1, \ldots, U_n \subseteq H \backslash X$, the union $\bigcup_{i=1}^{n} \{x \in X : H(x) \cap U_i \neq \varnothing\}$ has empty interior;

(4) *topologically principal* if the set of points of X with trivial isotropy is dense or, equivalently, the set $\{x \in X : H(x)\backslash X \neq \varnothing\}$ has empty interior.

An action of an inverse semigroup on a topological space X or on a C*-algebra A is called *topologically free* if the transformation groupoid $X \rtimes S$ or the dual groupoid $\widehat{A} \rtimes S$ is topologically free, and similarly for effective, AS topologically free and topologically principal actions.

Remark 2.21. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group on a C*-algebra A. The transformation groupoid $\widehat{A} \rtimes G$ for the dual action is AS topologically free if and only if, for any $t_1, \ldots, t_n \in G\backslash\{1\}$, the set $\bigcup_{i=1}^{n}\{x \in \widehat{A} : \widehat{\alpha}_{t_i}(x) = x\}$ has empty interior in $\widehat{A}$. This definition is due to Archbold and Spielberg (see [4, Definition 1]), and this is what "AS" in Definition 2.20.(3) stands for.

Remark 2.22. Topological freeness for groupoids as defined in Definition 2.20.(2) has not yet received as much attention as it deserves. This condition appears, for instance, in [5, Lemma 3.1.(3)] and in [25, 14.15(ii)], where it is related to the conditions that we call "effective" and "topologically principal", respectively.

We are going to describe the above properties in terms of an inverse semigroup action $h\colon S \to \Pi(X)$. For $t \in S$, define

$$\mathrm{Fix}(h_t) := \{x \in X_t\backslash X_{1,t} : h_t(x) = x\}, \qquad \underline{\mathrm{Fix}}(h_t) := \{x \in X_t\backslash\overline{X_{1,t}} : h_t(x) = x\}.$$

By definition, $\underline{\mathrm{Fix}}(h_t) = \mathrm{Fix}(h_t) = \varnothing$ if $t \in E(S)$. If S is a group and $t \in S\backslash\{1\}$, then $\mathrm{Fix}(h_t) = \underline{\mathrm{Fix}}(h_t) = \{x \in X : h_t(x) = x\}$.

LEMMA 2.23. *Let $h\colon S \to \Pi(X)$ be an inverse semigroup action and $H :=X \rtimes S$.*

(1) *If H is effective, then H is topologically free. The converse holds if X is closed in H.*

(2) *H is topologically free if and only if $\mathrm{Fix}(h_t)$ has empty interior for any $t \in S$, if and only if $\underline{\mathrm{Fix}}(h_t)$ has empty interior for any $t \in S$.*

(3) *H is AS topologically free if and only if $\bigcup_{i=1}^{n} \underline{\mathrm{Fix}}(h_{t_i})$ has empty interior for any $t_1, \ldots, t_n \in S$.*

(4) *H is topologically principal if and only if $\bigcup_{t\in S} \mathrm{Fix}(h_t)$ has empty interior.*

Proof. The construction of $H = X \rtimes S$ implies

$$\{x \in X : H(x)\backslash X \neq \varnothing\} = \bigcup_{t\in S}\mathrm{Fix}(h_t).$$

This readily implies (4).

For a subset $A \subseteq X$, let $\mathrm{Int}(A)$ denote its interior in X. If $t \in S$, then $\mathrm{Int}(X_t\backslash X_{1,t}) = X_t\backslash\overline{X_{1,t}}$ and hence

$$\mathrm{Int}\left(\mathrm{Fix}(h_t)\right) = \mathrm{Int}\left(\underline{\mathrm{Fix}}(h_t)\right).$$

Thus the second equivalence in (2) is valid. The subset $V_t :=$ $\{[t, x] : x \in X_t \backslash \overline{X_{1,t}}\}$ is a bisection contained in $H \backslash X$, and

$$\{x \in X : H(x) \cap V_t \neq \varnothing\} = \underline{\mathrm{Fix}}(h_t). \tag{2.1}$$

Hence, if H is effective or topologically free, then $\mathrm{Int}\,(\mathrm{Fix}(h_t)) = \mathrm{Int}\,(\underline{\mathrm{Fix}}(h_t)) = \varnothing$. Conversely, if H is not topologically free, then there is a bisection $U \subseteq H \backslash X$ such that the interior V of $\{x \in X : H(x) \cap U \neq \varnothing\}$ is non-empty. Let $x \in V$. Then $x \in s(U)$ and there is a unique $\gamma \in U$ with $s(\gamma) = x$ because U is a bisection. And $r(\gamma) = x$ because $H(x) \cap U \neq \varnothing$. There is $t \in S$ with $\gamma = [t, s(\gamma)]$. Since U is open, it contains an open neighbourhood U_2 of γ, which we may take of the form $U_2 = \{[t, x] : x \in V_2\}$ for an open subset $V_2 \subseteq X_t$. Then $V \cap V_2$ is an open neighbourhood of $s(\gamma)$ such that $[t, x] \in U$ and $H(x) \cap U \neq \varnothing$ for all $x \in V \cap V_2$. Thus $r([t, x]) = s([t, x])$ for all $x \in V \cap V_2$. So the interior of $\mathrm{Fix}(h_t)$ is non-empty. This finishes the proof of (2).

As we noticed above, if H is effective, then $\mathrm{Int}\,(\mathrm{Fix}(h_t)) = \varnothing$ for all $t \in S$, and hence H is topologically free by (2). Suppose now that H is not effective. So there is a bisection $U \subseteq H$ with $r|_U = s|_U$ that is not contained in X. If X is closed in H, then $V := U \backslash X$ is a bisection contained in $H \backslash X$, and it is still non-empty. Since $s(V) = \{x \in X : H(x) \cap V \neq \varnothing\}$, the groupoid H is not topologically free. Thus (1) is proved.

The 'only if' part in (3) follows from (2.1). For the 'if' part, suppose that there are bisections $U_1, \ldots, U_n$ contained in $H \backslash X$ and a non-empty open subset

$$W \subseteq \bigcup_{i=1}^{n} \{x \in X : H(x) \cap U_i \neq \varnothing\}. \tag{2.2}$$

We may assume without loss of generality that $W \cap s(U_i) \neq \varnothing$ for $i = 1, \ldots, n$. Since U_1 is open and contained in $H \backslash X$ there are open sets $W_t \subseteq X_t \backslash X_{1,t}$, $t \in S$, with $U_1 = \bigcup_{t \in S} \{[t, x] : x \in W_t,\ t \in S\}$. If $t_1 \in S$ is such that $W_{t_1} \cap W \neq \varnothing$, then replacing W by $W \cap W_{t_1}$ and U_1 by $\{[t_1, x] : x \in W_{t_1}\}$, the inclusion (2.2) remains valid. Proceeding in this way, we may arrange for the sets U_i to be of the form $U_i = \{[t_i, x] : x \in W_i\}$ for some $t_i \in S$ and some open subsets W_i of $X_{t_i} \backslash X_{1,t_i}$ for $i = 1, \ldots, n$. Being open, these subsets are even contained in $X_t \backslash \overline{X_{1,t}}$. And

$$W \subseteq \bigcup_{i=1}^{n} \{x : H(x) \cap U_i \neq \varnothing\} = \bigcup_{i=1}^{n} \{x \in W_{t_i} : h_{t_i}(x) = x\} \subseteq \bigcup_{i=1}^{n} \underline{\mathrm{Fix}}(h_{t_i}).$$

This finishes the proof of (3). $\qquad\qquad\square$

Lemma 2.23 implies the following relations between the properties in Definition 2.20:

$$\text{(effective)} \Rightarrow \text{(top. free)} \Leftarrow \text{(AS top. free)} \Leftarrow \text{(top. principal)}.$$

And topological freeness and effectiveness are equivalent if the unit space is closed. Example 6.4 below exhibits a topologically principal action of an inverse semigroup S on $[0,1]$ that is not effective (see also [13, Example 5.1] for such an example). There are situations when topological freeness implies topological principality. The following result of this nature is essentially due to Renault (see [56, Proposition 3.6.(ii)]).

PROPOSITION 2.24. *Let X be a Hausdorff space. An action h of an inverse semigroup S on X is topologically free if and only if it is AS topologically free. If, in addition, X is a Baire space and S is countable, then h is topologically free if and only if it is topologically principal.*

Proof. Assume h to be topologically free. We are going to prove that h is AS topologically free. This implies the first statement. Let $t \in S$. Lemma 2.23 shows that $\mathrm{Fix}(h_t)$ has empty interior in X. Equivalently, the set of $x \in X_t$ with $h_t(x) \neq x$ is dense in X_t. This subset is open because X is Hausdorff. Therefore, the open subset Y_t of all $x \in X$ with either $x \notin \overline{X_t}$ or $x \in X_t$ and $h_t(x) \neq x$ is dense in X. No point in Y_t can belong to the closure of $\mathrm{Fix}(h_t)$. Therefore, the interior of $\overline{\mathrm{Fix}(h_t)}$ is empty or, equivalently, $\mathrm{Fix}(h_t)$ is nowhere dense in X. This is inherited by the subset $\underline{\mathrm{Fix}}(h_t)$ of $\mathrm{Fix}(h_t)$. A finite union of nowhere dense subsets is again nowhere dense. Hence every finite union $\bigcup_{i=1}^n \underline{\mathrm{Fix}}(h_{t_i})$ for $t_1, \ldots, t_n \in S$ is nowhere dense and hence has empty interior. Thus h is AS topologically free by Lemma 2.23. If, in addition, S is countable, and X Baire, then the countable union $\bigcup_{t \in S} \mathrm{Fix}(h_t)$ is still nowhere dense in X. Hence Lemma 2.23 implies that h is topologically principal. $\square$

In Theorem 6.13, we will prove an analogue of Proposition 2.24 for the dual groupoid $H = \widehat{A} \rtimes S$ of an inverse semigroup action on a separable C*-algebra A, so $\widehat{A}$ need not be Hausdorff. Some further assumption besides X being Baire and S countable is needed for this converse implication. For instance, the action of the permutation group S_3 on the three-element set X with the chaotic topology $\{\varnothing, X\}$ is topologically free, but not AS topologically free.

The following lemma allows to relax the assumptions in Proposition 2.24 slightly:

LEMMA 2.25. *Let H be an étale groupoid and X its space of units. Let $X' \subseteq X$ be an open dense subset of X. Then $H' := s^{-1}(X') \cap r^{-1}(X')$ is an open dense subgroupoid of H, and*

(1) *H is topologically free if and only if H' is topologically free.*

(2) *H is AS topologically free if and only if H' is AS topologically free.*

(3) *H is topologically principal if and only if H' is topologically principal.*

Proof. Clearly, H' is an open subgroupoid of H. It is dense in H because for every bisection $U \subseteq H$ the intersection $U \cap H'$ is open and dense in U. For any $x \in X'$ the isotropy groups in H and H' are the same. This immediately

gives (3). If U is a bisection in H, then $U \subseteq H \backslash X$ if and only if $U \cap H' \subseteq H' \backslash X'$. This readily implies (1). Concerning (2), it is easy to see that H' is AS topologically free if H is. Conversely, let H' be AS topologically free and let $U_1, \ldots, U_n \subseteq H \backslash X$ be bisections of H. Let $U'_j := U_j \cap H'$ for $j = 1, \ldots, n$. Then

$$\bigcup_{i=1}^{n} \{x \in X : H(x) \cap U_i \neq \varnothing\} \cap X' = \bigcup_{i=1}^{n} \{x \in X' : H'(x) \cap U'_i \neq \varnothing\}.$$

The set on the right has empty interior because H' is AS topologically free. Since X' is dense in X, it follows that $\bigcup_{i=1}^{n} \{x \in X : H(x) \cap U_i \neq \varnothing\}$ has empty interior. Thus H is AS topologically free. $\qquad \square$

COROLLARY 2.26. *Suppose that the space of units X of an étale groupoid H contains an open dense subset X' which is Hausdorff. Then H is topologically free if and only if H is AS topologically free. If X' is Baire and H has a countable cover by bisections, then H is topologically free if and only if it is topologically principal.*

Proof. Combine Proposition 2.24 and Lemma 2.25. $\qquad \square$

3 Full and reduced crossed products

We first establish some basic notation about generalised conditional expectations. Then we construct the full and reduced crossed products for inverse semigroup actions and prove that the canonical weak expectation on the reduced crossed product is faithful.

3.1 Generalised expectations

Conditional expectations are crucial tools in the study of crossed products for group actions. For an inverse semigroup action, the reduced crossed product is defined in [10] using a "weak conditional expectation" $E \colon A \rtimes S \to A''$, which takes values in the bidual von Neumann algebra A''. Pseudo-expectations, which take values in the injective hull of A, have been studied in [52, 53, 61]. To define the essential crossed product, we will use expectations with values in the local multiplier algebra (see Definition 4.2 below). The following definition covers all these cases:

DEFINITION 3.1. Let $A \subseteq B$ be a C*-inclusion. A *generalised expectation* consists of another C*-inclusion $A \subseteq \tilde{A}$ and a completely positive, contractive map $B \to \tilde{A}$ that restricts to the identity map on A. If $\tilde{A} = A$, $\tilde{A} = A''$, or $\tilde{A}$ is the injective envelope of A, then we speak of a *conditional expectation*, a *weak conditional expectation*, or a *pseudo-expectation*, respectively.

LEMMA 3.2. *Any generalised expectation is an A-bimodule map.*

Proof. The unique unital, linear extension $E^+\colon B^+ \to \tilde{A}^+$ is still completely positive and contractive (see, for instance, [6, Subsection 2.2]). And it is the identity map on A^+. Then E^+ is A^+-bilinear by Choi's Theorem, [12, Theorem 3.1]. $\qquad\square$

Any idempotent linear contraction $E\colon B \to A$ is a completely positive bimodule map and thus a conditional expectation. This result is due to Tomiyama [60].

Example 3.3. For any C*-inclusion $A \subseteq B$, the identity map on B is a generalised expectation with values in B.

Example 3.4 ([52]). The identity map on A extends to a completely positive map from B to the injective hull of A. Thus, any C*-inclusion $A \subseteq B$ has a pseudo-expectation.

We will see more examples of generalised expectations in the definitions of the reduced and essential crossed products for inverse semigroup actions.

DEFINITION 3.5. Let $E\colon B \to \tilde{A} \supseteq A$ be a generalised expectation. Let $\mathcal{N}_E$ be the closed linear span of all $J \in \mathbb{I}(B)$ with $J \subseteq \ker E$. This is the largest two-sided ideal in B that is contained in $\ker E$. Let $B_\mathrm{r} := B/\mathcal{N}_E$ and let $\Lambda\colon B \to B_\mathrm{r}$ be the quotient map. Since $E|_{\mathcal{N}_E} = 0$, the expectation E descends to a map $E_\mathrm{r}\colon B_\mathrm{r} \to \tilde{A} \supseteq A$, called the *reduced generalised expectation* of E.

Since $E|_A = \mathrm{Id}_A$ and $E|_{\mathcal{N}_E} = 0$, it follows that $A \cap \mathcal{N}_E = 0$. Hence the composite map $A \to B \to B/\mathcal{N}_E$ is injective. The map E_r is a generalised expectation for the inclusion $A \hookrightarrow B_\mathrm{r}$.

PROPOSITION 3.6. *Let $E\colon B \to \tilde{A} \supseteq A$ be a generalised expectation. Let*

$$\mathcal{LN}_E := \{b \in B : E(b^*b) = 0\},$$
$$\mathcal{RN}_E := \{b \in B : E(bb^*) = 0\}.$$

Then $\mathcal{LN}_E$ and $\mathcal{RN}_E$ are the largest left and right ideals in B contained in $\ker E$, respectively. Hence $(\mathcal{LN}_E)^* = \mathcal{RN}_E$ and $\mathcal{N}_E \subseteq \mathcal{LN}_E \cap \mathcal{RN}_E$. And

$$\mathcal{N}_E = \{b \in B : E((bc)^*bc) = 0 \text{ for all } c \in B\} = \{b \in B : b \cdot B \subseteq \mathcal{LN}_E\}$$
$$= \{b \in B : E(xby) = 0 \text{ for all } x, y \in B\}.$$

The third description of $\mathcal{N}_E$ was pointed out to us by Ruy Exel.

Proof. If necessary, adjoin units to B and $\tilde{A}$ to make them unital. The unique unital extension of E is unital and completely positive. It is already observed in [12, Remark 3.4] that $\mathcal{LN}_E$ is the largest left ideal contained in $\ker E$; this is a general feature of 2-positive maps. The main point is the Schwarz inequality $E(b^*)E(b) \leqslant E(b^*b)$ for all $b \in B$ (see [12, Corollary 2.8]). Since $\mathcal{RN}_E = (\mathcal{LN}_E)^*$, it follows that $\mathcal{RN}_E$ is the largest right ideal contained in $\ker E$. Thus $\mathcal{N}_E$ is contained in $\mathcal{LN}_E$ and $\mathcal{RN}_E$. If $b \in \mathcal{N}_E$, then $b \cdot c \in \mathcal{N}_E \subseteq \mathcal{LN}_E$ for all $c \in B$. Thus $E((bc)^*bc) = 0$ for all $c \in B$. Conversely, if $E((bc)^*bc) = 0$

for all $c \in B$, then $\overline{b \cdot B} \subseteq \mathcal{LN}_E$; and then $\overline{B \cdot b \cdot B} \subseteq \mathcal{LN}_E$ because $\mathcal{LN}_E$ is a left ideal. This implies $b \in \mathcal{N}_E$ because $\overline{B \cdot b \cdot B}$ is a two-sided ideal that contains b. Thus $b \in \mathcal{N}_E$ if and only if $E((bc)^*bc) = 0$ for all $c \in B$, if and only if $b \cdot B \subseteq \mathcal{LN}_E$. And $E(xby) = 0$ for all $x, y \in B$ if and only if $B \cdot b \cdot B \subseteq \ker E$, if and only if the closed two-sided ideal generated by b is contained in $\ker E$, if and only if $b \in \mathcal{N}_E$. □

DEFINITION 3.7. A generalised expectation $E \colon B \to \tilde{A} \supseteq A$ is *faithful* if $E(b^*b) = 0$ for some $b \in B$ implies $b = 0$. It is *almost faithful* if $E((bc)^*bc) = 0$ for all $c \in B$ and some $b \in B$ implies $b = 0$. It is *symmetric* if $E(b^*b) = 0$ for some $b \in B$ implies $E(bb^*) = 0$.

The concept of an almost faithful positive map plays an important role in the theory of Exel's crossed products (see [7, Definition 4.1] and [42, Subsection 2.1]).

COROLLARY 3.8. *Let* $E \colon B \to \tilde{A} \supseteq A$ *be a generalised expectation.*

(1) E *is symmetric if and only if* $\mathcal{LN}_E = \mathcal{RN}_E = \mathcal{N}_E$;

(2) E *is faithful if and only if* $\mathcal{LN}_E = \mathcal{RN}_E = \mathcal{N}_E = 0$;

(3) E *is almost faithful if and only if* $\mathcal{N}_E = 0$, *if and only if there are no non-zero ideals in* B *contained in* $\ker E$;

(4) E *is faithful if and only if* E *is almost faithful and symmetric.*

Proof. This readily follows from Proposition 3.6. □

COROLLARY 3.9. *If* A *detects ideals in* B, *then* E *is almost faithful.*

Proof. The intersection $\mathcal{N}_E \cap A$ is 0 because E is the identity map on A. Therefore, if A detects ideals in B, then $\mathcal{N}_E = \{0\}$, that is, E is almost faithful. □

LEMMA 3.10. *The reduced generalised expectation* E_r *is almost faithful. It is faithful if and only if* E *is symmetric.*

Proof. Proposition 3.6 implies $\mathcal{LN}_{E_\mathrm{r}} = \Lambda(\mathcal{LN}_E)$, $\mathcal{RN}_{E_\mathrm{r}} = \Lambda(\mathcal{RN}_E)$ and $\mathcal{N}_{E_\mathrm{r}} = \Lambda(\mathcal{N}_E) = 0$. This together with Corollary 3.8 implies all the statements. □

LEMMA 3.11. *Let* $E \colon B \to \tilde{A} \supseteq A$ *be a generalised expectation, and* $\pi \colon B \to C$ *a *-homomorphism. The following are equivalent:*

(1) $\ker \pi \subseteq \mathcal{N}_E$;

(2) *there is a *-homomorphism* $\varphi \colon \pi(B) \to B/\mathcal{N}_E$ *with* $\varphi \circ \pi = \Lambda \colon B \to B/\mathcal{N}_E$;

(3) *there is a map* $E_\pi \colon \pi(B) \to \tilde{A}$ *with* $E_\pi \circ \pi = E$.

*If these conditions hold, then $\pi|_A$ is injective, φ and E_π are unique and, identifying A with $\pi(A)$, E_π is a generalised expectation for the inclusion $\pi(A) \subseteq \pi(B)$. The *-homomorphism φ in (2) is faithful if and only if E_π is almost faithful.*

Proof. Since $\mathcal{N}_E = \ker \Lambda$, the *-homomorphism Λ descends to a *-homomorphism from $\pi(B) \cong B/\ker \pi$ to $B/\mathcal{N}_E$ if and only if $\ker \pi \subseteq \mathcal{N}_E$. Thus (1) is equivalent to (2). A map E_π as in (3) exists if and only if $E(\ker \pi) = 0$ or, equivalently, $\ker \pi \subseteq \ker E$. Since $\ker \pi$ is an ideal, this is equivalent to $\ker \pi \subseteq \mathcal{N}_E$. Thus (3) is equivalent to (1). Since $\pi\colon B \to \pi(B)$ is surjective, the maps φ and E_π are unique if they exist. Then E_π is automatically a completely positive contraction. And E_π is almost faithful if and only if $\ker \pi = \mathcal{N}_E$, if and only if φ is injective. □

Lemma 3.11 is a key point in the proof of gauge-equivariant uniqueness theorems. An action of a compact group such as $\mathbb{T}$ defines a conditional expectation onto the fixed-point algebra by averaging, and $\mathbb{T}$-equivariant maps intertwine these expectations. Hence a conditional expectation as in 3.11.(3) exists in the situation of gauge-equivariant uniqueness theorems.

3.2 Full and reduced crossed products

We fix an action $\mathcal{E} = \big((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t,u \in S}\big)$ of a unital inverse semigroup S on a C*-algebra A as in Definition 2.8. We recall how the algebraic crossed product $A \rtimes_{\mathrm{alg}} S$, the full crossed product $A \rtimes S$ and the reduced crossed product $A \rtimes_{\mathrm{r}} S$ of $\mathcal{E}$ are defined in [10]. The algebraic crossed products defined in [10] and in [20] differ. The definition in [10] has the merit that the canonical maps from $A \rtimes_{\mathrm{alg}} S$ to $A \rtimes S$ and $A \rtimes_{\mathrm{r}} S$ are injective.

For any $t \in S$, let $r(\mathcal{E}_t)$ and $s(\mathcal{E}_t)$ be the ideals in A generated by the left and right inner products of elements in $\mathcal{E}_t$, respectively. Thus $\mathcal{E}_t$ is an $r(\mathcal{E}_t)$-$s(\mathcal{E}_t)$-imprimitivity bimodule. These ideals satisfy

$$s(\mathcal{E}_t) = s(\mathcal{E}_{t^**t}) = r(\mathcal{E}_{t^**t}) = r(\mathcal{E}_{t^*}).$$

If $v \leqslant t$, then the inclusion map $j_{t,v}$ restricts to a Hilbert bimodule isomorphism from $\mathcal{E}_v$ onto $r(\mathcal{E}_v) \cdot \mathcal{E}_t = \mathcal{E}_t \cdot s(\mathcal{E}_v)$. For $t, u \in S$ and $v \leqslant t, u$, this gives Hilbert bimodule isomorphisms

$$\vartheta_{u,t}^v \colon \mathcal{E}_t \cdot s(\mathcal{E}_v) \xrightarrow[\cong]{j_{t,v}^{-1}} \mathcal{E}_v \xrightarrow[\cong]{j_{u,v}} \mathcal{E}_u \cdot s(\mathcal{E}_v).$$

Let

$$I_{t,u} := \overline{\sum_{v \leqslant t,u} s(\mathcal{E}_v)} \tag{3.1}$$

be the closed ideal generated by $s(\mathcal{E}_v)$ for $v \leqslant t, u$. This is contained in $s(\mathcal{E}_t) \cap s(\mathcal{E}_u)$, and the inclusion may be strict. There is a unique Hilbert bimodule isomorphism

$$\vartheta_{u,t} \colon \mathcal{E}_t \cdot I_{t,u} \xrightarrow{\sim} \mathcal{E}_u \cdot I_{t,u} \tag{3.2}$$

that restricts to $\vartheta_{u,t}^v$ on $\mathcal{E}_t \cdot s(\mathcal{E}_v)$ for all $v \leqslant t, u$ by [10, Lemma 2.4].

Let $A \rtimes_{\mathrm{alg}} S$ be the quotient vector space of $\bigoplus_{t \in S} \mathcal{E}_t$ by the linear span of $\vartheta_{u,t}(\xi)\delta_u - \xi\delta_t$ for all $t, u \in S$ and $\xi \in \mathcal{E}_t \cdot I_{t,u}$. The multiplication maps $\mu_{t,u}$ and the involutions $\mathcal{E}_t^* \to \mathcal{E}_{t*}$ turn $A \rtimes_{\mathrm{alg}} S$ into a *-algebra.

Definition 3.12 ([10]). The (*full*) *crossed product* $A \rtimes S$ of the action $\mathcal{E}$ is the maximal C*-completion of the *-algebra $A \rtimes_{\mathrm{alg}} S$ described above.

The C*-algebra $A \rtimes S$ is canonically isomorphic to the full section C*-algebra of the Fell bundle over S corresponding to $\mathcal{E}$ introduced in [20]. It is also characterised by a universal property.

Definition 3.13. A *representation* of $\mathcal{E}$ in a C*-algebra B is a family of linear maps $\pi_t \colon \mathcal{E}_t \to B$ for $t \in S$ such that $\pi_{tu}(\mu_{t,u}(\xi \otimes \eta)) = \pi_t(\xi)\pi_u(\eta)$, $\pi_t(\xi_1)^*\pi_t(\xi_2) = \pi_1(\langle \xi_1 \,|\, \xi_2 \rangle)$ and $\pi_t(\xi_1)\pi_t(\xi_2)^* = \pi_1(\langle\!\langle\!\langle \xi_1 \,|\, \xi_2 \rangle\!\rangle)$ for all $t, u \in S$, $\xi, \xi_1, \xi_2 \in \mathcal{E}_t$, $\eta \in \mathcal{E}_u$. Here $\langle \cdot \,|\, \cdot \rangle$ and $\langle\!\langle \cdot \,|\, \cdot \rangle\!\rangle$ denote the right and left inner products, respectively.

Remark 3.14. A representation π of $\mathcal{E}$ in B induces a *-homomorphism $A \rtimes S \to B$ and, conversely, every *-homomorphism $A \rtimes S \to B$ is of this form for a unique representation π (compare [10, Proposition 2.9]). This universal property determines $A \rtimes S$ uniquely up to isomorphism. Let B be a C*-algebra with an S-grading $(B_t)_{t \in S}$. Then $(B_t)_{t \in S}$ is a Fell bundle over S, and the inclusion maps $B_t \to B$ are a representation of this Fell bundle. The induced *-homomorphism $A \rtimes S \to B$ is surjective because $\sum B_t$ is dense in B. And its restriction to $A \subseteq A \rtimes S$ is injective by construction. Conversely, any surjective *-homomorphism $A \rtimes S \to B$ that restricts to an injective map on $A \subseteq A \rtimes S$ comes from a unique S-grading on B because $A \rtimes S$ is S-graded by the images of the Hilbert bimodules $\mathcal{E}_t$ for $t \in S$.

The reduced section C*-algebra of a Fell bundle over S is introduced in [20]. An equivalent definition appears in [10], where it is called the reduced crossed product $A \rtimes_{\mathrm{r}} S$ of the action $\mathcal{E}$. The main ingredient in the construction in [10] is a canonical weak conditional expectation $E \colon A'' \rtimes_{\mathrm{alg}} S \to A''$, involving the unique normal extension of the S-action $\mathcal{E}$ to A''. It is defined in [10, Lemma 4.5] through the formula

$$E(\xi\delta_t) = \operatorname{s-lim}_i \vartheta_{1,t}(\xi \cdot u_i) \tag{3.3}$$

for $\xi \in \mathcal{E}_t$ and $t \in S$, where (u_i) is an approximate unit for $I_{1,t}$ and s-lim denotes the limit in the strict topology on $\mathcal{M}(I_{1,t}) \subseteq A''$. In fact, this net also converges in the strong topology on A''. Let $x \in A \rtimes_{\mathrm{alg}} S$. Then $E(x^*x) \geqslant 0$, and $E(x^*x) = 0$ implies $x = 0$ (see [10, Proposition 3.6]). Thus $A'' \rtimes_{\mathrm{alg}} S$ may be completed to a Hilbert A''-module $\ell^2(S, A'')$ using the inner product $\langle x \,|\, y \rangle := E(x^*y)$. The action of $A \rtimes_{\mathrm{alg}} S$ on $A'' \rtimes_{\mathrm{alg}} S$ by left multiplication extends to a non-degenerate representation of $A \rtimes S$ on $\ell^2(S, A'')$.

Definition 3.15 ([10]). The *reduced crossed product* $A \rtimes_r S$ of the action $\mathcal{E}$ is the image of $A \rtimes S$ in $\mathbb{B}(\ell^2(S, A''))$. Let $\Lambda \colon A \rtimes S \to A \rtimes_r S$ be the canonical map.

This definition agrees with the one in [10] by [10, Remark 4.2].

Proposition 3.16. *The map $E \colon A \rtimes_{\mathrm{alg}} S \to A''$ extends to a weak conditional expectation $A \rtimes S \to A''$, which we also denote by E. Moreover, $\mathcal{N}_E = \ker \Lambda$. Thus*

$$A \rtimes_r S \cong (A \rtimes S)/\mathcal{N}_E$$

is the reduced quotient of $A \rtimes S$ for the generalised expectation E, and E descends to an almost faithful weak conditional expectation $E_r \colon A \rtimes_r S \to A''$.

Proof. The unit element of A'' gives an element 1 of $A'' \rtimes_{\mathrm{alg}} S \subseteq \mathbb{B}(\ell^2(S, A''))$, and $\langle 1 \mid b \cdot 1 \rangle = E(b)$ for all $b \in A \rtimes S$. This provides the unique extension of E to a completely positive, contractive map $A \rtimes S \to A''$, which we also denote by E. By construction, an element $b \in A \rtimes S$ satisfies $\Lambda(b) = 0$ if and only if $b \cdot c = 0$ for all $c \in \ell^2(S, A'')$. This is equivalent to $E(c^*b^*bc) = \langle b \cdot c \mid b \cdot c \rangle = 0$ for all $c \in \ell^2(S, A'')$. This follows once it holds for all c in the dense subspace $A'' \rtimes_{\mathrm{alg}} S$. Since E is normal, we may further reduce to the weakly dense subspace $A \rtimes_{\mathrm{alg}} S$. This is norm dense in $A \rtimes S$. So $\Lambda(b) = 0$ if and only if $E(c^*b^*bc) = 0$ for all $c \in A \rtimes S$. Then Proposition 3.6 implies $\ker \Lambda = \mathcal{N}_E$. Hence the weak conditional expectation on $A \rtimes S$ descends to one on $A \rtimes_r S$, which is almost faithful. $\qquad\square$

Remark 3.17. The canonical map from $A \rtimes_{\mathrm{alg}} S$ to $A \rtimes_r S$ is injective by [10, Proposition 4.3].

Remark 3.18. Let $(\mathcal{A}_t)_{t \in S}$ be a non-saturated Fell bundle over a unital inverse semigroup S. It is turned into a saturated Fell bundle $(\mathcal{E}_t)_{t \in \tilde{S}}$ over another inverse semigroup $\tilde{S}$ in [9]. This construction does not change the full and reduced section C*-algebras by [9, Theorem 7.2]. Therefore, we usually restrict attention to saturated Fell bundles, which we replace by inverse semigroup actions as in Definition 2.8.

We recall some known conditions for $E \colon A \rtimes S \to A''$ to be A-valued, that is, a genuine conditional expectation.

Proposition 3.19 ([10, Proposition 6.3]). *The map E is A-valued if and only if the ideal $I_{1,t}$ defined in (3.1) is complemented in the larger ideal $s(\mathcal{E}_t)$ for each $t \in S$.*

There is a largest ideal $J \in \mathbb{I}(A)$ with $J \cap I_{1,t} = \{0\}$, namely,

$$I_{1,t}^\perp := \{x \in A : x \cdot I_{1,t} = 0\}. \tag{3.4}$$

Therefore, $I_{1,t}$ is complemented in $s(\mathcal{E}_t)$ if and only if $s(\mathcal{E}_t) = I_{1,t} \oplus (s(\mathcal{E}_t) \cap I_{1,t}^\perp)$. Then

$$\mathcal{E}_t = (\mathcal{E}_t \cdot I_{1,t}) \oplus (\mathcal{E}_t \cdot I_{1,t}^\perp) \xrightarrow[\cong]{\vartheta_{1,t} \oplus \mathrm{Id}} I_{1,t} \oplus (\mathcal{E}_t \cdot I_{1,t}^\perp), \tag{3.5}$$

and $E|_{\mathcal{E}_t}$ is the orthogonal projection onto the summand $I_{1,t}$ by (3.3).

Proposition 3.20. *Let $\mathcal{E}$ be an action of S on A. The following are equivalent:*

(1) *the weak conditional expectation $E\colon A \rtimes S \to A''$ is A-valued;*

(2) *the subset of units $\check{A}$ is closed in $\check{A} \rtimes S$;*

(3) *the action is closed, that is, the dual S-action on $\widehat{A}$ is closed.*

Proof. The equivalence of the first two statements is [10, Theorem 6.5]. The map $\widehat{A} \ni [\pi] \mapsto \ker[\pi] \in \check{A}$ intertwines the inverse semigroup actions $(\widehat{\mathcal{E}}_t)_{t \in S}$ and $(\check{\mathcal{E}}_t)_{t \in S}$. It is continuous, open and closed because the spaces $\check{A}$ and $\widehat{A}$ both have $\mathbb{I}(A)$ as their lattice of open subsets. So the map above extends to a continuous, open and closed groupoid homomorphism

$$\kappa\colon \widehat{A} \rtimes S \to \check{A} \rtimes S, \qquad [t, [\pi]] \mapsto [t, \ker \pi].$$

Since $\kappa^{-1}(\check{A}) = \widehat{A}$, the space of units $\widehat{A}$ is closed in $\widehat{A} \rtimes S$ if and only if $\check{A}$ is closed in $\check{A} \rtimes S$. Now (2) and (3) are equivalent by Lemma 2.6. $\square$

3.3 The reduced weak conditional expectation is faithful

It is claimed in [10] that the weak conditional expectation E becomes faithful on $A \rtimes_r S$. But in [10, Proposition 3.6], this is shown only for its restriction to $A \rtimes_{\mathrm{alg}} S$. Here we fill this gap.

Lemma 3.21. *Let $\varrho\colon A'' \to \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ be the projection to the direct sum of all irreducible representations in the universal representation of A''. Then ϱ is isometric on the C*-algebra generated by the range of E in A''. Hence $\|E(x)\| = \|\varrho \circ E(x)\|$ for all $x \in A \rtimes S$ and, in particular, $\ker E = \ker(\varrho \circ E)$.*

Proof. A family of representations $(\pi_i)_{i \in I}$ of A is called E-*faithful* in [10, Definition 4.9] if the extension of the representation $\bigoplus_{i \in I} \pi_i$ to A'' restricts to a faithful representation on the C*-subalgebra that is generated by the image of E. The family of all irreducible representations of A is E-faithful by [10, Theorem 7.4]. $\square$

Theorem 3.22. *The weak conditional expectation $E\colon A \rtimes S \to A''$ is symmetric. Equivalently, $E_r\colon A \rtimes_r S \to A''$ is faithful.*

Proof. By Proposition 3.6 and Corollary 3.8, it suffices to show that if $b \in A \rtimes S$ satisfies $E(b^*b) = 0$, then $E(c^*b^*bc) = 0$ for all $c \in A \rtimes S$. Since $c \in A \rtimes S$ satisfies $E(c^*b^*bc) = 0$ if and only if $b \cdot c \in \mathcal{LN}_E$, the space of c with this property is a closed linear subspace. Hence it suffices to prove $E(c^*b^*bc) = 0$ for $c \in \mathcal{E}_t$ for some $t \in S$. Here we embed the Hilbert A-bimodules $\mathcal{E}_t$ of the S-action on A into $A \rtimes_{\mathrm{alg}} S$ in the usual way. So we may fix $t \in S$ and $c \in \mathcal{E}_t$. We must show that $E(b^*b) = 0$ implies $E(c^*b^*bc) = 0$.
Let $\pi \in \widehat{A}$. The tensor product $\mathcal{E}_t \otimes_A \mathcal{H}_\pi$ is non-zero if and only if π belongs to $\widehat{s(\mathcal{E}_t)}$, and then the left multiplication action of A on $\mathcal{E}_t \otimes_A \mathcal{H}_\pi$ is the irreducible

representation $\omega := \widehat{\mathcal{E}}_t(\pi)$. The element $c \in \mathcal{E}_t$ induces an adjointable linear map

$$T_c \colon \mathcal{H}_\pi \to \mathcal{E}_t \otimes_A \mathcal{H}_\pi, \qquad \xi \mapsto c \otimes \xi.$$

Its adjoint is given by $T_c^*(e \otimes \xi) = \pi(c^*e)\xi$ for $e \in \mathcal{E}_t$, $\xi \in \mathcal{H}_\pi$. Let π'' and ω'' be the weakly continuous extensions of π and ω to A''. We claim that

$$\pi''(E(c^*dc)) = \begin{cases} T_c^*\omega''(E(d))T_c & \text{if } \pi \in \widehat{s(\mathcal{E}_t)}, \\ 0 & \text{otherwise}, \end{cases} \qquad (3.6)$$

for all $d \in A \rtimes S$. We are interested, of course, in $d = b^*b$. If $\pi \notin \widehat{s(\mathcal{E}_t)}$, then $\pi''(E(c^*dc)) = 0$. So it suffices to consider the case when $\pi \in \widehat{s(\mathcal{E}_t)}$. The set of $d \in A \rtimes S$ for which (3.6) holds is a closed linear subspace because the two sides of the equality are bounded linear operators of d. Therefore, it suffices to prove (3.6) for $d \in \mathcal{E}_u$ with some $u \in S$. It is more convenient to work in the bidual A'' for this computation and to allow $d \in \mathcal{E}_u''$. Let $[I]$ for an ideal I in A denote the support projection of I, that is, the weak limit in A'' of an approximate unit in I. There is a canonical Hilbert bimodule isomorphism

$$\Theta_u \colon \mathcal{E}_u'' \cdot [I_{1,u}] \xrightarrow{\sim} I_{1,u}'',$$

and $E(d) = \Theta_u(d \cdot [I_{1,u}]) = E(d \cdot [I_{1,u}])$. Similarly, $E(c^*dc) = \Theta_{t^**ut}(c^*dc \cdot [I_{1,t^**ut}])$ because $c^*dc \in \mathcal{E}_{t^**ut}''$. The Rieffel correspondence for $\mathcal{E}_t''$ implies that $\mathcal{E}_t'' \cdot [I] = (\mathcal{E}_t \cdot I)'' = (J \cdot \mathcal{E}_t)'' = [J] \cdot \mathcal{E}_t''$, where $\widehat{J} = \widehat{\mathcal{E}}_t(\widehat{I})$. Therefore, $c \cdot [I_{1,t^**ut}] = [J] \cdot c$, where $\widehat{J} = \widehat{\mathcal{E}}_t(\widehat{I_{1,t^**ut}})$. Since

$$\widehat{\mathcal{E}}_t(\widehat{I_{1,t^**ut}}) = \widehat{I_{tt^*,u}} = I_{1,u} \widehat{\cap\, r}(\mathcal{E}_t),$$

we get $[J] \cdot c = [I_{1,u}] \cdot c$. So

$$E(c^*dc) = \Theta_{t^**ut}(c^*d \cdot ([I_{1,u}] \cdot c)) = E(c^*(d \cdot [I_{1,u}])c).$$

Therefore, the two sides in (3.6) do not change when we replace d by $d \cdot [I_{1,u}]$. But $\Theta_u(d \cdot [I_{1,u}]) \in I_{1,u}'' \subseteq A''$ and $d \in \mathcal{E}_u'' \cdot [I_{1,u}] \subseteq \mathcal{E}_u''$ are identified in $A'' \rtimes S$. Since the two sides in (3.6) only depend on the image of d in $A'' \rtimes S$, we are reduced to the case $d \in A''$. Then $c^*dc \in A''$ as well, and so $E(d) = d$ and $E(c^*dc) = c^*dc$. And (3.6) follows because $T_c^*\omega''(d)T_c = \pi''(c^*dc)$. This finishes the proof of (3.6).

Now $E(b^*b) = 0$ is equivalent to $\omega''(E(b^*b)) = 0$ for all $\omega \in \widehat{A}$ by Lemma 3.21. This implies $\pi(E(c^*b^*bc)) = T_c^*\omega''(E(b^*b))T_c = 0$ for all $\pi \in \widehat{A}$ by (3.6). This is equivalent to $E(c^*b^*bc) = 0$ by Lemma 3.21. This proves the claim about E being symmetric. And this is equivalent to E_r being faithful by Lemma 3.10. $\qquad\square$

4 The essential crossed product

We are going to define a variant of the reduced crossed product that is based on a generalised expectation with values in the local multiplier algebra.

4.1 The definition of the essential crossed product

Definition 4.1 ([3]). Let A be a C*-algebra. The essential ideals in A form a directed set for the relation $\supseteq$ because $I \cap J \in \mathbb{I}(A)$ is essential if $I, J \in \mathbb{I}(A)$ are essential. If $I \subseteq J \subseteq A$ are essential ideals, then a multiplier of J restricts to a multiplier of I. This defines a canonical, unital *-homomorphism $\mathcal{M}(J) \to \mathcal{M}(I)$, which is injective because I is essential in J. The *local multiplier algebra* $\mathcal{M}_{\mathrm{loc}}(A)$ of A is the inductive limit C*-algebra of this inductive system.

The canonical map $A \to \mathcal{M}_{\mathrm{loc}}(A)$ is injective because $A \to \mathcal{M}(I)$ is injective for any essential ideal I in A. We identify A with its image in $\mathcal{M}_{\mathrm{loc}}(A)$.

Definition 4.2. For any C*-inclusion $A \subseteq B$, a generalised expectation with values in $\mathcal{M}_{\mathrm{loc}}(A)$ is called an $\mathcal{M}_{\mathrm{loc}}$-*expectation*.

Now let us fix an action $\mathcal{E}$ of a unital inverse semigroup S on a C*-algebra A. We modify the construction of the reduced crossed product by replacing the weak conditional expectation $E \colon A \rtimes S \to A''$ by an $\mathcal{M}_{\mathrm{loc}}$-expectation $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$. We first define EL on the dense subalgebra $A \rtimes_{\mathrm{alg}} S \subseteq A \rtimes S$. This is spanned by $\mathcal{E}_t$ for $t \in S$. Let $t \in S$ and define $I_{1,t}$ as in (3.1). Recall that the ideal $I_{1,t}^\perp$ defined in (3.4) is the largest ideal $J \in \mathbb{I}(A)$ with $J \cap I_{1,t} = \{0\}$. The ideal $J_t := I_{1,t} \oplus I_{1,t}^\perp \subseteq A$ is essential by construction. And

$$\mathcal{E}_t \cdot J_t \cong (\mathcal{E}_t \cdot I_{1,t}) \oplus (\mathcal{E}_t \cdot I_{1,t}^\perp) \xrightarrow[\cong]{\vartheta_{1,t} \oplus \mathrm{Id}} I_{1,t} \oplus (\mathcal{E}_t \cdot I_{1,t}^\perp)$$

as in (3.5). Any $\xi \in \mathcal{E}_t$ defines a map $J_t \to \mathcal{E}_t \cdot J_t$, $a \mapsto \xi \cdot a$. Composing with the orthogonal projection to $I_{1,t}$ defines an adjointable operator $J_t \to I_{1,t} \subseteq J_t$ or, equivalently, a multiplier of J_t. We let $EL(\xi) \in \mathcal{M}(J_t) \subseteq \mathcal{M}_{\mathrm{loc}}(A)$ be this multiplier of J_t.

Proposition 4.3. *The map* $\bigoplus_{t \in S} \mathcal{E}_t \ni \xi \mapsto EL(\xi) \in \mathcal{M}_{\mathrm{loc}}(A)$ *factors through to a map* $A \rtimes_{\mathrm{alg}} S \to \mathcal{M}_{\mathrm{loc}}(A)$, *which extends to an* $\mathcal{M}_{\mathrm{loc}}$-*expectation* $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ *that satisfies* $\|EL(x)\| \leqslant \|E(x)\|$ *for all* $x \in A \rtimes S$.

Proof. If $\xi \in \mathcal{E}_t \cdot J_t$, then the above definition gives $EL(\xi) \in I_{1,t} \subseteq A$. In this case, $EL(\xi) = E(\xi)$ because the strict limit in (3.3) is equal to a norm limit. More generally, let $\xi \in A \rtimes_{\mathrm{alg}} S$. Then there are a finite subset $F \subseteq S$ and $\xi_t \in \mathcal{E}_t$ for $t \in F$ with $\xi = \sum_{t \in F} \xi_t \in A \rtimes_{\mathrm{alg}} S$. The ideal $J := \bigcap_{t \in F} J_t$ in A is essential as a finite intersection of essential ideals. The arguments above imply

$$E(\xi)a = \sum_{t \in F} E(\xi_t a) = \sum_{t \in F} EL(\xi_t a) = \left(\sum_{t \in F} EL(\xi_t) \right) a \in J$$

for every $a \in J$. Hence $\sum_{t \in F} EL(\xi_t) \in \mathcal{M}(J) \subseteq \mathcal{M}_{\mathrm{loc}}(A)$ coincides with the multiplier given by multiplication by $E(\xi)$. This implies that $EL(\xi) := \sum_{t \in F} EL(\xi_t)$ is well defined on $A \rtimes_{\mathrm{alg}} S$ (because E is). The norm of $EL(\xi)$

in $\mathcal{M}_{\mathrm{loc}}(A)$ is equal to its norm in $\mathcal{M}(J)$. If $a \in J$ satisfies $\|a\| \leqslant 1$, then $\|EL(\xi) \cdot a\| = \|E(\xi) \cdot a\| \leqslant \|E(\xi)\|$. Hence $\|EL(\xi)\| \leqslant \|E(\xi)\|$. Thus EL extends to a contractive linear map $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ because $E \colon A \rtimes S \to A''$ is contractive. The above considerations also imply $EL|_A = \mathrm{Id}_A$.

To see that EL is completely positive, let $x \in \mathbb{M}_n(A \rtimes_{\mathrm{alg}} S)$. There are a finite subset $F \subseteq S$ and $x_t \in \mathcal{E}_t \otimes \mathbb{M}_n$ for $t \in F$ such that $x = \sum_{t \in F} x_t$. Then $J := \bigcap_{s,t \in F} J_{s^* t}$ is an essential ideal in A. One checks as above that multiplication by $E(x^*x)$ on $\mathbb{M}_n(J)$ coincides with $EL(x^*x) \in \mathcal{M}(\mathbb{M}_n(J)) = \mathbb{M}_n(\mathcal{M}(J)) \subseteq \mathbb{M}_n(\mathcal{M}_{\mathrm{loc}}(A))$. Therefore, if $y \in \mathbb{M}_n(J)$, then

$$y^* \cdot EL(x^*x) \cdot y = y^* \cdot E(x^*x) \cdot y = E((xy)^*xy) \geqslant 0$$

because E is completely positive. Therefore, $EL(x^*x)$ is positive in $\mathbb{M}_n(\mathcal{M}_{\mathrm{loc}}(A))$. So EL is completely positive on $A \rtimes_{\mathrm{alg}} S$. It follows that EL is completely positive on $A \rtimes S$. So EL is an $\mathcal{M}_{\mathrm{loc}}$-expectation. $\qquad\square$

DEFINITION 4.4. The *essential crossed product* $A \rtimes_{\mathrm{ess}} S$ is the reduced quotient $(A \rtimes S)/\mathcal{N}_{EL}$ for the generalised expectation EL.

Remark 4.5. Since $\|EL(x)\| \leqslant \|E(x)\|$ for all $x \in A \rtimes S$, it follows that $\ker EL \supseteq \ker E$. Thus $A \rtimes_{\mathrm{ess}} S$ is also a quotient of $A \rtimes_{\mathrm{r}} S$. By construction, EL descends to an almost faithful $\mathcal{M}_{\mathrm{loc}}$-expectation $A \rtimes_{\mathrm{ess}} S \to \mathcal{M}_{\mathrm{loc}}(A)$ for the inclusion $A \hookrightarrow A \rtimes_{\mathrm{ess}} S$.

Remark 4.6. If the action is closed, then $EL = E$ and $A \rtimes_{\mathrm{ess}} S = A \rtimes_{\mathrm{r}} S$ by the construction and (3.5). Thus essential crossed products give something new only for non-closed actions.

Example 4.7. We show by an example that $A \rtimes_{\mathrm{ess}} S$ may differ from $A \rtimes_{\mathrm{r}} S$. Let G be an amenable discrete group. View G as an inverse semigroup and adjoin a zero element to it. This gives the inverse semigroup $S := G \cup \{0\}$ with $g \cdot 0 = 0 \cdot g = 0$ for all $g \in S$ and such that $g \cdot h$ for $g, h \in G$ is the usual product in G. Let G act on $A := \mathrm{C}[0,1]$ by $\mathcal{E}_g := \mathrm{C}[0,1]$ for $g \in G$ and $\mathcal{E}_0 := \mathrm{C}_0(0,1)$, equipped with the usual involution and multiplication maps. If $g \in G \setminus \{1\}$, then $I_{g,1} = \mathrm{C}_0(0,1)$ because $v \in S$ satisfies $v \leqslant 1, g$ if and only if $v = 0$. The full crossed product for this action is a $\mathrm{C}[0,1]$-C^*-algebra with fibres $\mathrm{C}^*(G)$ at 0 and $\mathbb{C}$ at $x \in (0,1]$. The reduced crossed product is equal to the full one here because G is amenable. (If G is non-amenable, then the fibre of the reduced crossed product at 0 is $\mathbb{C} \oplus \mathrm{C}_{\mathrm{r}}^*(G)$ and not $\mathrm{C}_{\mathrm{r}}^*(G)$. We restrict to amenable groups to avoid discussing this issue further.) The essential crossed product in this case reduces to $A = \mathrm{C}[0,1]$. Indeed, let $\xi_g \in \mathcal{E}_g = \mathrm{C}[0,1]$. Let $\xi_1 \in \mathcal{E}_1$ be the same element of $\mathrm{C}[0,1]$. Then $EL(\xi_g) = EL(\xi_1)$ because the ideal $\mathrm{C}_0(0,1)$ in A is essential. Thus $\ker EL$ contains the kernel of the *-homomorphism $A \rtimes_{\mathrm{alg}} S \to A$ that applies the trivial representation of the group in each fibre. It follows that $A \rtimes_{\mathrm{ess}} S \cong A$.

In this example, EL takes values in A although the action is not closed. And the map $A \rtimes_{\mathrm{alg}} S \to A \rtimes_{\mathrm{ess}} S$ is not injective. In contrast, the map $A \rtimes_{\mathrm{alg}} S \to A \rtimes_{\mathrm{r}} S$ is always injective.

Remark 4.8. Unlike the reduced and full crossed products, the essential crossed product is not functorial for *-homomorphisms. This is because the local multiplier algebra is not functorial. The problem is very visible for quotient maps. If $I \subseteq A$ is an essential ideal, then a local multiplier of A does not descend to anything on A/I. For instance, consider the restriction map $C[0,1] \to \mathbb{C}$, $f \mapsto f(0)$, in the situation of Example 4.7. This is S-equivariant, where S acts on $\mathbb{C}$ by the trivial action, with 0 acting by the ideal $\{0\}$. Now $\mathbb{C} \rtimes S \cong C^*(G)$, and the weak conditional expectation $C^*(G) \to \mathbb{C}$ is the usual trace, which has values in $\mathbb{C} = \mathbb{C}'' = \mathcal{M}_{\mathrm{loc}}(\mathbb{C})$. Hence the essential and reduced crossed products are the same for the action on $\mathbb{C}$. But the fibre restriction map $A \rtimes_{\mathrm{r}} S \to C^*_{\mathrm{r}}(G)$ does not factor through $A \rtimes_{\mathrm{ess}} S = A$.

In order to relate $EL \colon A \rtimes_{\mathrm{ess}} S \to \mathcal{M}_{\mathrm{loc}}(A)$ to $E \colon A \rtimes S \to A''$, we use Lemma 3.21. It says that the *-homomorphism $\varrho \colon A'' \to \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ is isometric on $E(A \rtimes S)$. We shall embed $\mathcal{M}_{\mathrm{loc}}(A)$ into a quotient of $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$.

DEFINITION 4.9. A subset of $\widehat{A}$ is *nowhere dense* if its closure has empty interior. It is *meagre* if it may be written as a countable union of nowhere dense subsets. It is *comeagre* if its complement is meagre. For $(f_\pi)_{\pi \in \widehat{A}} \in \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$, define

$$\left\| (f_\pi)_{\pi \in \widehat{A}} \right\|_{\mathrm{ess}} := \inf \left\{ \sup_{\pi \in R} \|f_\pi\| : R \subseteq \widehat{A} \text{ comeagre} \right\}.$$

By definition, a subset is meagre if and only if it is contained in a union of countably many closed subsets with empty interior. Thus a subset is comeagre if and only if it contains an intersection of countably many dense open subsets. Any countable intersection of comeagre subsets is again comeagre, and a subset that contains a comeagre subset is also comeagre. It follows that the infimum in Definition 4.9 is a minimum. And the set of comeagre subsets of $\widehat{A}$ is directed by $\supseteq$. The values $\sup_{\pi \in R} \|f_\pi\|$ for comeagre subsets $R \subseteq \widehat{A}$ form a monotone net indexed by this directed set. Thus

$$\|(f_\pi)\|_{\mathrm{ess}} = \min_{\substack{R \subseteq \widehat{A} \\ \text{comeagre}}} \sup_{\pi \in R} \|f_\pi\| = \lim_{\substack{R \subseteq \widehat{A} \\ \text{comeagre}}} \sup_{\pi \in R} \|f_\pi\|.$$

PROPOSITION 4.10. *The function $\|\cdot\|_{\mathrm{ess}}$ on $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ is a C*-seminorm. Let D be the quotient of $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ by the null space of $\|\cdot\|_{\mathrm{ess}}$. The faithful representation $A \hookrightarrow \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ factors through to $A \hookrightarrow D$, and the latter extends to an isometric *-homomorphism $\iota \colon \mathcal{M}_{\mathrm{loc}}(A) \hookrightarrow D$ such that the following diagram commutes:*

$$
\begin{array}{ccccc}
A \rtimes S & \xrightarrow{\ E\ } & A'' & \xrightarrow{\ \varrho\ } & \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi) \\
\downarrow{\scriptstyle EL} & & & & \downarrow{\scriptstyle q} \\
\mathcal{M}_{\mathrm{loc}}(A) & & \xrightarrow{\hspace{3cm} \iota \hspace{3cm}} & & D
\end{array}
$$

If $b \in A \rtimes S$, then $EL(b) = 0$ if and only if $\{\pi \in \widehat{A} : \pi''(E(b)) = 0\}$ is comeagre in $\widehat{A}$.

Proof. If $R \subseteq \widehat{A}$ is a comeagre subset, then the supremum of $\|f_\pi\|$ for $\pi \in R$ is a C*-seminorm on $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$. As the pointwise limit of these C*-seminorms, $\|\cdot\|_{\mathrm{ess}}$ is a C*-seminorm as well. Since $\prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi)$ is already a C*-algebra, the quotient D by the null space of this C*-seminorm is complete, hence a C*-algebra.

We define ι. Let $J \subseteq A$ be an essential ideal. Then $\widehat{J} \subseteq \widehat{A}$ is a dense open subset, hence comeagre. If $\pi \in \widehat{J}$, then π extends uniquely to a representation $\bar{\pi}$ of $\mathcal{M}(J)$, and so $\bar{\pi}(a) \in \mathbb{B}(\mathcal{H}_\pi)$ is defined for all $a \in \mathcal{M}(J)$. This defines a unital *-homomorphism $\iota_J \colon \mathcal{M}(J) \to D$. If $K \subseteq J$, then the composite of ι_K with the restriction map $\mathcal{M}(J) \to \mathcal{M}(K)$ is equal to ι_J. Hence the maps ι_J for all essential ideals $J \subseteq A$ combine to a unital *-homomorphism $\iota \colon \mathcal{M}_{\mathrm{loc}}(A) \to D$. To see that ι is isometric, it suffices to prove that $\|a\| \leqslant \|\iota_J(a)\|$ holds for all essential ideals $J \subseteq A$ and all $a \in \mathcal{M}(J)$. Let $\varepsilon > 0$ and $C := \|a\|_{\mathcal{M}(J)}$. The function $\widehat{\mathcal{M}(J)} \ni \bar{\pi} \to \|\bar{\pi}(a)\|$ has the supremum $\|a\|$ and is lower semicontinuous (see [17, Proposition 3.3.2]), and $\widehat{J}$ is an open dense subset in $\widehat{\mathcal{M}(J)}$. Therefore, $U := \{\pi \in \widehat{J} \colon \|\bar{\pi}(a)\| > C - \varepsilon\}$ is a non-empty open subset of $\widehat{J} \subseteq \widehat{A}$. Since $\widehat{A}$ is a Baire space, every comeagre subset $R \subseteq \widehat{A}$ is also dense in $\widehat{A}$. Thus $R \cap U \neq \varnothing$, and the supremum of $\|\bar{\pi}(a)\|$ for $\pi \in R$ is at least $C - \varepsilon$. Then $\|\iota_J(a)\| > C - \varepsilon$ follows. Since $\varepsilon > 0$ is arbitrary, this implies $\|\iota_J(a)\| \geqslant C$. Hence $\iota \colon \mathcal{M}_{\mathrm{loc}}(A) \to D$ is isometric.

Now we show that the diagram in the proposition commutes. Let $b \in A \rtimes_{\mathrm{alg}} S$. Then $b = \sum_{t \in F} \xi_t \delta_t$ for a finite subset $F \subseteq S$ and $\xi_t \in \mathcal{E}_t$. Let $J_t := I_{1,t} \oplus I_{1,t}^\perp$ and $J = \bigcap_{t \in F} J_t$. These are essential ideals in A. We have already seen that $E(b) \cdot a \in J$ for all $a \in J$, and $EL(b)$ is the resulting multiplier of J. If $\pi \in \widehat{J} \subseteq \widehat{A}$, then $\pi''(E(b)) = \bar{\pi}(EL(b))$. Since $\widehat{J}$ is comeagre, q maps $\varrho \circ E(b)$ to $\iota(EL(b))$. Hence the diagram commutes on all elements of $A \rtimes_{\mathrm{alg}} S$. Since $A \rtimes_{\mathrm{alg}} S$ is dense in $A \rtimes S$ and all maps involved are contractive, it commutes on all of $A \rtimes S$. $\qquad\square$

THEOREM 4.11. *The $\mathcal{M}_{\mathrm{loc}}$-expectation $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ is symmetric and thus descends to a faithful $\mathcal{M}_{\mathrm{loc}}$-expectation $A \rtimes_{\mathrm{ess}} S \to \mathcal{M}_{\mathrm{loc}}(A)$.*

Proof. The two statements in the assertion are equivalent by Lemma 3.10. Let $\mathcal{E}_t$ be the Hilbert A-bimodules that define the S-action on A. We embed these into $A \rtimes_{\mathrm{ess}} S$ as usual. Arguing as in the beginning of the proof of Theorem 3.22, it suffices to show that if $b \in A \rtimes_{\mathrm{ess}} S$ satisfies $EL(b^*b) = 0$, then $EL(c^*b^*bc) = 0$ for all $c \in \mathcal{E}_t$, $t \in S$. Thus let $b \in A \rtimes_{\mathrm{ess}} S$ be such that $EL(b^*b) = 0$. By Proposition 4.10, the set of $\omega \in \widehat{A}$ with $\omega''(E(b^*b)) \neq 0$ is meagre. Equation (3.6) shows that $\pi''(E(c^*b^*bc)) \neq 0$ can only happen if $\pi \in s(\widehat{\mathcal{E}_t})$ and $\omega(E(b^*b)) \neq 0$ for $\omega = \widehat{\mathcal{E}_t}(\pi)$. Therefore, the set

$$\left\{\pi \in \widehat{A} \colon \pi''(E(c^*b^*bc))) \neq 0\right\} \subseteq \widehat{\mathcal{E}_t}^{-1}\left(\left\{\omega \in \widehat{r(\mathcal{E}_t)} \colon \omega''(E(b^*b)) \neq 0\right\}\right)$$

is meagre as a subset of a meagre set. Thus $EL(c^*b^*bc) = 0$ by Proposition 4.10. $\qquad\square$

COROLLARY 4.12. $\mathcal{N}_{EL} = \left\{ b \in A \rtimes S : \{\pi \in \widehat{A} : \pi''(E(b^*b)) = 0\} \text{ is comeagre} \right\}$.

4.2 Topological gradings

The following definition is an inverse semigroup analogue of [18, Definition 3.4]. We choose to use the essential instead of the reduced crossed product here.

DEFINITION 4.13. Let B be a C*-algebra with an S-grading $(B_t)_{t\in S}$. Let $\pi \colon A \rtimes S \to B$ be the canonical *-epimorphism as in Remark 3.14. The grading is called *topological* and B is called *topologically S-graded* if $\ker \pi$ is contained in the kernel $\mathcal{N}_{EL}$ of the quotient map $A \rtimes S \to A \rtimes_{\mathrm{ess}} S$.

By definition, a grading is topological if and only if there is a (surjective) *-homomorphism

$$\varphi \colon B \to A \rtimes_{\mathrm{ess}} S$$

with $\varphi \circ \pi = \Lambda$. So B lies between the full and the essential crossed products and may be called an *exotic crossed product*. Another equivalent characterisation when a grading is topological is $\|\pi(x)\| \geqslant \|x\|_{\mathrm{ess}}$ for all $x \in A \rtimes S$ or all $x \in A \rtimes_{\mathrm{alg}} S$, where $\|x\|_{\mathrm{ess}}$ denotes the norm in the essential crossed product $A \rtimes_{\mathrm{ess}} S$. Equivalently, B is isomorphic to a completion of the *-algebra $A \rtimes_{\mathrm{alg}} S$ for a C*-norm that lies between the maximal and the essential C*-norm:

PROPOSITION 4.14. *Let B be an S-graded C*-algebra. View the grading as an S-action on its unit fibre A, and let $\pi \colon A \rtimes S \to B$ be the induced surjective *-homomorphism as in Remark 3.14. Let $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ be the canonical $\mathcal{M}_{\mathrm{loc}}$-expectation. The following are equivalent:*

(1) *the S-grading on B is topological;*

(2) *there is an $\mathcal{M}_{\mathrm{loc}}$-expectation $P \colon B \to \mathcal{M}_{\mathrm{loc}}(A)$ with $P \circ \pi = EL$;*

(3) $\|EL(x)\| \leqslant \|\pi(x)\|$ *for all $x \in A \rtimes_{\mathrm{alg}} S$.*

If P is an $\mathcal{M}_{\mathrm{loc}}$-expectation as in (2), then the canonical surjective *-homomorphism $B \to A \rtimes_{\mathrm{ess}} S$ is an isomorphism if and only if P is almost faithful, if and only if P is faithful.

Proof. The assertions mostly follow from Lemma 3.11. That P is almost faithful if and only if it is faithful follows from Theorem 4.11. $\square$

4.3 Coincidence of the reduced and essential crossed products

The essential crossed product has the advantage of having the "right" ideal structure, even for non-Hausdorff groupoids and similar situations. A serious disadvantage is that it is not functorial (see Remark 4.8). This suggests that it is not the right object for K-theory computations. A question like exactness does not even make sense for it. So it is desirable to know when the reduced and

essential crossed products are the same, meaning that the canonical quotient map $A \rtimes_{\mathrm{r}} S \to A \rtimes_{\mathrm{ess}} S$ is an isomorphism.

Let $E \colon A \rtimes S \to A''$ be the canonical weak conditional expectation and let $EL \colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ be the canonical $\mathcal{M}_{\mathrm{loc}}$-expectation. By definition, $A \rtimes_{\mathrm{r}} S = A \rtimes_{\mathrm{ess}} S$ if and only if $\mathcal{N}_E = \mathcal{N}_{EL}$. A case when this is particularly clear is when E is a genuine conditional expectation, taking values in $A \subseteq A''$. Then EL also takes values in $A \subseteq \mathcal{M}_{\mathrm{loc}}(A)$ and $EL = E$. If E is not A-valued, then $\mathcal{N}_E \subseteq \mathcal{N}_{EL}$. The difference between $A \rtimes_{\mathrm{r}} S$ and $A \rtimes_{\mathrm{ess}} S$ is measured by the ideal

$$J_{\mathrm{sing}} := \Lambda(\mathcal{N}_{EL})$$

in $A \rtimes_{\mathrm{r}} S$ because

$$(A \rtimes_{\mathrm{r}} S)/J_{\mathrm{sing}} \cong A \rtimes_{\mathrm{ess}} S.$$

We call elements of J_{sing} *singular*. Proposition 4.10 describes the kernels of E and EL using the supremum and the essential supremum norm of functions on $\widehat{A}$, where the essential supremum is defined as the supremum over comeagre subsets. We use it to describe J_{sing}. If $a \in A''$, then we define

$$\nu_a \colon \widehat{A} \to [0, \infty), \qquad \pi \mapsto \|\pi''(a)\|,$$

where $\pi'' \colon A'' \to \mathbb{B}(\mathcal{H}_\pi)$ denotes the unique extension of π to A''. If $b \in A \rtimes S$, then

$$\|E(b)\| = \|\varrho \circ E(b)\| = \sup_{\pi \in \widehat{A}} \nu_{E(b)}(\pi) = \|\nu_{E(b)}\|_\infty$$

by Lemma 3.21. Similarly, Proposition 4.10 gives

$$\|EL(b)\| = \|\iota \circ EL(b)\| = \min_{\substack{R \subseteq \widehat{A} \\ \mathrm{comeagre}}} \sup_{\pi \in R} \nu_{E(b)}(\pi) =: \|\nu_{E(b)}\|_{\mathrm{ess}}.$$

In particular, $EL(b) = 0$ if and only if $\{\pi \in \widehat{A} : \nu_{E(b)}(\pi) \neq 0\}$ is meagre.

PROPOSITION 4.15. *Let $b \in A \rtimes S$. The function $\nu_{E(b)} \colon \widehat{A} \to [0, \infty)$ is lower semicontinuous on a comeagre subset. The following are equivalent:*

(1) $\{\pi \in \widehat{A} : \nu_{E(b)}(\pi) \neq 0\}$ *is meagre;*

(2) $\{\pi \in \widehat{A} : \nu_{E(b)}(\pi) \neq 0\}$ *has empty interior;*

(3) *the subsets $\{\pi \in \widehat{A} : \nu_{E(b)}(\pi) > \varepsilon\}$ have empty interior for all $\varepsilon > 0$.*

Proof. Assume first that $b = \sum_{t \in F} \xi_t \delta_t \in A \rtimes_{\mathrm{alg}} S$. In the last part of the proof of Proposition 4.10, we have defined an essential ideal J in A such that $EL(b) \in \mathcal{M}(J)$ and $\pi''(E(b)) = \bar{\pi}(EL(b))$ for all $\pi \in \widehat{J} \subseteq \widehat{A}$, where $\bar{\pi}$ denotes the unique extension of π to $\mathcal{M}(J)$. The map $\pi \mapsto \bar{\pi}$ identifies $\widehat{J}$ with an open subset in $\widehat{\mathcal{M}(J)}$, and the function $\omega \to \|\omega(EL(b))\|$ is lower semicontinuous on $\widehat{\mathcal{M}(J)}$ by [17, Proposition 3.3.2]. Hence $\nu_{E(b)}$ is lower semicontinuous on $\widehat{J}$,

which is a dense open subset in $\widehat{A}$. Now let $b \in A \rtimes S$. Then there is a sequence $(b_n)_{n \in \mathbb{N}}$ in $A \rtimes_{\mathrm{alg}} S$ with $\lim b_n = b$. Since E is contractive, the sequence of functions $\nu_{E(b_n)}$ converges uniformly towards $\nu_{E(b)}$. For each $n \in \mathbb{N}$, there is an essential ideal J_n as above such that ν_{b_n} is lower semicontinuous on $\widehat{J_n}$. The intersection $Y := \bigcap \widehat{J_n}$ is comeagre. If $\pi \in Y$, then the functions $\nu_{E(b_n)}$ are lower semicontinuous at π. Since they converge uniformly to $\nu_{E(b)}$, the latter is also lower semicontinuous at π.

The statement (1) implies (2) because $\widehat{A}$ is a Baire space (see [17, Proposition 3.4.13]). And (2) clearly implies (3). To complete the proof, we show that not (1) implies not (3). Let Y be the comeagre subset of $\widehat{A}$ defined above. If $\{\pi \in \widehat{A} : \nu_{E(b)}(\pi) \neq 0\}$ is not meagre, then it cannot be contained in the complement of Y. So there is $\pi \in Y$ with $\nu_{E(b)}(\pi) \neq 0$. Since $\nu_{E(b)}$ is lower semicontinuous at π, it follows that $\nu_{E(b)}(\omega) > \nu_{E(b)}(\pi)/2 > 0$ for all ω in some neighbourhood of π. $\qquad\square$

COROLLARY 4.16. *An element $b \in A \rtimes_{\mathrm{r}} S$ belongs to J_{sing} if and only if the subset $\{\pi \in \widehat{A} : \nu_{E_{\mathrm{r}}(b^*b)}(\pi) \neq 0\}$ is meagre in $\widehat{A}$, if and only if this subset has empty interior, if and only if $\{\pi \in \widehat{A} : \nu_{E_{\mathrm{r}}(b^*b)}(\pi) > \varepsilon\}$ has empty interior in $\widehat{A}$ for all $\varepsilon > 0$.*

Proof. Combine Corollary 4.12 and Proposition 4.15. $\qquad\square$

COROLLARY 4.17. *$A \rtimes_{\mathrm{ess}} S = A \rtimes_{\mathrm{r}} S$ if and only if for every $b \in (A \rtimes_{\mathrm{r}} S)^+ \backslash \{0\}$ there is $\varepsilon > 0$ such that $\{\pi \in \widehat{A} : \nu_{E_r(b)}(\pi) > \varepsilon\}$ has non-empty interior, if and only if for every $b \in (A \rtimes_{\mathrm{r}} S)^+ \backslash \{0\}$ the set $\{\pi \in \widehat{A} : \nu_{E_r(b)}(\pi) \neq 0\}$ is not meagre.*

COROLLARY 4.18. *If there is $\varepsilon > 0$ with $\|\nu_{E(b)}\|_{\mathrm{ess}} \geqslant \varepsilon \cdot \|\nu_{E(b)}\|_\infty$ for all positive elements $b \in (A \rtimes_{\mathrm{alg}} S)^+$, then $A \rtimes_{\mathrm{ess}} S = A \rtimes_{\mathrm{r}} S$.*

Proof. By Proposition 4.10, the assumption implies $\|EL(b^*b)\| \geqslant \varepsilon \|E(b^*b)\|$ for all $b \in A \rtimes_{\mathrm{alg}} S$. This inequality extends by continuity to all $b \in A \rtimes S$ and then implies $\mathcal{LN}_{EL} = \mathcal{LN}_E$. Since both E and EL are symmetric, the latter is equivalent to $\mathcal{N}_{EL} = \mathcal{N}_E$ and then to $A \rtimes_{\mathrm{ess}} S = A \rtimes_{\mathrm{r}} S$. $\qquad\square$

The condition in Corollary 4.18 has the advantage to involve only elements of $A \rtimes_{\mathrm{alg}} S$, making it more checkable. It is unclear, however, whether it is necessary.

Remark 4.19. The reader may readily rephrase the above results to characterise when $\ker EL = \ker E$. Namely, this is equivalent to conditions as above for all elements in $A \rtimes_{\mathrm{r}} S$, not only in the positive cone $(A \rtimes_{\mathrm{r}} S)^+$.

4.4 The local multiplier algebra and the injective hull

Let $I(A)$ be the injective hull of A (see [29]). The canonical embedding $A \hookrightarrow I(A)$ factors through the inclusion $A \subseteq \mathcal{M}_{\mathrm{loc}}(A)$ and an embedding $\iota \colon \mathcal{M}_{\mathrm{loc}}(A) \hookrightarrow I(A)$ by [26, Theorem 1]. And this is an isomorphism if A is

commutative. Hence every $\mathcal{M}_{\mathrm{loc}}$-expectation is a pseudo-expectation, and both the kernel and the ideal $\mathcal{N}_E$ do not change when we view an $\mathcal{M}_{\mathrm{loc}}$-expectation as a pseudo-expectation. The two notions are exactly the same if A is commutative.

Let $A = \mathrm{C}_0(X)$ be commutative. Let $\mathfrak{B}(X) \supseteq A$ be the C*-algebra of all bounded Borel functions on X. The subset

$$\mathfrak{M}(X) := \{f \in \mathfrak{B}(X) \colon f \text{ vanishes on a comeagre set}\}$$

is an ideal in $\mathfrak{B}(X)$ with $\mathrm{C}_0(X) \cap \mathfrak{M}(X) = 0$. So $\mathrm{C}_0(X)$ embeds into $\mathfrak{B}(X)/\mathfrak{M}(X)$. Gonshor has identified $I(\mathrm{C}_0(X))$ with the algebra $\mathfrak{B}(X)/\mathfrak{M}(X)$ (see [28, Theorem 1]). In fact, this follows from a much earlier result of Dixmier [16]. So

$$\mathcal{M}_{\mathrm{loc}}(\mathrm{C}_0(X)) \cong \mathfrak{B}(X)/\mathfrak{M}(X) \cong I(\mathrm{C}_0(X))$$

for any locally compact Hausdorff space X. For Gonshor, injectivity means that if $A \to B$ is an injective *-homomorphism to another commutative C*-algebra B such that A detects ideals in B, then there is an injective *-homomorphism $B \hookrightarrow \mathcal{M}_{\mathrm{loc}}(A)$ that respects the inclusions of A. Roughly speaking, $\mathcal{M}_{\mathrm{loc}}(A)$ is the largest commutative C*-algebra in which A detects ideals.

If A is simple, then $\mathcal{M}_{\mathrm{loc}}(A) = \mathcal{M}(A)$. In particular, if A is simple and unital, then $\mathcal{M}_{\mathrm{loc}}(A) = A$. A simple unital C*-algebra need not be injective; an example is the Calkin algebra, see [29]. So $I(A)$ differs from $\mathcal{M}_{\mathrm{loc}}(A)$ in general.

5 Aperiodic inclusions and generalised expectations

In this section, we fix a general C*-inclusion $A \subseteq B$. More generally, we often treat an injective *-homomorphism $A \hookrightarrow B$ as if it were an inclusion. Recall that $\mathbb{I}(B)$ denotes the lattice of (closed, two-sided) ideals in B. We say that A *detects ideals in* B if $J \cap A \neq 0$ for any ideal $J \in \mathbb{I}(B)$ with $J \neq 0$. This is a fundamental property in the study of the ideal structure of reduced crossed products (see [4, 43, 44, 58]). The usual assumptions that guarantee it imply a stronger property, namely, that the full crossed product has a unique quotient in which the coefficient algebra detects ideas. In this section, we study this generalised intersection property. We introduce the concept of aperiodicity for the C*-inclusion $A \subseteq B$, which implies the generalised intersection property. If, in addition, $E\colon B \to \tilde{A}$ is a generalised expectation, then we find a criterion when A detects ideals in $B/\mathcal{N}_E$: this happens if the conditional expectation is supportive. We show that any $\mathcal{M}_{\mathrm{loc}}$-expectation has this property. This gives general simplicity and pure infiniteness criteria for B when E is almost faithful.

5.1 Generalised intersection property and hidden ideal

Before we begin with the study of aperiodicity, we prove some elementary results about detection of ideals in quotients of B. Let

$$\mathbb{I}_0(B) := \{J \in \mathbb{I}(B) : J \cap A = 0\}.$$

So A detects ideals in B if and only if $\mathbb{I}_0(B) = \{0\}$.

LEMMA 5.1. *Let $J \in \mathbb{I}(B)$. The composite map $A \to B \to B/J$ is injective if and only if $J \in \mathbb{I}_0(B)$. If this is the case, then the inclusion $A \hookrightarrow B/J$ detects ideals if and only if J is a maximal element of $\mathbb{I}_0(B)$.*

Proof. The first claim holds because the kernel of the composite map $A \to B/J$ is $A \cap J$. To prove the second claim, we use that any ideal in B/J is the image of an ideal $K \in \mathbb{I}(B)$ with $J \subseteq K$. And $A \cap K = 0$ is the preimage in A of the ideal in B/J corresponding to K. So A does not detect ideals in B/J if and only if there is $K \in \mathbb{I}_0(B)$ with $J \subsetneq K$. $\square$

Example 5.2. Consider the unital inclusion $\mathbb{C} \subseteq \mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}(\mathbb{T})$. An ideal $J \in \mathbb{I}(\mathrm{C}(\mathbb{T}))$ satisfies $J \cap \mathbb{C} = 0$ if and only if it is proper. Thus $\mathbb{I}_0(\mathrm{C}(\mathbb{T}))$ has many different maximal elements, namely, all the maximal ideals $\mathrm{C}_0(\mathbb{T}\backslash\{z\})$ for $z \in \mathbb{T}$. The resulting quotients are all isomorphic to $\mathbb{C}$, in which $\mathbb{C}$ detects ideals. There is, however, no canonical way to choose one of these quotients.

LEMMA 5.3. *We have $0 \in \mathbb{I}_0(B)$. If $J_1, J_2 \in \mathbb{I}(B)$ satisfy $J_1 \subseteq J_2$ and $J_2 \in \mathbb{I}_0(B)$, then $J_1 \in \mathbb{I}_0(B)$. If $X \subseteq \mathbb{I}_0(B)$ with the partial order $\subseteq$ is directed, then the supremum of X in $\mathbb{I}(B)$ belongs to $\mathbb{I}_0(B)$. Any element of $\mathbb{I}_0(B)$ is contained in a maximal element of $\mathbb{I}_0(B)$.*

Proof. The claim about subideals and $0 \in \mathbb{I}_0(B)$ are obvious. Let $X \subseteq \mathbb{I}_0(B)$ be directed. Its supremum in $\mathbb{I}(B)$ is the closure K of the union $\bigcup_{J \in X} J$, which is equal to the closed linear span because X is directed. If $a \in A$ and $J \in X$, then the distance between a and J is $\|a\|$ because $A \hookrightarrow B/J$ is injective. Hence the distance between a and K is $\|a\|$. So $K \in \mathbb{I}_0(B)$. It follows that any chain of ideals in $\mathbb{I}_0(B)$ has a supremum in $\mathbb{I}_0(B)$. So the set of elements in $\mathbb{I}_0(B)$ containing a given $J \in \mathbb{I}_0(B)$ has a maximal element by Zorn's Lemma. This element remains maximal in $\mathbb{I}_0(B)$. $\square$

PROPOSITION 5.4. *The subset $\mathbb{I}_0(B)$ has a unique maximal element if and only if $J_1 + J_2 \in \mathbb{I}_0(B)$ for all $J_1, J_2 \in \mathbb{I}_0(B)$. If $\mathcal{N}$ is this unique maximal element, then $\mathbb{I}_0(B) = \{J \in \mathbb{I}(B) : J \subseteq \mathcal{N}\}$.*

Proof. By Lemma 5.3, $J_1 + J_2 \in \mathbb{I}_0(B)$ holds for all $J_1, J_2 \in \mathbb{I}_0(B)$ once it holds for all maximal elements of $\mathbb{I}_0(B)$. If $J_1 \neq J_2$ for two maximal elements of $\mathbb{I}_0(B)$, then $J_1 + J_2 \supsetneq J_1, J_2$ and hence $J_1 + J_2 \notin \mathbb{I}_0(B)$ because otherwise J_1, J_2 would not be maximal. Hence $J_1 + J_2 \in \mathbb{I}_0(B)$ for all $J_1, J_2 \in \mathbb{I}_0(B)$ if and only if $\mathbb{I}_0(B)$ has a unique maximal element. Since $\mathbb{I}(B)$ is a complete lattice, a subset of $\mathbb{I}(B)$ is of the form $\{J \in \mathbb{I}(B) : J \subseteq \mathcal{N}\}$ for some $\mathcal{N} \in \mathbb{I}(B)$ if and only if it has $\mathcal{N}$ as a unique maximal element. $\square$

DEFINITION 5.5. The inclusion $A \subseteq B$ has the *generalised intersection property* if there is a unique maximal ideal $\mathcal{N} \in \mathbb{I}(B)$ with $\mathcal{N} \cap A = 0$. Then we call $\mathcal{N}$ the *hidden ideal*, and we define $B_{\text{ess}} := B/\mathcal{N}$.

Proposition 5.4 shows that the hidden ideal exists if and only if the sum of two ideals in $\mathbb{I}_0(B)$ remains in $\mathbb{I}_0(B)$, if and only if $\mathbb{I}_0(B) = \{J \in \mathbb{I}(B) : J \subseteq \mathcal{N}\}$ for some $\mathcal{N} \in \mathbb{I}(B)$, and then $\mathcal{N}$ is the hidden ideal.

DEFINITION 5.6. A C*-subalgebra $A \subseteq B$ is called B-*minimal* if $\overline{BIB} = B$ for all $0 \neq I \in \mathbb{I}(A)$.

In the following proposition, and in the whole paper, we assume $A \neq 0$.

PROPOSITION 5.7. *A C*-inclusion $A \subseteq B$ has the generalised intersection property if and only if there is a unique quotient $B/\mathcal{N}$ of B such that $A \cap \mathcal{N} = 0$ and the image of A detects ideals in $B/\mathcal{N}$. Then $\mathcal{N}$ is the hidden ideal and $B_{\text{ess}} = B/\mathcal{N}$.*
If the C-inclusion $A \subseteq B$ has the generalised intersection property and A is B-minimal, then B_{ess} is the unique simple quotient of B for which the quotient map is injective on A. Moreover, B is simple if and only if A detects ideals in B and A is B-minimal.*

Proof. The first part of the assertion follows from Lemma 5.1 and Proposition 5.4. Assume that the C*-inclusion $A \subseteq B$ has the generalised intersection property and that A is B-minimal. Since A detects ideals in $B_{\text{ess}} = B/\mathcal{N}$, every non-zero ideal in B_{ess} is the image of an ideal $J \in \mathbb{I}(B)$, with $\mathcal{N} \subseteq J$ and $J \cap A \neq 0$. Then $B = B(J \cap A)B \subseteq J$ by minimality. So the ideal in question is B_{ess}. That is, B_{ess} is simple. If J is any ideal in $\mathbb{I}_0(B)$ such that B/J is simple, then A detects ideals in B/J and hence $B/J = B_{\text{ess}}$ by the first part. As a result, if A detects ideals in B and A is B-minimal, then $B = B_{\text{ess}}$ is simple. Conversely, if A does not detect ideals in B, then there is an ideal $J \subseteq B$ with $J \neq 0$ and $J \cap A = 0$. It cannot be 0 or B. So B is not simple. If A is not B-minimal, then there is $0 \neq I \in \mathbb{I}(A)$ with $\overline{BIB} \neq B$. Then $\overline{BIB} \in \mathbb{I}(B)$ is not B and not 0 because it contains I. So B is not simple. $\square$

PROPOSITION 5.8. *Let $A \subseteq B$ be a C*-inclusion with a generalised expectation $E : B \to \tilde{A} \supseteq A$. The following are equivalent:*

(1) $\mathcal{N}_E$ *is the hidden ideal for the inclusion $A \subseteq B$;*

(2) $B/\mathcal{N}_E$ *is the unique quotient of B for which the induced map $A \to B/\mathcal{N}_E$ is injective and detects ideals;*

(3) *if $J \in \mathbb{I}(B)$ satisfies $J \cap A = 0$, then $J \subseteq \ker E$;*

(4) *if $J \in \mathbb{I}(B)$ satisfies $J \cap A = 0$, then $J \subseteq \mathcal{N}_E$;*

(5) *for every C*-algebra C and *-homomorphism $\pi : B \to C$ that is injective on A, there is a *-homomorphism $\varphi : \pi(B) \to B/\mathcal{N}_E$ with $\varphi \circ \pi = \Lambda : B \to B/\mathcal{N}_E$;*

(6) *for every C*-algebra C and *-homomorphism $\pi\colon B \to C$ that is injective on A, there is a generalised expectation $E_\pi\colon \pi(B) \to \tilde{A}$ with $E_\pi \circ \pi = E$.*

In particular, A detects ideals in B if and only if the statements above hold and E is almost faithful.

Proof. Statements (3) and (4) are equivalent by the definition of $\mathcal{N}_E$. Since $\mathcal{N}_E \cap A = 0$, (4) is equivalent to (1). And (1) and (2) are equivalent by Proposition 5.7. Any ideal J in B is the kernel of a *-homomorphism $B \to B/J$. And a *-homomorphism π of B is faithful on A if and only if its kernel $J := \ker \pi$ satisfies $J \cap A = 0$. Hence (4)–(6) are equivalent by Lemma 3.11.

If A detects ideals in B, then $\mathcal{N}_E = 0$ is the hidden ideal and E is almost faithful. Conversely, if $\mathcal{N}_E$ is the hidden ideal and E is almost faithful, then $\mathcal{N}_E = 0$ and A detects ideals in B by (2). $\qquad\square$

5.2 Aperiodic inclusions

Kishimoto's condition for automorphisms was extended to bimodules in [43] in order to generalise the known criteria for detection of ideals in reduced crossed products for group actions to Fell bundles over groups. Here we rename Kishimoto's condition, speaking more briefly of *aperiodicity*.

Let $\mathbb{H}(A)$ denote the set of *non-zero*, hereditary C*-subalgebras of A. Let A^+ be the cone of positive elements in A.

DEFINITION 5.9 ([43]). Let X be a normed A-bimodule. We say that $x \in X$ satisfies *Kishimoto's condition* if, for all $D \in \mathbb{H}(A)$ and $\varepsilon > 0$, there is $a \in D^+$ with $\|a\| = 1$ and $\|axa\| < \varepsilon$. We call X *aperiodic* if Kishimoto's condition holds for all $x \in X$.

LEMMA 5.10. *Consider the C*-algebra A as an A-bimodule. No non-zero positive element of A satisfies Kishimoto's condition.*

Proof. Given $b \in A^+$ with $\|b\| = 1$, [43, Lemma 2.9] provides a hereditary subalgebra $D_0 \subseteq A$ such that $\|xbx\| \geqslant \|x^2\| - \|xbx - x^2\| > (1 - \varepsilon)\|x\|^2$ for all $x \in D_0^+$. $\qquad\square$

LEMMA 5.11 ([43, Lemma 4.2]). *The subset of elements in a normed A-bimodule that satisfy Kishimoto's condition is a closed vector subspace.*

LEMMA 5.12. *Subbimodules, quotient bimodules, extensions, finite direct sums, and inductive limits of aperiodic normed A-bimodules remain aperiodic. If $f\colon X \to Y$ is a bounded A-bimodule homomorphism with dense range and X is aperiodic, then so is Y. If $D \in \mathbb{H}(A)$, then an aperiodic A-bimodule is also aperiodic as a D-bimodule. If $J \in \mathbb{I}(A)$ is an essential ideal and X an A-bimodule, then JXJ is aperiodic as a J-bimodule if and only if X is aperiodic as an A-bimodule.*

Proof. The claims about subbimodules and about aperiodicity as a D-bimodule are trivial. If $f\colon X \to Y$ is a bounded A-bimodule homomorphism and $x \in X$, then $f(x)$ inherits Kishimoto's condition from x. Hence the second statement follows from Lemma 5.11. This implies the claim about quotient bimodules. The claim about extensions is proved as in the proof of [43, Lemma 4.2]. Namely, let $M_1 \rightarrowtail M_2 \twoheadrightarrow M_3$ be an extension of A-bimodules such that M_1 and M_3 are aperiodic. Let $x \in M_2$, $\varepsilon > 0$, and $D \in \mathbb{H}(A)$. We may assume without loss of generality that $\|x\| = 1$. Since M_3 is aperiodic, there is $a_0 \in D^+$ with $\|a_0\| = 1$ and $\|a_0 x a_0 + M_1\|_{M_3} < \varepsilon/2$. Hence $a_0 x a_0 = x_1 + x_2$ with $x_1 \in M_1$ and $\|x_2\| < \varepsilon$. By [43, Lemma 2.9], there is $D_0 \in \mathbb{H}(A)$ such that $D_0 \subseteq D$ and $\|a - a \cdot a_0\| \leqslant \varepsilon\|a\|$ for all $a \in D_0$. Kishimoto's condition for x_1 gives $a \in D_0^+ \subseteq D^+$ with $\|a\| = 1$ and $\|a x_1 a\| < \varepsilon$. Then $\|a x a\| < 4\varepsilon$. Thus M_2 is aperiodic.

A direct sum of two aperiodic normed A-bimodules is also an extension, hence inherits aperiodicity. By induction, this remains true for direct sums of finitely many aperiodic normed A-bimodules; here the norm should be one that defines the product topology. The claim about inductive limits follows from Lemma 5.11.

Now let $J \in \mathbb{I}(A)$ be an essential ideal and X an A-bimodule. The claims in the lemma already proven show that JXJ is aperiodic as a J-bimodule if X is aperiodic as an A-bimodule. Conversely, assume JXJ to be aperiodic as a J-bimodule. The Cohen–Hewitt Factorisation Theorem shows that JX, XJ and JXJ are closed J-subbimodules in X. There are extensions $JXJ \rightarrowtail JX \twoheadrightarrow JX/JXJ$ and $JX \rightarrowtail X \twoheadrightarrow X/JX$. The quotients in both are aperiodic as J-bimodules because they satisfy $xa = 0$ or $ax = 0$ for all $a \in J$, respectively. Hence the claim about extensions shows that X is also aperiodic as a J-bimodule. Let $D \in \mathbb{H}(A)$, $x \in X$, and $\varepsilon > 0$. Since J is essential, the intersection $D \cap J$ is still non-zero. It is a hereditary C*-subalgebra in J, and Kishimoto's condition for x gives $a \in (D \cap J)^+$ with $\|a\| = 1$ and $\|a x a\| < \varepsilon$. This witnesses that X is aperiodic as an A-bimodule. $\qquad\square$

Remark 5.13. An infinite direct sum of aperiodic normed bimodules inherits aperiodicity when it is given a norm that defines the product topology on each finite sub-sum. This follows from Lemma 5.12 by viewing it as an inductive limit of finite direct sums.

For any C*-inclusion $A \subseteq B$, both A and B are naturally normed A-bimodules. So is the quotient Banach space B/A with the quotient norm.

Definition 5.14. A C*-inclusion $A \subseteq B$ is *aperiodic* if the Banach A-bimodule B/A is aperiodic.

Proposition 5.15. *If $A \subseteq B$ is aperiodic and $A \subseteq C \subseteq B$, then the inclusion $A \subseteq C$ is aperiodic. If $A \subseteq B$ is aperiodic and $J \in \mathbb{I}(B)$ satisfies $J \cap A = 0$, then the induced inclusion $A \hookrightarrow B/J$ is aperiodic. Let $I \in \mathbb{I}(A)$. If $A \subseteq B$ is aperiodic, then $I \subseteq IBI$ is aperiodic; conversely, $A \subseteq B$ is aperiodic if $I \subseteq IBI$ is aperiodic and I is essential.*

Proof. This follows from Lemma 5.12 because C/A and IBI/I are isometrically isomorphic to A-subbimodules of B/A and $(B/J)/A \cong B/(J + A)$ is isometrically isomorphic to a quotient bimodule of B/A. $\qquad \square$

PROPOSITION 5.16. *Let $A \subseteq B$ be aperiodic and $J \in \mathbb{I}(B)$. Then $J \cap A = 0$ if and only if J is an aperiodic A-bimodule.*

Proof. First assume $J \cap A \neq 0$. Then there is $b \in J \cap A$ with $b \neq 0$. Then b^*b is a non-zero positive element of $J \cap A$. By Lemma 5.10, it does not satisfy Kishimoto's condition. Then J is not aperiodic. Another proof goes as follows. The intersection $J \cap A$ is an A-subbimodule in J and an ideal in A. Lemma 5.10 shows that $J \cap A$ is not aperiodic as a $J \cap A$-bimodule. Then Lemma 5.12 implies that J is not aperiodic as an A-bimodule.

Now assume that $J \cap A = 0$. Then the composite *-homomorphism $A \hookrightarrow B \twoheadrightarrow B/J$ is injective, hence isometric. So its image is closed. Thus the map $A \oplus J \to B$, $(a, x) \mapsto a + x$, is a continuous bijection onto a closed subspace of B. It follows that it is a topological isomorphism. Then the injective map $J \hookrightarrow B/A$ is also a topological isomorphism onto its image. Since B/A is aperiodic by assumption, so is J by Lemma 5.12. $\qquad \square$

THEOREM 5.17. *Every aperiodic inclusion $A \subseteq B$ has the generalised intersection property. The hidden ideal is the largest ideal that is aperiodic as an A-bimodule.*

Proof. Let $\mathbb{I}_a(B)$ be the set of all ideals in B that are aperiodic as an A-bimodule. Certainly, $0 \in \mathbb{I}_a(B)$. Let $J_1, J_2 \in \mathbb{I}_a(B)$. Then there is a Banach A-bimodule extension $J_1 \rightarrowtail J_1 + J_2 \twoheadrightarrow J_2/(J_1 \cap J_2)$. So Lemma 5.12 implies $J_1 + J_2 \in \mathbb{I}_a(B)$. Hence $\mathbb{I}_a(B) \subseteq \mathbb{I}(B)$ is closed under finite joins. Lemma 5.12 implies that $\mathbb{I}_a(B)$ is also closed under increasing unions. Hence it is closed under arbitrary joins. Let $\mathcal{N}$ be the join of all ideals in $\mathbb{I}_a(B)$. Then $J \in \mathbb{I}(B)$ satisfies $J \subseteq \mathcal{N}$ if and only if $J \in \mathbb{I}_a(B)$. Proposition 5.16 says that $\mathbb{I}_a(B) = \mathbb{I}_0(B)$. So $\mathcal{N}$ is the hidden ideal. $\qquad \square$

Remark 5.18. Let $A \subseteq B$ be a C*-inclusion that need not be aperiodic. The proof of Theorem 5.17 still gives $\mathcal{N} \in \mathbb{I}(B)$ such that $J \in \mathbb{I}(B)$ is aperiodic as an A-bimodule if and only if $J \subseteq \mathcal{N}$. That is, $\mathcal{N}$ is the largest aperiodic ideal in B. The proof of Proposition 5.16 still shows that $\mathcal{N} \cap A = 0$. So we get an induced inclusion $A \hookrightarrow B/\mathcal{N}$. Lemma 5.12 implies that no non-zero ideal in $B/\mathcal{N}$ is aperiodic as an A-bimodule. But A need not detect ideals in $B/\mathcal{N}$. For instance, in Example 5.2, the largest aperiodic ideal is 0 because a unital $\mathbb{C}$-bimodule is never aperiodic.

5.3 Supportive generalised expectations

Now assume the inclusion $A \subseteq B$ to be aperiodic and let $E \colon B \to \tilde{A} \supseteq A$ be a generalised expectation. Then $A \subseteq B$ has the generalised intersection property by Theorem 5.17. The hidden ideal $\mathcal{N}$ contains $\mathcal{N}_E$ because $\mathcal{N}_E \cap A = 0$.

When is $\mathcal{N}$ equal to $\mathcal{N}_E$? We cannot expect this for all generalised expectations. For instance, for the trivial generalised expectation in Example 3.3 we always have $\mathcal{N}_E = 0$ independently of $\mathcal{N}$. More importantly, there are examples of aperiodic inclusions coming from non-Hausdorff groupoids where $\mathcal{N}_E \neq \mathcal{N}$ for the canonical weak conditional expectation $E\colon B \to A''$. We are going to identify an extra property of generalised conditional expectations that implies that the hidden ideal is $\mathcal{N}_E$. Even more, it implies that the positive elements in A support all positive elements in $B/\mathcal{N}_E$.

DEFINITION 5.19. A generalised expectation $E\colon B \to \tilde{A} \supseteq A$ is called *supportive* if, for any $b \in B^+$ with $E(b) \neq 0$, there are $\delta > 0$ and a hereditary C*-subalgebra $D \in \mathbb{H}(A)$ such that $\|xE(b)x\| \geqslant \delta$ for all $x \in D^+$ with $\|x\| = 1$.

Remark 5.20. By definition, E is supportive if and only if no non-zero element of $E(B^+)$ satisfies Kishimoto's condition. Then any aperiodic ideal in B is contained in $\ker E$ and hence in $\mathcal{N}_E$. If $A \subseteq B$ is aperiodic, then Proposition 5.16 implies that $\mathcal{N}_E$ is aperiodic because $\mathcal{N}_E \cap A = 0$; so the unique maximal aperiodic ideal is $\mathcal{N}_E$ if $A \subseteq B$ is aperiodic and E is supportive. This short argument justifies our definition of supportive conditional expectations. Theorem 5.28 will prove the stronger statement that A supports B.

Remark 5.21. Since the property of being supportive depends only on $E(B^+)$, a generalised expectation $E\colon B \to \tilde{A} \supseteq A$ is supportive if and only if the corresponding reduced generalised expectation $B/\mathcal{N}_E \to \tilde{A}$ is supportive.

PROPOSITION 5.22. *Any $\mathcal{M}_{\mathrm{loc}}$-expectation $E\colon B \to \mathcal{M}_{\mathrm{loc}}(A)$ and, in particular, any genuine conditional expectation $E\colon B \to A$ is supportive.*

Proof. Let $b \in B^+$ satisfy $E(b) \neq 0$. Then there is $\varepsilon > 0$ with $\|E(b)\| > \varepsilon$ and $(1-\varepsilon)^2 > 1/2$. By the definition of $\mathcal{M}_{\mathrm{loc}}(A)$ as an inductive limit, there are an essential ideal $I \subseteq A$ and $c \in \mathcal{M}(I) \subseteq \mathcal{M}_{\mathrm{loc}}(A)$ with $\|E(b) - c\| < \varepsilon/4$. Since $E(b) \geqslant 0$, we may assume without loss of generality that $c \geqslant 0$. Then $\|c\| > 3\varepsilon/4$. Hence there is $a \in I$ with $0 \leqslant a \leqslant 1$, $\|a\| = 1$, and $\|ac^{1/2}\| > (3\varepsilon/4)^{1/2}$. Let $d := (c^{1/2}a^2c^{1/2})^{1/2} \in I$. [43, Lemma 2.9] gives a hereditary C*-subalgebra $D \in \mathbb{H}(A)$ such that $\|xd - x\| < \varepsilon\|x\| \cdot \|d\|$ and $\|xd\| > (1-\varepsilon)\|x\| \cdot \|d\|$ for all $x \in D$. Let $x \in D^+$ satisfy $\|x\| = 1$. Then $xcx \geqslant xc^{1/2}a^2c^{1/2}x = xd(xd)^*$. Hence

$$\|xcx\| \geqslant \|xd\|^2 > (1-\varepsilon)^2\|d\|^2 > (1-\varepsilon)^2 \cdot 3\varepsilon/4 > 3\varepsilon/8.$$

Then

$$\|xE(b)x\| \geqslant \|xcx\| - \|x\|^2\|E(b) - c\| > 3\varepsilon/8 - \varepsilon/4 = \varepsilon/8.$$

Since this holds for all $x \in D^+$ with $\|x\| = 1$, E is supportive. $\square$

LEMMA 5.23. *A generalised expectation $E\colon B \to \tilde{A} \supseteq A$ is supportive if, for any $b \in B^+$ with $E(b) \neq 0$, there is $a \in A^+\backslash\{0\}$ with $a \leqslant E(b)$.*

Proof. Choose $0 < \delta < \|a^{1/2}\|$ and let $\varepsilon = 1 - \delta/\|a^{1/2}\|$. The element $\|a\|^{-1/2}a^{1/2} \in A^+$ has norm 1, and [43, Lemma 2.9] gives a hereditary subalgebra $D \in \mathbb{H}(A)$ with $\|a^{1/2}x\| > (1-\varepsilon)\|a^{1/2}\|\|x\| = \delta\|x\|$ for all $x \in D$. Let

$x \in D$. Since $x^*E(b)x \geqslant x^*ax$, we may estimate $\|x^*E(b)x\| \geqslant \|x^*ax\| = \|a^{1/2}x\|^2 \geqslant \delta^2\|x\|^2$ as desired. $\qquad\square$

DEFINITION 5.24 ([41, Definition 2.39]). We say that A^+ *supports* B if, for every $b \in B^+\backslash\{0\}$, there is $a \in A^+\backslash\{0\}$ with $a \precsim b$ in the Cuntz preorder (see [14]); that is, for every $\varepsilon > 0$, there is $x \in B$ with $\|a - x^*bx\| < \varepsilon$.

DEFINITION 5.25 ([46, Lemma 2.1]). An element $a \in B^+\backslash\{0\}$ is *infinite* in B if and only if there is $b \in B^+\backslash\{0\}$ such that for all $\varepsilon > 0$ there are $x, y \in aB$ with $\|x^*x - a\| < \varepsilon$, $\|y^*y - b\| < \varepsilon$ and $\|x^*y\| < \varepsilon$.

PROPOSITION 5.26. *If B is simple, then B is purely infinite if and only if $A^+ \subseteq B$ supports B and all elements of $A^+\backslash\{0\}$ are infinite in B.*

Proof. Since B is simple, every infinite element in B is properly infinite by [35, Proposition 3.14]. Hence by [35, Theorem 4.16], B is purely infinite if and only if all elements of $B^+\backslash\{0\}$ are infinite in B. If B is purely infinite, then elements of $A^+\backslash\{0\}$ are infinite in B, and A^+ supports B by [35, Definition 4.1] and the simplicity of B. Conversely, assume that A^+ supports B and that all elements of $A^+\backslash\{0\}$ are infinite in B. Let $b \in B^+\backslash\{0\}$. Then there is $a \in A^+\backslash\{0\}$ with $a \precsim b$. Since B is simple, $b \in \overline{BaB} = B$. This implies $b \precsim a$ by [35, Proposition 3.5]. Hence a and b are Cuntz equivalent. So b is infinite. $\qquad\square$

LEMMA 5.27. *If $\mathcal{N} \in \mathbb{I}_0(B)$ is such that for every $b \in B^+$ with $b \notin \mathcal{N}$, there is $a \in A^+\backslash\{0\}$ with $a \precsim b$, then $\mathcal{N}$ is the hidden ideal for $A \subseteq B$ and A^+ supports $B/\mathcal{N}$. In particular, if A^+ supports B, then A detects ideals in B.*

Proof. Let $J \in \mathbb{I}(B)$ satisfy $J \nsubseteq \mathcal{N}$. Then there is $b \in J^+\backslash\mathcal{N}$. By assumption, there is $a \in A^+\backslash\{0\}$ with $a \precsim b$. This implies $a \in BbB \subseteq J$, and so $J \cap A \neq \{0\}$. Therefore, if $J \cap A = \{0\}$ for $J \in \mathbb{I}(B)$, then $J \subseteq \mathcal{N}$. The converse also holds because $\mathcal{N} \cap A = 0$. So $\mathcal{N}$ is the hidden ideal. And $a \precsim b$ in B implies $a \precsim q(b)$, where $q\colon B \to B/\mathcal{N}$ is the quotient map. Hence A^+ supports $B/\mathcal{N}$. $\qquad\square$

THEOREM 5.28. *Let $A \subseteq B$ be an aperiodic C^*-inclusion. Let E be an $\mathcal{M}_{\mathrm{loc}}$-expectation or, more generally, a supportive generalised expectation $E\colon B \to \tilde{A} \supseteq A$. Then*

(1) *for every $b \in B^+$ with $b \notin \mathcal{N}_E$, there is $a \in A^+\backslash\{0\}$ with $a \precsim b$;*

(2) *A^+ supports $B/\mathcal{N}_E$;*

(3) *A detects ideals in $B/\mathcal{N}_E$;*

(4) *$\mathcal{N}_E$ is the hidden ideal for the inclusion $A \subseteq B$, and so all the equivalent statements in Proposition 5.8 hold;*

(5) *B is simple if and only if A is B-minimal and E is almost faithul;*

(6) *if B is simple, then B is purely infinite if and only if all elements of $A^+\backslash\{0\}$ are infinite in B.*

Proof. (1) implies (2)–(4) by Lemma 5.27. (4) implies (5) by Proposition 5.7, and (2) implies (6) by Proposition 5.26. So everything follows once we show (1). Moreover, $\mathcal{M}_{\mathrm{loc}}$-expectations are supportive by Proposition 5.22. So we may assume E to be a supportive generalised expectation.

Let $b \in B^+$ with $b \notin \mathcal{N}_E$. Then $b^{1/2} \notin \mathcal{N}_E$. Proposition 3.6 gives $x \in B^+$ with $E(x^*bx) \neq 0$. Since $x^*bx \precsim b$, we may replace b by x^*bx and assume without loss of generality that $E(b) \neq 0$. Since E is supportive, there are $\delta > 0$ and $D \in \mathbb{H}(A)$ such that $\|hE(b)h\| \geqslant \delta$ for all $h \in D^+$ with $\|h\| = 1$. Choose $\varepsilon := \delta/4$. Since B/A is aperiodic, there is $h \in D^+$ with $\|h\| = 1$ and $\|hbh\|_{B/A} < \varepsilon$. That is, there is $c \in A$ with $\|hbh - c\| < \varepsilon$. Since hbh is self-adjoint, even positive, we may replace c by its real part $(c^* + c)/2$. This makes c self-adjoint and does not increase $\|hbh - c\|$. Next, decompose c into its positive and negative parts, $c = c_+ - c_-$ with $c_\pm \geqslant 0$ and $c_+ \cdot c_- = c_- \cdot c_+ = 0$. We claim that $\|hbh - c_+\| < 2\varepsilon$. First, $\|hbh - c\| < \varepsilon$ implies $\varepsilon \geqslant hbh - c \geqslant -c = c_- - c_+$ because $hbh \geqslant 0$. Since $c_\pm$ are orthogonal, this implies $\varepsilon \geqslant \|c_-\| = \|c - c_+\|$. So $\|hbh - c_+\| \leqslant \|hbh - c\| + \|c - c_+\| < 2\varepsilon$. Now we estimate

$$\|c_+\| = \|E(c_+)\| \geqslant \|hE(b)h\| - \|E(hbh - c_+)\| > \delta - 2\varepsilon = 2\varepsilon.$$

Hence $(c_+ - \varepsilon)_+ \neq 0$. Let $a := (c_+ - \varepsilon)_+ \in A^+ \backslash \{0\}$. Since $\|hbh - c_+\| < 2\varepsilon$, [36, Lemma 2.2] gives a contraction $y \in B$ with $a = y^*hbhy$. Thus $a \precsim b$. $\qquad \square$

By Proposition 5.15, the reduced inclusion $A \hookrightarrow B/\mathcal{N}_E$ is aperiodic if $A \subseteq B$ is aperiodic. The converse is false:

Example 5.29. Embed $A = \mathbb{C}$ diagonally into $B = \mathbb{C} \oplus \mathbb{C}$ and define $E \colon B \to A$, $E(x, y) := x$. Then the reduced inclusion is an isomorphism $A \cong B/\mathcal{N}_E$ and hence aperiodic. But the inclusion $A \subseteq B$ is not aperiodic.

6   Aperiodicity for inverse semigroup actions

In this section, we characterise when the inclusion $A \subseteq A \rtimes S$ for an inverse semigroup action is aperiodic. In this case, $A \rtimes_{\mathrm{ess}} S$ is the unique quotient of $A \rtimes S$ in which A embeds and detects ideals. Even more, A^+ supports $A \rtimes_{\mathrm{ess}} S$. Using previous results in [43], we relate aperiodicity of an inverse semigroup action to topological freeness of the dual groupoid and pure outerness of the action. Following Archbold and Spielberg [4], we also show directly that a variant of topological freeness implies detection of ideals.

6.1   Aperiodic inverse semigroup actions

DEFINITION 6.1. An inverse semigroup action $\mathcal{E}$ is called *aperiodic* if the Hilbert A-bimodules $\mathcal{E}_t \cdot I_{1,t}^\perp$ are aperiodic for all $t \in S$, where $I_{1,t}^\perp$ is defined in (3.4).

Recall the inverse semigroup $\mathcal{S}(A, B)$ for a regular inclusion defined in Proposition 2.12.

Lemma 6.2. *Let $A \subseteq B$ be a regular C*-inclusion and let $\mathcal{S} \subseteq \mathcal{S}(A, B)$ be a subset with closed linear span B; for instance, $\mathcal{S} = \mathcal{S}(A, B)$. Let $(N \cap A)^{\perp}$ for $N \in \mathcal{S}(A, B)$ be the annihilator of the ideal $N \cap A$ in A. The inclusion $A \subseteq B$ is aperiodic if and only if the image of $N \cdot (N \cap A)^{\perp}$ in B/A is an aperiodic A-bimodule for all $N \in \mathcal{S}$.*

Proof. Since $\sum N$ is dense in B, the images of the A-submodules $N \subseteq B$ in B/A are linearly dense. By Lemma 5.11, B/A is aperiodic if and only if these images, equipped with the quotient norm from B/A, are all aperiodic. Fix $N \in \mathcal{S}$ and put $I := N \cap A$. Let $J := I + I^{\perp}$. This is an essential ideal in A and $N \cdot J = N \cdot I \oplus N \cdot I^{\perp} = N \cap A \oplus N \cdot I^{\perp}$. By Lemma 5.12, the image of N in B/A is an aperiodic A-bimodule if and only if the image of $J \cdot N \cdot J$ in B/A is an aperiodic J-bimodule, if and only if the image of $J \cdot N \cdot J$ in B/A is an aperiodic A-bimodule. And the same proof works with $N \cdot J$ instead of $J \cdot N \cdot J$. Finally, the subspace $N \cdot J$ has the same image in B/A as $N \cdot (N \cap A)^{\perp}$. $\square$

Proposition 6.3. *Let B be an S-graded C*-algebra with a grading $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$. Let $A := \mathcal{E}_1 \subseteq B$ and turn $\mathcal{E}$ into an S-action on A. If this action is aperiodic, then the inclusion $A \subseteq B$ is aperiodic. The converse holds if the grading is topological as in Definition 4.13.*

Proof. Suppose first that $\mathcal{E}$ is aperiodic. Any S-grading satisfies $\mathcal{E}_t \subseteq \mathcal{E}_u$ for $t \leq u$ in S. Hence the ideal $I_{1,t}$ used in Definition 6.1 is contained in $\mathcal{E}_t \cap A$ for all $t \in S$. So $\mathcal{E}_t \cdot (\mathcal{E}_t \cap A)^{\perp}$ is a subbimodule of $\mathcal{E}_t \cdot I_{1,t}^{\perp}$ and hence inherits aperiodicity from the latter by Lemma 5.12. Thus $A \subseteq B$ is aperiodic by Lemma 6.2.

Conversely, assume that $A \subseteq B$ is aperiodic. Let $t \in S$. Using that the grading is topological, we are going to prove that the seminorm on $\mathcal{E}_t I_{1,t}^{\perp}$ induced by the quotient norm on B/A is equal to the usual norm in B. Hence $\mathcal{E}_t I_{1,t}^{\perp}$ inherits aperiodicity from B/A by Lemma 5.12. It is clear that $\|x\|_{B/A} \leq \|x\|_B$ for all $x \in B$. It remains to prove the opposite inequality for $x \in \mathcal{E}_t I_{1,t}^{\perp}$. Since $\|x\|_B^2 = \|x^*x\|_A$, it follows that $\|x\|_B = \|x\|_{A \rtimes_{\mathrm{ess}} S}$. By Proposition 4.14, $\|x\|_{B/A} \geq \|x\|_{(A \rtimes_{\mathrm{ess}} S)/A}$. So we may assume without loss of generality that $B = A \rtimes_{\mathrm{ess}} S$. By definition of EL, $EL(a^*x) = 0$ for all $a \in A$. Hence $x \odot 1$ and $a \odot 1$ in $(A \rtimes_{\mathrm{alg}} S) \odot \mathcal{M}_{\mathrm{loc}}(A)$ are orthogonal. The following norms are computed in the Hilbert module completion $\mathcal{F}$ of $(A \rtimes_{\mathrm{alg}} S) \odot \mathcal{M}_{\mathrm{loc}}(A)$ or in $\mathcal{M}_{\mathrm{loc}}(A)$:

$$\|(x + a) \odot 1\|^2 \geq \|x \odot 1\|^2 = \|\langle x \odot 1 \mid x \odot 1 \rangle\| = \|EL(x^*x)\| = \|x^*x\| = \|x\|^2.$$

The first inequality follows because $x \odot 1$ and $a \odot 1$ are orthogonal, and $EL(x^*x) = x^*x$ because $x^*x \in A$. Hence the norm of the operator of left multiplication by $x + a$ on $\mathcal{F}$ is at least $\|x\|$ for all $a \in A$. This implies $\|x\|_{(A \rtimes_{\mathrm{ess}} S)/A} \geq \|x\|_B$ as desired. $\square$

Example 6.4. We are going to build a regular inclusion $A \subseteq B$ with two different wide gradings, such that one grading gives an aperiodic action and the other not. By Proposition 6.3, the inclusion $A \subseteq B$ is aperiodic although it can be

equipped with a wide grading which is not aperiodic. So the equivalence in the
second part of Proposition 6.3 fails for non-topological gradings, even if they
are wide.

Let B be the C*-algebra of all bounded functions $f \colon [0,1] \to \mathbb{C}$ such that f is
continuous at irrational numbers and, at rational numbers in $[0,1]$, still satisfies
$f(t) := \lim_{x \nearrow t} f(x)$ and that the limit from the right $\lim_{x \searrow t} f(x)$ exists. Let
$A := \mathrm{C}([0,1]) \subseteq B$. A character on B maps $f \colon [0,1] \to \mathbb{C}$ to the value $f(t)$
for some $t \in [0,1]$ or to the right limit $f(t^+) := \lim_{x \searrow t} f(x)$ for some $t \in \mathbb{Q}$.
The set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers is dense in the spectrum of B. This implies
that A detects ideals in B. The inclusion $A \subseteq B$ is unital, and it is regular by
Lemma 2.13 because B is commutative. If $u \in B$ is unitary, then $u \cdot A \subseteq B$
belongs to $\mathcal{S}(A, B)$ (see the proof of Lemma 2.13). These elements generate a
subgroup of $\mathcal{S}(A, B)$ that is isomorphic to the quotient group $\Gamma := \mathrm{U}(B)/\mathrm{U}(A)$.
Let $\Gamma_0 \subseteq \Gamma$ be the subgroup of all unitaries that are discontinuous at only
finitely many points of $[0,1]$. These separate the characters of B and hence
generate B as a C*-algebra by the Stone–Weierstraß Theorem. So B is graded
by the group Γ_0. And then it is also graded by the larger group Γ. If $u \in \Gamma_0$,
then $f \in A$ satisfies $f \cdot u \in A$ if and only if $f(t) = 0$ at all $t \in [0,1]$ where u
is not continuous. So $A \cap u \cdot A = \mathrm{C}_0([0,1]\backslash F)$ for a finite set $F \subseteq \mathbb{Q}$. In
order for this to be useful, the subgroup Γ_0 of $\mathcal{S}(A, B)$ must be enlarged to
the wide inverse subsemigroup that it generates. Or we may as well take
$\bar{\Gamma}_0 := \{u \cdot J : u \in \Gamma_0, \ J \lhd A\}$. Since $[0,1]\backslash F$ is dense in $[0,1]$, the ideals $I_{t,1}^\perp$ are
zero for all $t \in \bar{\Gamma}_0$. Hence the $\bar{\Gamma}_0$-action on A defined by the $\bar{\Gamma}_0$-grading of B is
aperiodic.

This action is topologically principal as well because its isotropy is trivial in
$[0,1]\backslash\mathbb{Q}$. At the same time, this action is trivial: any slice $u \cdot J$ in $\bar{\Gamma}_0$ is
isomorphic to J as an A-bimodule, and the multiplication in the Fell bundle
over $\bar{\Gamma}_0$ is also the usual multiplication in A. The Γ_0-grading on B is not
aperiodic, however, because for any $t \in \Gamma_0\backslash\{1\}$, there is no element in Γ_0 below
1 and t, so that $I_{1,t} = 0$. We thank Jonathan Taylor for pointing this out to
us.

Let $\bar{\Gamma} := \{u \cdot J : u \in \Gamma, \ J \lhd A\}$. Then B carries a wide $\bar{\Gamma}$-grading. There is a
unitary $u \in B$ that is discontinuous at all rational points in $[0,1]$. Then $u \cdot f$
for $f \in A$ is only continuous if $f = 0$. So $u \cdot A \cap A = 0$. Then $I_{A,u\cdot A} = 0$ and
aperiodicity of the $\bar{\Gamma}$-grading on B would ask for $u \cdot A$ to be aperiodic as an
A-bimodule. This contradicts Lemma 5.10 because $u \cdot A$ carries the standard
A-bimodule structure. Hence the $\bar{\Gamma}$-action on A defined by the $\bar{\Gamma}$-grading on B
is not aperiodic.

6.2   Properties of aperiodic inverse semigroup actions

The following theorem specialises Theorem 5.28 to aperiodic inverse semigroup
actions:

Theorem 6.5. *Let $\mathcal{E}$ be an aperiodic action of a unital inverse semigroup S on a C^*-algebra A. Let $EL\colon A \rtimes S \to \mathcal{M}_{\mathrm{loc}}(A)$ be the canonical $\mathcal{M}_{\mathrm{loc}}$-expectation.*

(1) *The ideal $\mathcal{N}_{EL}$ is the hidden ideal of the C^*-inclusion $A \subseteq A \rtimes S$.*

(2) *A^+ supports $A \rtimes_{\mathrm{ess}} S$.*

(3) *A detects ideals in $A \rtimes_{\mathrm{ess}} S$, and $A \rtimes_{\mathrm{ess}} S$ is the unique quotient of $A \rtimes S$ with this property.*

(4) *For any representation π of $(\mathcal{E}_t)_{t \in S}$ in a C^*-algebra B that is faithful on A, the image of the induced $*$-homomorphism $A \rtimes S \to B$ is an exotic crossed product, that is, $\pi(B)$ is topologically graded by $(\mathcal{E}_t)_{t \in S}$.*

Proof. Proposition 6.3 implies that the inclusion $A \subseteq A \rtimes S$ is aperiodic. Then (1)–(3) follow from Theorem 5.28. In the situation of (4), the theorem implies $\ker \pi \subseteq \mathcal{N}_{EL}$. And then (4) follows easily. $\qquad\square$

Theorem 6.6. *Let B be an S-graded C^*-algebra and turn the grading into an action $\mathcal{E}$ of S on A. Then B is simple if and only if A detects ideals in B and the action $\mathcal{E}$ is minimal. In particular, if the S-action on A is aperiodic and minimal, then $A \rtimes_{\mathrm{ess}} S$ is simple.*

Proof. It is shown in [44, Section 6.3] that an ideal in A is of the form $J \cap A$ for $J \in \mathbb{I}(B)$ if and only if it is invariant. Therefore, the minimality assumption here is equivalent to the B-minimality assumption in Proposition 5.7, and the first claim follows. The second claim follows because A detects ideals in $A \rtimes_{\mathrm{ess}} S$ for any aperiodic action (Theorem 6.5). $\qquad\square$

Corollary 6.7. *Let $\mathcal{E}$ be an aperiodic, minimal action of a unital inverse semigroup S on a C^*-algebra A. Then $A \rtimes_{\mathrm{ess}} S$ is simple and purely infinite if and only if every element in $A^+\backslash\{0\}$ is infinite in $A \rtimes_{\mathrm{ess}} S$.*

Proof. This follows from the previous theorem and Proposition 5.26. $\qquad\square$

Proposition 6.8. *Let $\mathcal{E}$ be an aperiodic action of a unital inverse semigroup S on a C^*-algebra A. Let $\|\cdot\|_{\min}$ be the infimum of all C^*-seminorms on $A \rtimes_{\mathrm{alg}} S$ that restrict to the given C^*-norm on A. This is a C^*-seminorm on $A \rtimes_{\mathrm{alg}} S$. The Hausdorff completion of $A \rtimes_{\mathrm{alg}} S$ in this seminorm is canonically isomorphic to $A \rtimes_{\mathrm{ess}} S$.*

Proof. Any C^*-seminorm on $A \rtimes_{\mathrm{alg}} S$ is of the form $\|\pi(x)\|$ for a representation x. And π is injective on A if and only if $\|\pi(x)\| = \|x\|$ for all $x \in A$. Hence Theorem 6.5.(4) says that the C^*-norm of $A \rtimes_{\mathrm{ess}} S$, restricted to $A \rtimes_{\mathrm{alg}} S$, is the minimal C^*-seminorm on $A \rtimes_{\mathrm{alg}} S$ that restricts to the given C^*-norm on A. $\qquad\square$

6.3   Aperiodicity, topological freeeness and pure outerness

Aperiodicity of single Hilbert bimodules is shown in [43] to be equivalent to several other properties, under some mild assumptions. This allows us to compare aperiodicity of inverse semigroup actions to the non-triviality conditions in Definition 2.20.

DEFINITION 6.9. A Hilbert A-bimodule $\mathcal{H}$ over a C*-algebra A is *purely outer* if there is no non-zero ideal $J \in \mathbb{I}(A)$ with $\mathcal{H} \cdot J \cong J$ as a Hilbert bimodule (see [43]). An action $\mathcal{E}$ of an inverse semigroup on a C*-algebra A is *purely outer* if the Hilbert A-bimodules $\mathcal{E}_t \cdot I_{1,t}^\perp$ are purely outer for all $t \in S$.

DEFINITION 6.10 ([43]). A Hilbert A-bimodule $\mathcal{H}$ over a C*-algebra A is *topologically non-trivial* if the set $\left\{ [\pi] \in \widehat{A} : \widehat{\mathcal{H}}[\pi] = [\pi] \right\}$ has empty interior. Here $\widehat{\mathcal{H}}[\pi] = [\pi]$ means that the irreducible representations $\mathcal{H} \otimes_A \pi$ and π are unitarily equivalent.

THEOREM 6.11 ([43, Theorem 8.1]). *Let A be a C*-algebra and let $\mathcal{H}$ be a Hilbert A-bimodule. Consider the following conditions:*

  (1) *$\mathcal{H}$ is aperiodic;*

  (2) *the partial homeomorphism $\widehat{\mathcal{H}}$ of $\widehat{A}$ is topologically non-trivial;*

  (3) *$\mathcal{H}$ is purely outer.*

Then (1) or (2) implies (3). If A contains a separable essential ideal, then (1) and (2) are equivalent. If A contains a simple, essential ideal, then (1) and (3) are equivalent. If A contains an essential ideal of Type I, then (1)–(3) are equivalent.

LEMMA 6.12 ([43, Proposition 9.7]). *Assume that A contains an essential ideal which is separable or whose spectrum is Hausdorff. Let $(\mathcal{H}_i)_{i \in I}$ be a countable family of Hilbert A-bimodules. The following are equivalent:*

  (1) *$\widehat{\mathcal{H}}_i$ is topologically non-trivial for every $i \in I$;*

  (2) *the union $\bigcup_{i \in I} \left\{ [\pi] \in \widehat{A} : \widehat{\mathcal{H}}_i[\pi] = [\pi] \right\}$ has empty interior.*

Proof. We only need to show that (1) implies (2). By [43, Corollary 6.9] we may assume that A is separable or that $\widehat{A}$ is Hausdorff (compare Corollary 2.26). If $\widehat{A}$ is Hausdorff, then the sets $\left\{ [\pi] \in \widehat{A} : \widehat{\mathcal{H}}_i[\pi] = [\pi] \right\}$ for $i \in I$ are closed in $\widehat{A}$. Hence (1) implies (2) because $\widehat{A}$ is a Baire space.

Now assume A to be separable. We will reduce our considerations to the case when the Hilbert bimodules $\mathcal{H}_i$ come from automorphisms of A. We may assume that A and $(\mathcal{H}_i)_{i \in I}$ are embedded into a C*-algebra B in such a way that all the structure of these objects is inherited from B. In addition, we may

assume that the C*-algebra B is generated by A and $(\mathcal{H}_i)_{i\in I}$. Let $\mathbb{F}$ be the free group on the set I. We define an $\mathbb{F}$-grading $(\mathcal{E}_t)_{t\in\mathbb{F}}$ on B with unit fibre A. Let

$$\mathcal{E}_t := \mathcal{H}_t \quad \text{if } t \in I, \quad \text{and} \quad \mathcal{E}_t := \mathcal{H}_t^* \quad \text{if } t \in I^{-1}.$$

If $t_1 \cdots t_n \in \mathbb{F}$ is a reduced word, that is, $t_k \in I \cup I^{-1}$ for $1 \leqslant k \leqslant n$ and $t_k \neq t_{k+1}^{-1}$ for $1 \leqslant k < n$, then

$$\mathcal{E}_{t_1 \cdots t_n} := \mathcal{E}_{t_1} \mathcal{E}_{t_2} \cdots \mathcal{E}_{t_n}.$$

One readily sees that $(\mathcal{E}_t)_{t\in\mathbb{F}}$ is indeed an $\mathbb{F}$-grading on B and hence a Fell bundle over $\mathbb{F}$. Let $\gamma\colon \mathbb{F} \to \operatorname{Aut}(C)$ be the Morita globalisation of $(\mathcal{E}_t)_{t\in\mathbb{F}}$ in [43, Proposition 7.1]. So C is a separable C*-algebra and for each $i \in I$, the Hilbert bimodules $(\mathcal{E}_{tit^{-1}})_{t\in\mathbb{F}}$ cover C_{γ_i} up to Morita equivalence. For each $t \in \mathbb{F}$, either $\mathcal{E}_{tit^{-1}} = \mathcal{E}_t \mathcal{H}_i \mathcal{E}_{t^{-1}}$ or $\mathcal{E}_{tit^{-1}} = \mathcal{E}_u \mathcal{H}_i \mathcal{E}_{u^{-1}}$ with $u = ti^{\pm 1}$. Hence $\mathcal{E}_{tit^{-1}}$ is topologically non-trivial if $\mathcal{H}_i$ is. Thus by [43, Proposition 6.8.(2)] γ_i is topologically non-trivial, that is, the sets $\left\{ [\pi] \in \widehat{C} : \widehat{C_{\gamma_i}}[\pi] = [\pi] \right\} = \left\{ [\pi] \in \widehat{C} : [\pi \circ \gamma_i] = [\pi] \right\}$ for $i \in I$ have empty interiors. As in the proof of [49, Proposition 4.4], one shows that the union $\bigcup_{i\in I} \left\{ [\pi] \in \widehat{C} : [\pi \circ \gamma_i] = [\pi] \right\}$ has empty interior. Since $\gamma\colon \mathbb{F} \to \operatorname{Aut}(C)$ is a Morita globalisation of $(\mathcal{E}_t)_{t\in\mathbb{F}}$, the partial action $(\widehat{\mathcal{E}}_t)_{t\in\mathbb{F}}$ may be identified with the restriction of $\widehat{\gamma}$ to an open subset. Hence the union $\bigcup_{i\in I} \left\{ [\pi] \in \widehat{A} : \widehat{\mathcal{H}}_i[\pi] = [\pi] \right\}$ has empty interior. $\qquad\square$

THEOREM 6.13. *Let A be a C*-algebra, S a unital inverse semigroup, and $\mathcal{E}$ an S-action on A by Hilbert bimodules. Consider the following conditions:*

(1) *the action $\mathcal{E}$ is aperiodic;*

(2) *for each $t \in S$ and $0 \neq K \in \mathbb{I}(I_{1,t}^{\perp})$ there is $[\pi] \in \widehat{K}$ with $\widehat{\mathcal{E}}_t[\pi] \neq [\pi]$;*

(3) *the dual groupoid $\widehat{A} \rtimes S$ is topologically free;*

(4) *the dual groupoid $\widehat{A} \rtimes S$ is AS topologically free;*

(5) *the dual groupoid $\widehat{A} \rtimes S$ is topologically principal;*

(6) *the action $\mathcal{E}$ is purely outer.*

If A contains an essential ideal which is separable or of Type I, then (1)–(4) are equivalent. If, in addition, S is countable, then (1)–(5) are equivalent. In general, each of the conditions (1)–(5) implies (6). Conversely, (6) implies (1) if A contains an essential ideal which is of Type I or simple.

Proof. By definition, the action $\mathcal{E}$ has one of the properties in Definition 2.20 if and only if the dual groupoid $\widehat{A} \rtimes S$ has that property. The groupoid $\widehat{A} \rtimes S$

is the union of the bisections $\widehat{\mathcal{E}}_t$. And $\widehat{\mathcal{E}}_t \cap \widehat{A} = \widehat{I_{1,t}}$. The restriction of $\widehat{\mathcal{E}}_t$ to $\widehat{I_{1,t}^\perp}$ is the complement of the closure of $\widehat{\mathcal{E}}_t \cap \widehat{A}$ in $\widehat{\mathcal{E}}_t$. So

$$\left\{ [\pi] \in \widehat{I_{1,t}^\perp} : \widehat{\mathcal{E}}_t[\pi] = [\pi] \right\} = \underline{\mathrm{Fix}}(\widehat{\mathcal{E}}_t) \tag{6.1}$$

in the notation of Lemma 2.23. Condition (2) for $t \in S$ makes precise what it means for the partial homeomorphism of $\widehat{A}$ induced by $\mathcal{E}_t \cdot I_{1,t}^\perp$ to be topologically non-trivial. So (2) and (3) are equivalent by Lemma 2.23. If A contains an essential ideal which is separable or of Type I, then (1) is equivalent to (2) by Theorem 6.11. By Lemma 6.12, condition (2) holds if and only if countable unions of the subsets $\underline{\mathrm{Fix}}(\widehat{\mathcal{E}}_t)$ have empty interior. Accordingly, (2) and (4) are equivalent by Lemma 2.23.(3). Also by Lemma 2.23, conditions (2) and (5) are equivalent if S is countable. The implications concerning condition (6) follow from Theorem 6.11. $\square$

Following Archbold and Spielberg [4], we prove that A detects ideals in $A \rtimes_{\mathrm{ess}} S$ for AS topologically free actions, without going through aperiodicity and thus without separability assumptions on A:

THEOREM 6.14. *Let A be a C*-algebra with an AS topologically free action $\mathcal{E}$ of a unital inverse semigroup S. Then $A \rtimes_{\mathrm{ess}} S$ is the unique quotient of $A \rtimes S$ in which A embeds and detects ideals.*

Proof. By Proposition 5.8 it suffices to show that, for any representation $\pi \colon A \rtimes S \to \mathbb{B}(H)$ which is injective on A, there is a map $E_\pi \colon \pi(A \rtimes S) \to \mathcal{M}_{\mathrm{loc}}(A)$ such that $E_\pi \circ \pi = EL \circ \pi$. This is equivalent to $\|EL(a)\| \leqslant \|\pi(a)\|$ for all $a \in A \rtimes S$. It suffices to check this on the dense *-subalgebra $A \rtimes_{\mathrm{alg}} S$. So take $a \in A \rtimes_{\mathrm{alg}} S$ and write it as $a = \sum_{t \in F} a_t \delta_t$ for a finite subset $F \subseteq S$ and $a_t \in \mathcal{E}_t$ for $t \in F$. Define $J_t := I_{1,t} \oplus I_{1,t}^\perp$ and $J := \bigcap_{t \in S} J_t$ as in the proof of Proposition 4.3. These are essential ideals in A, and $\|EL(a)\|$ is equal to the supremum of $\|EL(ax)\|$ for $x \in J$ with $\|x\| \leqslant 1$. So $\|EL(a)\| \leqslant \|\pi(a)\|$ follows if $\|EL(ax)\| \leqslant \|\pi(ax)\|$ holds for all $x \in J$. So we may assume without loss of generality that $a_t \in \mathcal{E}_t \cdot J$ for all $t \in F$.
Then we may further decompose $a_t = EL(a_t \delta_t) + a_t'$ with $EL(a_t \delta_t) \in I_{1,t}$ and $a_t' \in \mathcal{E}_t^\perp := \mathcal{E}_t \cdot I_{1,t}^\perp$. Thus $a = EL(a) + \sum_{t \in F} a_t' \delta_t$ with $EL(a) \in A$. To simplify, we may assume without loss of generality that $\|EL(a)\| = 1$. Let $0 < \varepsilon < 1$. The set $U := \{[\sigma] \in \widehat{A} : \|\sigma(EL(a))\| > 1 - \varepsilon\}$ is open and non-empty because $\|EL(a)\| = 1$. For each $t \in S$, Equation (6.1) and Lemma 2.23.(3) show that there is $[\sigma] \in U$ with $\widehat{\mathcal{E}_t^\perp}[\sigma] \neq [\sigma]$ for all $t \in F$. There is a representation $\nu \colon \pi(A \rtimes S) \to \mathbb{B}(H_\nu)$ that extends the irreducible representation σ, that is, there is a closed subspace $H_\sigma \subseteq H_\nu$ on which $\nu \circ \pi$ is unitarily equivalent to σ. For each $t \in F$, let P_t be the orthogonal projection from H_ν onto the closed subspace $\nu(\pi(\mathcal{E}_t^\perp))H_\sigma$. Let P_σ be the orthogonal projection onto H_σ. The subspaces $P_t H_\nu$ are invariant for $\nu(\pi(A))$, and $\nu \circ \pi \colon A \to \mathbb{B}(P_t H_\nu)$ is either zero or an irreducible representation whose equivalence class is $\widehat{\mathcal{E}_t^\perp}[\sigma]$

(see [40, Lemma 1.3] or [1, Proposition 3.1]). Therefore, $\widehat{\mathcal{E}_t^\perp}[\sigma] \neq [\sigma]$ implies $P_\sigma P_t = 0$. Thus $P_\sigma \nu(\pi(a_t'))P_\sigma = P_\sigma P_t \nu(\pi(a_t'))P_\sigma = 0$ for all $t \in F$. Hence

$$1 - \varepsilon < \|\sigma(E(a))\| = \|P_\sigma \nu(\pi(E(a)))P_\sigma\| = \|P_\sigma \nu(\pi(a))P_\sigma\| \leqslant \|\pi(a)\|.$$

Since $\varepsilon \in (0,1)$ is arbitrary, this implies $\|EL(a)\| = 1 \leqslant \|\pi(a)\|$. Then EL factors through π, as desired. $\square$

COROLLARY 6.15. *Suppose that $\mathcal{E}$ is aperiodic or that $\widehat{A} \rtimes S$ is AS topologically free. Then $A \rtimes_{\mathrm{ess}} S$ is simple if and only if $\mathcal{E}$ is minimal, if and only if $\widehat{A} \rtimes S$ is minimal.*

Proof. Theorem 6.6 applies in both cases. $\square$

Remark 6.16. More recently, it is shown in [45] that topologically free actions are aperiodic. This significantly improves Theorem 6.14 and Corollary 6.15. First, aperiodicity is weaker than (AS) topological freeness and, secondly, it also gives results about pure infiniteness (see Corollary 6.7).

The aperiodicity assumption in Theorem 6.5 is not necessary for A to detect ideals in $A \rtimes_{\mathrm{ess}} S$. An easy counterexample is an irrational rotation algebra, viewed as the (essential) crossed product for a twisted action of the group $\mathbb{Z}^2$ on $A = \mathbb{C}$. An important case where aperiodicity is necessary for detection of ideals is when S is very special, namely, $\mathbb{Z}$ or $\mathbb{Z}_n$ with square-free $n \in \mathbb{N}_{\geqslant 1}$:

THEOREM 6.17. *Let the compact group Γ be either $\mathbb{T}$ or $\mathbb{Z}_n$ with square-free $n \in \mathbb{N}_{\geqslant 1}$. Let B be a C^*-algebra with a continuous action $\beta\colon \Gamma \to \mathrm{Aut}(B)$ and let $A = B^\beta$ be the fixed point algebra. Assume that A contains an essential ideal which is separable, simple, or of Type I. Then the following are equivalent:*

(1) *$A \subseteq B$ is aperiodic;*

(2) *A detects ideals in B;*

(3) *A^+ supports B.*

Proof. Let $G = \widehat{\Gamma}$ be the dual group, that is, $G = \mathbb{Z}$ or $G = \Gamma = \mathbb{Z}_n$. Let $\mathcal{B} = (B_g)_{g \in G}$ be the Fell bundle formed by the spectral subspaces of the action β. Then $B_0 = A^\beta = A$ and $B = \mathrm{C}_r^*(\mathcal{B}) = \mathrm{C}^*(\mathcal{B})$ by the Gauge-Equivariant Uniqueness Theorem. By Proposition 6.3, the inclusion $A \subseteq B$ is aperiodic if and only if the Fell bundle is aperiodic. Then [43, Theorem 9.12] implies that (1)–(3) are equivalent. $\square$

Remark 6.18. For an action of a discrete group on a separable C^*-algebra for which $A \subseteq A \rtimes_r G$ detects ideals, Kennedy and Schafhauser [32] introduce a cohomological obstruction whose vanishing implies aperiodicity (the condition they call proper outerness is aperiodicity).

7 Fell bundles over groupoids

Fell bundles over groups are studied, for instance, in [22]. We are going to describe their analogues over étale groupoids through inverse semigroup actions. Then we carry over our definitions and results for inverse semigroup actions to the realm of Fell bundles over étale groupoids. *Throughout this section, X is a locally compact Hausdorff space, H is an étale groupoid with unit space X, and $S \subseteq \mathrm{Bis}(H)$ is a unital, wide inverse subsemigroup of bisections of H.*

7.1 Groupoid Fell bundles and inverse semigroups actions

DEFINITION 7.1 ([8, Section 2]). A *Fell bundle* over H is an upper semicontinuous bundle $\mathcal{A} = (A_\gamma)_{\gamma \in H}$ of Banach spaces with a continuous involution $* \colon \mathcal{A} \to \mathcal{A}$ and a continuous multiplication

$$\cdot \colon \{(a,b) \in \mathcal{A} \times \mathcal{A} : a \in A_{\gamma_1}, \ b \in A_{\gamma_2}, \ \gamma_1, \gamma_2 \in H, \ s(\gamma_1) = r(\gamma_2)\} \to \mathcal{A},$$

which is associative whenever defined. In addition, the fibres A_x for $x \in X$ must be C*-algebras and the fibres A_γ for $\gamma \in H$ must be Hilbert $A_{r(\gamma)}$-$A_{s(\gamma)}$-bimodules for the left and right inner products $\langle\!\langle x \,|\, y \rangle\!\rangle := xy^*$ and $\langle x \,|\, y \rangle := x^*y$ for $x, y \in A_\gamma$. Then the multiplication map yields isometric Hilbert bimodule maps

$$\mu_{\gamma_1, \gamma_2} \colon A_{\gamma_1} \otimes_{A_{s(\gamma_1)}} A_{\gamma_2} \to A_{\gamma_1 \gamma_2}$$

for all $\gamma_1, \gamma_2 \in H$ with $s(\gamma_1) = r(\gamma_2)$. If these maps are surjective, the Fell bundle $\mathcal{A}$ is called *saturated*. This holds if and only if $\mu_{\gamma, \gamma^{-1}}$ is surjective for all $\gamma \in H$.

Example 7.2. An action α of H on a C*-algebra as in [54, Section 3] yields a saturated Fell bundle over H. Namely, let $A_\gamma := A_{r(\gamma)}$ for $\gamma \in H$ and define the multiplication maps and involutions by $A_\eta \times A_\gamma \to A_{\eta\gamma}$, $(a,b) \mapsto a \cdot \alpha_\eta(b)$, and $A_\gamma \to A_{\gamma^{-1}}$, $a \mapsto \alpha_\gamma^{-1}(a^*)$.

We are going to turn a Fell bundle $\mathcal{A} = (A_\gamma)_{\gamma \in H}$ over H into an inverse semigroup action. Let A be the $\mathrm{C}_0(X)$-algebra corresponding to the bundle of C*-algebras $(A_x)_{x \in X}$. Let $S \subseteq \mathrm{Bis}(H)$ be any unital, wide inverse subsemigroup of bisections of H and let $U \in S$. This subset is always Hausdorff and locally compact because the source and range maps restrict to homeomorphisms from U onto the open subsets $s(U)$ and $r(U)$ in X. Let A_U be the space of C_0-sections of the restriction of $(A_\gamma)_{\gamma \in H}$ to U. The spaces $A_{r(U)}$ and $A_{s(U)}$ are closed ideals in $A = A_X$. And the formulas

$$(a \cdot \xi \cdot b)(\gamma) := a(r(\gamma))\xi(\gamma)b(s(\gamma)),$$
$$\langle \xi \,|\, \eta \rangle(s(\gamma)) := \xi(\gamma)^*\eta(\gamma),$$
$$\langle\!\langle \xi \,|\, \eta \rangle\!\rangle(r(\gamma)) := \xi(\gamma)\eta(\gamma)^*$$

for $a \in A_{r(U)}$, $\xi, \eta \in A_U$, $b \in A_{s(U)}$, and $\gamma \in U$ define on A_U the structure of a Hilbert $A_{r(U)}$-$A_{s(U)}$-bimodule. For $U, V \in S$, there is a unique isometric

Hilbert bimodule map $\mu_{U,V}\colon A_U \otimes_A A_V \to A_{UV}$ with

$$\mu_{U,V}(\xi \otimes \eta)(\gamma_1 \cdot \gamma_2) = \xi(\gamma_1)\eta(\gamma_2)$$

for all $\gamma_1 \in U$, $\gamma_2 \in V$. The involutions are the maps $A_U \to A_U^*$, $f^*(\gamma) := f(\gamma^{-1})^*$. This defines a Fell bundle over S (see [8, Example 2.9]). If $\mathcal{A}$ is saturated, then the maps $\mu_{U,V}$ are surjective, that is, the Fell bundle over S is saturated. Then the Fell bundle above defines an action of S on A by Hilbert bimodules.

If $\mathcal{A}$ is not saturated, then we have to modify S to make the Fell bundle saturated. Instead of using the general construction in [9] mentioned in Remark 3.18, we prefer a more concrete construction that depends on the Fell bundle $\mathcal{A}$. Let $\tilde{S}_U$ for $U \in S$ be the set of all Hilbert subbimodules of A_U and let $\tilde{S}$ be the disjoint union of the sets $\tilde{S}_U$. By the Rieffel correspondence, any element of $\tilde{S}_U$ is of the form $\mathcal{F} = A_U \cdot I = J \cdot A_U$, where I and J are the source and range ideals of $\mathcal{F}$. The involutions $A_U \to A_U^*$ and multiplication maps $\mu_{U,V}\colon A_U \otimes_A A_V \to A_{UV}$ map Hilbert subbimodules again to Hilbert subbimodules. So they define maps $\tilde{S}_U \to \tilde{S}_{U^*}$ and $\tilde{S}_U \times \tilde{S}_V \to \tilde{S}_{UV}$. These define an involution and a multiplication on $\tilde{S}$.

LEMMA 7.3. *The multiplication above makes $\tilde{S}$ a unital inverse semigroup. And $(\mathcal{F})_{U \in S, \mathcal{F} \in \tilde{S}_U}$ is a saturated Fell bundle over $\tilde{S}$. There is also a canonical inverse semigroup homomorphism $\tilde{S} \to S$ with fibres $\tilde{S}_U$. The Fell bundles over S and $\tilde{S}$ defined above have the same algebraic, full and reduced section C*-algebras.*

Proof. The equation $tt^*t = t$ holds for all $t \in \tilde{S}$ because $\mathcal{F} \otimes_A \mathcal{F}^* \otimes_A \mathcal{F} \cong \mathcal{F}$ holds for all Hilbert A-bimodules $\mathcal{F}$. To show that $\tilde{S}$ is an inverse semigroup, it suffices to prove that the idempotent elements in $\tilde{S}$ form a commutative subsemigroup. If $\mathcal{F} \in \tilde{S}_U$ is idempotent, then $U^2 = U$ and so A_U is an ideal in A. Then $\mathcal{F}$ is an ideal in A as well. The product of two ideals is their intersection. Since this operation on ideals is commutative and $E(S)$ is commutative, the idempotents in $\tilde{S}$ form a commutative semigroup. Therefore, $\tilde{S}$ is an inverse semigroup. The element $A \in \tilde{S}_1$ is an identity element in $\tilde{S}$.

We have defined the multiplication in $\tilde{S}$ so that the involutions and multiplication maps in our original Fell bundle over S restricted to the elements of $\tilde{S}$ give $(\mathcal{F})_{U \in S, \mathcal{F} \in \tilde{S}_U}$ the structure of a saturated Fell bundle over $\tilde{S}$. We only discuss the inclusion maps (which are redundant in the saturated case by the results in [11]). Let $\mathcal{F}_1 \subseteq A_{U_1}$ and $\mathcal{F}_2 \subseteq A_{U_2}$ be elements of $\tilde{S}$. Then $(U_1, \mathcal{F}_1) \leqslant (U_2, \mathcal{F}_2)$ holds in $\tilde{S}$ if and only if $U_1 \leqslant U_2$ in S and the multiplication map $A_{U_2} \otimes_A A_{s(U_1)} \hookrightarrow A_{U_1}$ maps $\mathcal{F}_2 \otimes_A s(\mathcal{F}_1)$ onto $\mathcal{F}_1$; this follows from our description of idempotent elements in $\tilde{S}$ above. In this situation, there is a canonial inclusion map $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes_A s(\mathcal{F}_1) \hookrightarrow \mathcal{F}_2$. It agrees with the restriction of the map $A_{U_1} \hookrightarrow A_{U_2}$. And $(U_1, \mathcal{F}_1) \leqslant (U_2, \mathcal{F}_2)$ holds if and only if the map $A_{U_1} \hookrightarrow A_{U_2}$ maps $\mathcal{F}_1$ into $\mathcal{F}_2$.

A section of the Fell bundle over $\tilde{S}$ is a finite formal linear combination of elements of $\mathcal{F}$ for $(U, \mathcal{F}) \in \tilde{S}$. The relations are defined in [8] using only the

inclusion maps $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ for $\mathcal{F}_1 \subseteq A_{U_1}$, $\mathcal{F}_2 \subseteq A_{U_2}$ with $(U_1, \mathcal{F}_1) \leqslant (U_2, \mathcal{F}_2)$. More relations are used in [10], giving a variant of the algebraic section algebra; but this is worked out only in the saturated case. The relations in [8] already suffice to show that any element of $\mathcal{F}$ for $(U, \mathcal{F}) \in \tilde{S}$ is identified in the algebraic section C*-algebra with the corresponding element of (U, A_U). And these are subjected to the same relations and *-algebra structure as in the section C*-algebra of the Fell bundle over S. So the section *-algebras of the Fell bundles as defined in [8] are canonically isomorphic. The same would be true for the definition in [10] if that were carried over to the non-saturated case. The full section C*-algebras are defined as the maximal C*-completions of the section *-algebras. Hence these are also canonically isomorphic. The reduced section C*-algebra of a non-saturated Fell bundle is defined in [20] using all representations of the full section C*-algebra that are obtained by inducing irreducible representations of A in a certain way. When we identify the full section C*-algebras as above, we get the same induced representations for both of them. Hence the reduced section C*-algebras are also isomorphic. For saturated Fell bundles, the definition of the reduced section C*-algebra through a weak conditional expectation is equivalent to Exel's definition by Lemma 3.21.   □

Let $\mathcal{A}$ be a Fell bundle over H. Then H acts naturally on the space $\widehat{A}$ of all irreducible representations of the unit fibre $A = A_X$. First, any irreducible representation of A factors through the evaluation map $A \to A_x$ for some $x \in X$, and this defines a continuous map $\psi \colon \widehat{A} \to X$. Such a map is also equivalent to a $C_0(X)$-C*-algebra structure on A by the Dauns–Hofmann Theorem (see [48]). The map ψ is the anchor map of the H-action on $\widehat{A}$. Secondly, if $\gamma \in H$, then the Hilbert $A_{r(\gamma)}, A_{s(\gamma)}$-bimodule A_γ induces a partial homeomorphism $\widehat{A_\gamma}$ from $\widehat{A_{s(\gamma)}}$ to $\widehat{A_{r(\gamma)}}$. If the Fell bundle over H is not saturated, then the domain and codomain $\widehat{A_\gamma}$ may be smaller than $\widehat{A_{s(\gamma)}}$ and $\widehat{A_{r(\gamma)}}$, respectively. These partial homeomorphisms still form a continuous "partial" action of H on $\widehat{A}$, and this is enough to form a transformation groupoid $\widehat{A} \rtimes H$, just as for a partial group action on a space. The proof of Proposition 2.2 extends to this situation and shows that $\widehat{A} \rtimes H \cong \widehat{A} \rtimes S = \widehat{A} \rtimes \tilde{S}$.

Remark 7.4. If H is Hausdorff, then the unit space of H is closed in H. This implies that the unit space $\widehat{A}$ is closed in $\widehat{A} \rtimes H$. That is, all inverse semigroup actions associated to actions of H are closed (see [10, Example 6.7]).

The following theorem describes when a saturated Fell bundle over $\mathrm{Bis}(H)$ comes from a saturated Fell bundle over H. The extra ingredient is the map $\psi \colon \widehat{A} \to X$.

THEOREM 7.5 ([11, Theorem 6.1]). *Let H be an étale groupoid with locally compact, Hausdorff object space X. A saturated Fell bundle over $\mathrm{Bis}(H)$ comes from a saturated Fell bundle over H if and only if the map $U \mapsto A_U$ from open subsets in X to ideals in A commutes with suprema, if and only if there is a continuous map $\pi \colon \widehat{A} \to X$ such that $\widehat{A_U} = \pi^{-1}(U)$ for all open subsets $U \subseteq X$.*

A similar result holds for non-saturated Fell bundles over H and $\mathrm{Bis}(H)$. And if we replace $\mathrm{Bis}(H)$ by $S \subseteq \mathrm{Bis}(H)$, then a Fell bundle over S comes from a Fell bundle over H if and only if there is a continuous, S-equivariant map $\pi\colon \widehat{A} \to X$ such that $\widehat{A}_e = \pi^{-1}(e)$ for all $e \in E(S)$, identified with open subsets of X.

7.2 THE FULL GROUPOID CROSSED PRODUCT

If $U \subseteq H$ is a bisection, then U is Hausdorff and locally compact. Let $\mathrm{C_c}(U,\mathcal{A}) \subseteq A_U$ be the space of continuous sections of $\mathcal{A}|_U$ with compact support. Extend functions in $\mathrm{C_c}(U,\mathcal{A})$ by 0 to H. These extensions need not be continuous any more. Let $\mathfrak{S}(H,\mathcal{A})$ be the linear span of $\mathrm{C_c}(U,\mathcal{A})$ for all bisections $U \subseteq H$. We call sections in $\mathfrak{S}(H,\mathcal{A})$ *quasi-continuous*. The space $\mathfrak{S}(H,\mathcal{A})$ carries a convolution product and an involution given by

$$(f * g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta) \cdot g(\eta^{-1} \cdot \gamma), \qquad (f^*)(\gamma) := f(\gamma^{-1})^*$$

for all $f, g \in \mathfrak{S}(H,\mathcal{A})$, $\gamma \in H$. The *full section* C*-*algebra* $\mathrm{C}^*(H,\mathcal{A})$ of the Fell bundle $\mathcal{A}$ over H is defined as the maximal C*-completion of the *-algebra $\mathfrak{S}(H,\mathcal{A})$.

PROPOSITION 7.6. *Let H be an étale groupoid with locally compact and Hausdorff unit space X and let $\mathcal{A}$ be a Fell bundle over H. Turn $\mathcal{A}$ into a saturated Fell bundle over an inverse semigroup $\tilde{S}$ as in Lemma* 7.3 *above. Then* $\mathrm{C}^*(H,\mathcal{A}) \cong A \rtimes \tilde{S}$.

Proof. If the Fell bundle $\mathcal{A}$ is saturated, then the isomorphism $\mathrm{C}^*(H,\mathcal{A}) \cong A \rtimes S$ is [11, Corollary 5.6]. Under separability hypotheses, [8, Theorem 2.13] says that $\mathrm{C}^*(H,\mathcal{A})$ is isomorphic to the full section C*-algebra of the Fell bundle $(A_U)_{U \in S}$. This is isomorphic to $A \rtimes \tilde{S}$ by Lemma 7.3. We briefly sketch the proof to point out that the extra saturatedness or separability assumptions in these proofs are not needed. What makes the proof tricky is that the algebraic *-subalgebras used to define $\mathrm{C}^*(H,\mathcal{A})$ and $A \rtimes \tilde{S}$ are *not* the same. The proof shows that they have the same *-representations.

If $U \in \mathrm{Bis}(H)$, then any compact subset of U is covered by finitely many bisections in S. Using a partition of unity, one shows that functions in $\mathrm{C_c}(t,\mathcal{A})$ for $t \in S$ already span $\mathfrak{S}(H,\mathcal{A})$. This gives a surjective linear map $\bigoplus_{t \in \tilde{S}} \mathrm{C_c}(t,\mathcal{A}) \twoheadrightarrow \mathfrak{S}(H,\mathcal{A})$. The proof of Lemma 7.3 shows that $A \rtimes_{\mathrm{alg}} \tilde{S} \subseteq A \rtimes \tilde{S}$ is spanned by the subspaces A_t for $t \in S$, giving a surjective linear map $\bigoplus_{t \in S} A_t \twoheadrightarrow A \rtimes_{\mathrm{alg}} \tilde{S}$. We claim that the map $\bigoplus_{t \in S} \mathrm{C_c}(t,\mathcal{A}) \to \bigoplus_{t \in S} A_t$ defined by the inclusion maps $\mathrm{C_c}(t,\mathcal{A}) \to A_t$ for $t \in S$ descends to a well defined map $\mathfrak{S}(H,\mathcal{A}) \to A \rtimes_{\mathrm{alg}} \tilde{S}$. This follows from [11, Proposition B.2], which describes the kernel of the map $\bigoplus_{t \in S} \mathrm{C_c}(t,\mathcal{A}) \to \mathfrak{S}(H,\mathcal{A})$ in terms of the inclusion maps $\mathrm{C_c}(t,\mathcal{A}) \hookrightarrow \mathrm{C_c}(u,\mathcal{A})$ for $t \leqslant u$. The resulting map $\mathfrak{S}(H,\mathcal{A}) \to A \rtimes_{\mathrm{alg}} \tilde{S}$ is an injective *-algebra homomorphism, but not surjective.

If $U \subseteq X$, then any *-representation of $\mathrm{C_c}(U, \mathcal{A})$ is already bounded in the C*-norm on A_U because $\mathrm{C_c}(U, \mathcal{A})$ is a union of C*-subalgebras of A_U. Hence it extends uniquely to a *-representation of A_U. Then it follows that the restriction of a *-representation of $\mathfrak{S}(H, \mathcal{A})$ to $\mathrm{C_c}(t, \mathcal{A})$ for $t \in S$ is bounded in the norm of A_t and hence extends uniquely to a bounded linear map on A_t. These maps form a representation of the Fell bundle $(A_t)_{t \in S}$ over S. Thus $\mathfrak{S}(H, \mathcal{A})$ and $A \rtimes_{\mathrm{alg}} \tilde{S}$ have the same representations and hence the same maximal C*-completions. $\qquad\square$

Remark 7.7. A Fell line bundle over H is a continuous Fell bundle with $A_\gamma \cong \mathbb{C}$ as a vector space for all $\gamma \in H$. Such Fell bundles correspond to "twists" of H (the proof of [15, Theorem 5.6] still works for non-Hausdorff groupoids). The corresponding Fell bundles over inverse semigroups are studied in [8], where they are called *semi-Abelian*. The section C*-algebra of a Fell line bundle over H is a *twisted* groupoid C*-algebra of H. The usual groupoid C*-algebra $\mathrm{C}^*(H)$ corresponds to the "trivial" Fell line bundle, where all the multiplication maps are the usual multiplication map on $\mathbb{C}$.

7.3 The reduced section C*-algebra

Next we define the reduced section C*-algebra of the Fell bundle $\mathcal{A}$ over H. Let $x \in X$. Then

$$\mathrm{C_c}(H_x, \mathcal{A}) = \bigoplus_{s(\gamma)=x} A_\gamma$$

is a pre-Hilbert A_x-module for the obvious right multiplication and the standard inner product $\langle f \mid g \rangle := \sum_{s(\gamma)=x} f(\gamma)^* g(\gamma)$. Let $\ell^2(H_x, \mathcal{A})$ denote its Hilbert A_x-module completion. If $f \in \mathfrak{S}(H, \mathcal{A})$, $g \in \mathrm{C_c}(H_x, \mathcal{A})$, then define $\lambda_x(f)(g) \in \mathrm{C_c}(H_x, \mathcal{A})$ by

$$\lambda_x(f)(g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta) g(\eta^{-1}\gamma).$$

The operator $\lambda_x(f)$ extends uniquely to an adjointable operator on $\ell^2(H_x, \mathcal{A})$ with adjoint $\lambda_x(f^*)$, and this defines a non-degenerate *-representation of $\mathfrak{S}(H, \mathcal{A})$ on $\ell^2(H_x, \mathcal{A})$. It extends uniquely to a non-degenerate *-representation

$$\lambda_x \colon \mathrm{C}^*(H, \mathcal{A}) \to \mathbb{B}\big(\ell^2(H_x, \mathcal{A})\big).$$

Definition 7.8. The *reduced norm* on $\mathrm{C}^*(H, \mathcal{A})$ or $\mathfrak{S}(H, \mathcal{A})$ is defined by

$$\|f\|_{\mathrm{r}} := \sup_{x \in X} \|\lambda_x(f)\|.$$

The *reduced section* C*-*algebra* $\mathrm{C_r^*}(H, \mathcal{A})$ of the Fell bundle $\mathcal{A}$ over H is defined as the completion of $\mathfrak{S}(H, \mathcal{A})$ in the reduced norm. Equivalently, it is the quotient of $\mathrm{C}^*(H, \mathcal{A})$ by the ideal $\bigcap_{x \in X} \ker(\lambda_x)$, which is the null space of the reduced norm on $\mathrm{C}^*(H, \mathcal{A})$.

Proposition 7.9. *The isomorphism* $\mathrm{C}^*(H,\mathcal{A}) \cong A \rtimes \tilde{S}$ *in Proposition* 7.6 *descends to an isomorphism* $\mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \cong A \rtimes_{\mathrm{r}} \tilde{S}$.

Proof. For saturated Fell bundles over H, which correspond to saturated Fell bundles over S and thus actions of S by Hilbert bimodules, this is contained in [10, Theorem 8.11]. The same idea works in the non-saturated case. Each representation π of A may be induced to a representation $i(\pi)$ of $A \rtimes \tilde{S}$. The representation $\bigoplus_{\pi \in \hat{A}} i(\pi)$ of $A \rtimes \tilde{S}$ descends to a faithful representation of $A \rtimes_{\mathrm{r}} \tilde{S}$. This is how Lemma 3.21 is proven. Any irreducible representation of A factors through one of the fibres A_x for $x \in X$. The representation of $A \rtimes \tilde{S}$ induced by $\pi \in \widehat{A_x}$ corresponds to the representation $\lambda_x \otimes 1$ of $\mathrm{C}^*(H,\mathcal{A})$ on $\ell^2(H_x, \mathcal{A}) \otimes_{A_x} \mathcal{H}_\pi$. And $\|\lambda_x(f)\|$ is the supremum of $\|\lambda_x \otimes 1_{\mathcal{H}_\pi}(f)\|$ over all $\pi \in \widehat{A_x}$. So the reduced norm that defines $\mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A})$ corresponds to the supremum of $\|i(\pi)(f)\|$ over all $\pi \in \hat{A}$, which gives $A \rtimes_{\mathrm{r}} \tilde{S}$. $\square$

Let $\mathfrak{B}(H,\mathcal{A})$ denote the Banach space of bounded Borel sections of the Banach space bundle $\mathcal{A}$, and similarly for $\mathfrak{B}(X,\mathcal{A})$. So $\mathfrak{S}(H,\mathcal{A}) \subseteq \mathfrak{B}(H,\mathcal{A}) \subseteq \prod_{\gamma \in H} A_\gamma$ as vector spaces.

Proposition 7.10. *The embedding* $\mathfrak{S}(H,\mathcal{A}) \to \mathfrak{B}(H,\mathcal{A})$ *extends uniquely to an injective and contractive linear map* $j \colon \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \to \mathfrak{B}(H,\mathcal{A})$. *The map*

$$E_{\mathrm{r}} \colon \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \to \mathfrak{B}(X,\mathcal{A}), \qquad f \mapsto j(f)|_X,$$

is a faithful generalised expectation.

Proof. Let $\gamma \in H$. Then $A_\gamma \subseteq \ell^2(H_{s(\gamma)}, \mathcal{A})$ is a direct summand. Let $T_\gamma \colon A_\gamma \to \ell^2(H_{s(\gamma)}, \mathcal{A})$ be the isometric inclusion. Then T_γ^* is the orthogonal projection onto A_γ. If $f \in \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A})$, then define

$$j(f)(\gamma) := T_\gamma^* \lambda_x(f) T_{s(\gamma)} \colon A_{s(\gamma)} \to A_\gamma.$$

If $f \in \mathrm{C}_{\mathrm{c}}(U,\mathcal{A})$ for some bisection $U \subseteq H$, then $j(f)(\gamma)(a) = f(\gamma) \cdot a$ for all $a \in A_{s(\gamma)}$ and $\gamma \in H$. Thus $j(f)(\gamma)$ is the compact operator corresponding under the isomorphism $\mathbb{K}(A_{s(\gamma)}, A_\gamma) \cong A_\gamma$ to the element $f(\gamma) \in A_\gamma$. We simply write $j(f)(\gamma) = f(\gamma)$ for all $f \in \mathfrak{S}(H,\mathcal{A})$. Since $\|T_\gamma\| = 1$, we may estimate $\|j(f)(\gamma)\| \leqslant \|\lambda_x(f)\| \leqslant \|f\|_{\mathrm{r}}$ for all $f \in \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A})$. The section $j(f) \in \prod_{x \in X} A_x$ is Borel for all $f \in \mathrm{C}_{\mathrm{c}}(U,\mathcal{A})$ and hence for $f \in \mathfrak{S}(H,\mathcal{A})$. Since j is bounded, $\mathfrak{S}(H,\mathcal{A})$ is dense in $\mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A})$, and uniform limits of Borel functions are Borel, it follows that j is the unique contractive linear map $\mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \to \mathfrak{B}(H,\mathcal{A})$ extending j on $\mathfrak{S}(H,\mathcal{A})$.

If $x \in X$, then $f \mapsto j(f)(x) = T_x^* \lambda_x(f) T_x$ is a completely positive, contractive linear map $E_x \colon \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \to A_x$. Hence $E_{\mathrm{r}}(f) := j(f)|_X$ is completely positive and contractive as a map to $\prod_{x \in X} A_x$. Then it is a completely positive contraction $E_{\mathrm{r}} \colon \mathrm{C}_{\mathrm{r}}^*(H,\mathcal{A}) \to \mathfrak{B}(X,\mathcal{A})$ as well. Being the identity map on A, it is a generalised conditional expectation. We may further compose E_x with the

faithul representation $A_x \to \prod_{\pi \in \widehat{A_x}} \mathbb{B}(\mathcal{H}_\pi)$. Since $\widehat{A} = \bigsqcup_{x \in X} \widehat{A_x}$, this gives a generalised expectation

$$\tilde{E}_{\mathrm{r}} \colon \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A}) \to \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi), \qquad \tilde{E}_{\mathrm{r}}(f)(\pi) := \pi(j(f)|_X).$$

Let $E_{\mathrm{r}} \colon A \rtimes_{\mathrm{r}} \tilde{S} \to A''$ be the canonical weak conditional expectation (it should not be confused with $E_{\mathrm{r}} \colon \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A}) \to \mathfrak{B}(X, \mathcal{A})$ as the domain and codomain are different). The generalised expectation

$$\varrho \circ E_{\mathrm{r}} \colon A \rtimes_{\mathrm{r}} \tilde{S} \to \prod_{\pi \in \widehat{A}} \mathbb{B}(\mathcal{H}_\pi),$$

is faithful by Lemma 3.21 and Theorem 3.22. The isomorphism $\mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A}) \cong A \rtimes_{\mathrm{r}} \tilde{S}$ in Proposition 7.9 intertwines the generalised expectations $\tilde{E}_{\mathrm{r}}$ and $\varrho \circ E_{\mathrm{r}}$ because it does so on functions in $f \in \mathrm{C}_{\mathrm{c}}(t, \mathcal{A})$ for $t \in S$. Hence the generalised expectation $\tilde{E}_{\mathrm{r}}$ is faithful. Then so is $E_{\mathrm{r}} \colon \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A}) \to \mathfrak{B}(X, \mathcal{A})$.
If $f \in \mathfrak{S}(H, \mathcal{A})$, then we compute

$$E_{\mathrm{r}}(f^* * f)(x) = j(f^* * f)(x) = \sum_{s(\gamma) = x} f^*(\gamma^{-1}) f(\gamma) = \sum_{s(\gamma) = x} j(f)(\gamma)^* j(f)(\gamma).$$

The norm of the left hand side is bounded by $\|f\|_{\mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})}^2$. The norm of the right hand side is the square of the norm of $j(f)$ in $\ell^2(H_x, \mathcal{A})$. Hence $j(f)|_{H_x} \in \ell^2(H_x, \mathcal{A})$ and $\|j(f)|_{H_x}\|_{\ell^2(H_x, \mathcal{A})} \leqslant \|f\|_{\mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})}$ for all $f \in \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})$. By continuity, we get

$$E_{\mathrm{r}}(f^* * f)(x) = \sum_{s(\gamma) = x} j(f)(\gamma)^* j(f)(\gamma) \tag{7.1}$$

for all $f \in \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})$. So $E_{\mathrm{r}}(f^* * f) = 0$ is equivalent to $j(f) = 0$. Since E_{r} is faithful, this is equivalent to $f = 0$. Hence j is injective. $\qquad \square$

Remark 7.11. If $\mathcal{A}$ is a Fell line bundle, then its restriction to the unit space is the trivial bundle $X \times \mathbb{C}$. Hence identifying sections $j(f)|_X$ with scalar-valued functions on X, we may view E_{r} as a generalised expectation into the C*-algebra $\mathfrak{B}(X)$ of Borel functions on X. This expectation has been used already by Khoskham and Skandalis [33].

7.4   The essential groupoid crossed product

Definition 7.12. Let $\mathrm{C}_{\mathrm{ess}}^*(H, \mathcal{A})$ be the quotient of $\mathrm{C}^*(H, \mathcal{A})$ that corresponds to the quotient $A \rtimes_{\mathrm{ess}} \tilde{S}$ of $A \rtimes \tilde{S}$ under the isomorphism in Proposition 7.6.

So $\mathrm{C}_{\mathrm{ess}}^*(H, \mathcal{A})$ is the quotient of $\mathrm{C}^*(H, \mathcal{A})$ by the ideal $\mathcal{N}_{EL}$ for the canonical $\mathcal{M}_{\mathrm{loc}}$-expectation

$$EL \colon \mathrm{C}^*(H, \mathcal{A}) \xrightarrow{\sim} A \rtimes \tilde{S} \to \mathcal{M}_{\mathrm{loc}}(A).$$

Since EL is symmetric by Theorem 3.22, $b \in \mathrm{C}^*(H,\mathcal{A})$ belongs to $\mathcal{N}_{EL}$ if and only if $EL(b^* * b) = 0$. Since $A \rtimes_{\mathrm{ess}} \tilde{S}$ is a quotient of $A \rtimes_{\mathrm{r}} \tilde{S}$, Proposition 7.9 allows to identify $\mathrm{C}^*_{\mathrm{ess}}(H,\mathcal{A})$ with the quotient of $\mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$ by the image of $\mathcal{N}_{EL}$ in $\mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$. We denote this image by J_{sing} as in Section 4.3 and call its elements *singular*. We are going to describe J_{sing} in terms of the groupoid Fell bundle.

By Proposition 4.10, the essential multiplier algebra $\mathcal{M}_{\mathrm{loc}}(A)$ is embeds into the quotient of $\prod_{\pi \in \hat{A}} \mathbb{B}(\mathcal{H}_\pi)$ by the null space of the essential supremum norm $\|\cdot\|_{\mathrm{ess}}$, which takes the minimum of the supremum norms over comeagre subsets of $\hat{A}$. In the proof of Proposition 7.10, we have noticed that the canonical generalised expectation $E_{\mathrm{r}} : \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A}) \to \prod_{x \in X} A_x$ – composed with the standard faithful representations – gives a faithful generalised expectation

$$\tilde{E}_{\mathrm{r}} : \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A}) \to \prod_{x \in X} A_x \to \prod_{\pi \in \hat{A}} \mathbb{B}(\mathcal{H}_\pi),$$

which corresponds to $\varrho \circ E_{\mathrm{r}} : A \rtimes_{\mathrm{r}} \tilde{S} \to \prod_{\pi \in \hat{A}} \mathbb{B}(\mathcal{H}_\pi)$. So by Proposition 4.10, the norm of the image of $b \in \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$ in $\mathrm{C}^*_{\mathrm{ess}}(H,\mathcal{A})$ is the *essential* supremum of $\left\| \tilde{E}_{\mathrm{r}}(b)(\pi) \right\|$. Therefore, $b \in J_{\mathrm{sing}}$ if and only if the set of $\pi \in \hat{A}$ with $\tilde{E}_{\mathrm{r}}(b^* * b)(\pi) \neq 0$ is meagre. In general, this is the best we can say. Under extra assumptions, we are going to rewrite this in terms of the set of $x \in X$ with $E_{\mathrm{r}}(b^* * b)(x) \neq 0$, or the set of $\gamma \in H$ with $j(b)(\gamma) \neq 0$. The starting point is the following analogue of Proposition 4.15:

LEMMA 7.13. *Let $f \in \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$. There is a comeagre subset $C \subseteq H$ such that the section $j(f)$ of $\mathcal{A}$ is continuous in all points of C; that is, if $\gamma \in C$, then there is an open neighbourhood U of γ and a continuous section h of $\mathcal{A}|_U$ with $\lim_{\eta \to \gamma} \|f(\eta) - h(\eta)\| = 0$. Thus the section $E_{\mathrm{r}}(f)$ of $\mathcal{A}|_X$ is continuous in $C \cap X$, which is comeagre in X.*

Proof. First let $f \in \mathrm{C}_{\mathrm{c}}(U,A)$ for some bisection U. Then $j(f)$ is a continuous section on U and vanishes on the interior of $H \backslash U$. Thus it is continuous in all points of the dense open subset $H \backslash \partial U$. If $f \in \mathfrak{S}(H,\mathcal{A})$, then f is a finite linear combination of functions as above. Hence $j(f)$ is continuous in all points of a finite intersection of dense open subsets, which is again dense open. Finally, if $f \in \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$, then there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathfrak{S}(H,\mathcal{A})$ with $\lim \|f_n - f\| = 0$. Hence $j(f_n)$ converges uniformly towards $j(f)$. For each $n \in \mathbb{N}$, there is a dense open subset $Y_n \subseteq H$ where f_n is continuous. Let $C := \bigcap_{n \in \mathbb{N}} Y_n$. The subset C is comeagre as a countable intersection of dense open subsets. Its intersection with the closed subspace $X \subseteq H$ is comeagre in X.

We claim that $j(f)$ is a continuous section in all $\gamma \in C$. Thus $E_{\mathrm{r}}(f) = j(f)|_X$ is continuous in $C \cap X$. A continuous section h with $\lim_{\eta \to \gamma} \|f(\eta) - h(\eta)\| = 0$ is built as follows. Let f_n and Y_n for $n \in \mathbb{N}$ be as above. Define $f_{-1} := 0$. We may arrange that $\|f_n - f_{n-1}\|_\infty < 2^{-n}$ for all $n \in \mathbb{N}$. Let U be a bisection containing γ. Since Y_n is open, there is a function $w_n \in \mathrm{C}_{\mathrm{c}}(U \cap Y_n \cap Y_{n-1})$ with $w_n(\gamma) = 1$ and $\|w_n\|_\infty \leqslant 1$. Then $(f_n - f_{n-1}) \cdot w_n$ is a continuous section

of $\mathcal{A}|_U$ with $\|(f_n - f_{n-1}) \cdot w_n\| \leqslant 2^{-n}$. Hence $h := \sum_{n=0}^{\infty}(f_n - f_{n-1}) \cdot w_n$ is a well defined continuous section of $\mathcal{A}|_U$. The continuity of the functions w_n implies that $\lim_{\eta \to \gamma} \|f(\eta) - h(\eta)\| = 0$. $\qquad \square$

In order to compare our description of the essential crossed product to the one in [25], we need more information about the comeagre subset C in Lemma 7.13.

DEFINITION 7.14. Call $x \in X$ *dangerous* if there is a net (γ_n) in H that converges towards two different points $\gamma \neq \gamma' \in H$ with $s(\gamma) = s(\gamma') = x$.

LEMMA 7.15. *Let $f \in \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})$ and $\gamma \in H$. If $s(\gamma)$ is not dangerous, then $j(f)$ is continuous at γ. If $x \in X$ is dangerous, then the isotropy group $H(x)$ at x is non-trivial. If H is covered by countably many bisections, then the subset D of dangerous points is meagre, and so is $s^{-1}(D) \subseteq H$.*

Proof. First let $f \in \mathrm{C}_{\mathrm{c}}(U, \mathcal{A})$ for some $U \in \mathrm{Bis}(H)$ and assume that $j(f)$ is discontinuous at $\gamma \in H$. Then $\gamma \in H \backslash U$ and there is a net (γ_n) converging to γ with $f(\gamma_n) \nrightarrow 0$. So γ_n must lie in the support of f, which is a compact subset of U. Passing to a subnet, we may arrange that γ_n converges to some $\gamma' \in U$. This must be different from γ. Since X is Hausdorff, $s(\gamma) = \lim s(\gamma_n) = s(\gamma')$. So $s(\gamma)$ is dangerous. In other words, $j(f)$ for $f \in \mathrm{C}_{\mathrm{c}}(U, \mathcal{A})$ is continuous at all $\gamma \in H$ with $s(\gamma) \notin D$. This remains so for uniform limits of finite linear combinations of such $j(f)$, giving the first claim for all $f \in \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})$. The same argument shows that $r(\gamma) = r(\gamma')$. So $\gamma^{-1}\gamma'$ is a non-trivial element in the isotropy group of x and $H(x)$ is non-trivial if x is dangerous.

Now assume that H is covered by countably many bisections $S \subseteq \mathrm{Bis}(H)$. The arrows γ, γ' witnessing that some $x \in X$ is dangerous must belong to some bisections $U, V \in S$. Since a countable union of meagre subsets is meagre, it suffices to fix $U, V \in S$ and prove the meagreness of the set of all $x \in X$ for which there are $\gamma \in U$, $\gamma' \in V$ with $s(\gamma) = s(\gamma') = x$ and a net (γ_n) in H converging both to γ and γ'. Since U and V are open, we may restrict our net (γ_n) to a subnet that belongs to $U \cap V$. But $\gamma, \gamma' \notin U \cap V$. Since s restricts to homeomorphisms $U \cong s(U)$ and $V \cong s(V)$, it follows that $x \in \partial(s(U \cap V))$. This subset is closed and nowhere dense, hence meagre. The subset $s^{-1}(\partial(s(U \cap V)))$ is also closed and nowhere dense because s is continuous and open. Hence $s^{-1}(D) \subseteq H$ is meagre as well. $\qquad \square$

Example 7.16. If H is not covered by countably many bisections, then all $x \in X$ may be dangerous. To produce such an example, let $X = [0,1]$ and let the free group F on the set $[0,1]$ act identically on X. For each $t \in [0,1]$ we identify the arrow $s \to s$ given by the generator $t \in F$ with the identity arrow on s if $s \in [0, t]$. This extends to a congruence relation $\sim$ on the transformation groupoid $[0,1] \rtimes F$. By construction, each $t \in [0,1]$ is dangerous in $[0,1] \rtimes F/\sim$.

So far, we have assumed $\mathcal{A}$ to be an upper semicontinuous field. Then Lemma 7.13 implies that, for every $f \in \mathrm{C}_{\mathrm{r}}^*(H, \mathcal{A})$, the function

$$\nu_f \colon H \to [0, \infty), \qquad \gamma \mapsto \|j(f)(\gamma)\|_{A_\gamma},$$

is upper semicontinuous in a comeagre subset of X. This is useless, however. The applications of Proposition 4.15 in Section 4.3 need *lower* semicontinuity instead. Therefore, we assume $\mathcal{A}$ to be a continuous field of Banach spaces from now on. For brevity, we call $\mathcal{A}$ a *continuous Fell bundle*.

Remark 7.17. A Fell bundle $\mathcal{A}$ over H is continuous if and only if its restriction $\mathcal{A}|_X$ is a continuous field of C*-algebras over X. One implication is trivial, and the continuity of $\mathcal{A}|_X$ implies continuity of $\mathcal{A}$ because $\|a(\gamma)\|^2_{A_\gamma} = \|a^*a\|_{A_{s(\gamma)}}$ for all $a \in A_\gamma$. Recall also that a $C_0(X)$-C*-algebra structure on the C*-algebra A_X is equivalent to a continuous map $\psi \colon \widehat{A_X} \to X$. The field $\mathcal{A}|_X$ is a continuous field of C*-algebras if and only if ψ is open (see [48, Theorem 3.3]). A Fell bundle $\mathcal{A}$ is a line bundle if and only if $\mathcal{A}|_X$ is the trivial C*-algebra bundle over X with fibre $\mathbb{C}$. This implies that $\mathcal{A}$ is continuous. In fact, Fell line bundles are locally trivial.

We can now describe the ideal $J_{\mathrm{sing}} := \ker\left(\mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A}) \to \mathrm{C}^*_{\mathrm{ess}}(H,\mathcal{A})\right)$:

PROPOSITION 7.18. *Let H be an étale groupoid with locally compact and Hausdorff unit space X. Let $\mathcal{A}$ be a continuous Fell bundle over H; so the map $\psi \colon \widehat{A} \to X$ with $\psi^{-1}(x) = \widehat{A_x} \subseteq \widehat{A}$ for $x \in X$ is open. Assume H to be covered by countably many bisections. Let $f \in \mathrm{C}^*_{\mathrm{r}}(H,\mathcal{A})$ and $\varepsilon \geqslant 0$. Define*

$$s^\varepsilon_{\widehat{A}}(f) := \left\{ \pi \in \widehat{A} : \|\pi(\tilde{E}_{\mathrm{r}}(f^* * f))\| > \varepsilon \right\},$$
$$s^\varepsilon_X(f) := \{ x \in X : \|E_{\mathrm{r}}(f^* * f)(x)\| > \varepsilon \},$$
$$s^\varepsilon_H(f) := \{ \gamma \in H : \|j(f)(\gamma)\| > \varepsilon \}.$$

Let $D \subseteq X$ be the set of dangerous points. The following are equivalent:

(1) $f \in J_{\mathrm{sing}}$;

(2) $s^0_{\widehat{A}}(f) \subseteq \psi^{-1}(D)$;

(3) $s^0_{\widehat{A}}(f) \subseteq \widehat{A}$ *is meagre;*

(4) $s^0_{\widehat{A}}(f) \subseteq \widehat{A}$ *has empty interior;*

(5) $s^\varepsilon_{\widehat{A}}(f) \subseteq \widehat{A}$ *has empty interior for all $\varepsilon > 0$;*

(6) $s^0_X(f) \subseteq D$;

(7) $s^0_X(f) \subseteq X$ *is meagre;*

(8) $s^0_X(f) \subseteq X$ *has empty interior;*

(9) $s^\varepsilon_X(f) \subseteq X$ *has empty interior for all $\varepsilon > 0$;*

(10) $s^0_H(f) \subseteq s^{-1}(D)$;

(11) $s^0_H(f) \subseteq H$ *is meagre;*

(12) $s_H^0(f) \subseteq H$ *has empty interior;*

(13) $s_H^\varepsilon(f) \subseteq H$ *has empty interior for all $\varepsilon > 0$.*

In general, without any restriction on $\mathcal{A}$ and H, (1) is equivalent to each of the conditions (3)–(5). And if $\mathcal{A}$ is a Fell line bundle, then they are further equivalent to (7)–(9).

Proof. In general, without any restriction on $\mathcal{A}$ and H, (1) is equivalent to (3)–(5) by Corollary 4.16. If $\mathcal{A}$ is a line bundle, then $\psi \colon \widehat{A} \to X$ is a homeomorphism and $\psi(s_{\widehat{A}}^\varepsilon(f)) = s_X^\varepsilon(f)$ for all $\varepsilon \geqslant 0$. Thus (3)–(5) are equivalent to (7)–(9) in this case.

From now on, we assume $\mathcal{A}$ to be a continuous Fell bundle and H to be covered by countably many bisections. We first show that (10)–(13) are equivalent. The subset $s^{-1}(D) \subseteq H$ is meagre by Lemma 7.15. So (10) implies (11). Since H is a union of open sets that are Baire, H is a Baire space. Hence a meagre subset of H must have empty interior. So (11) implies (12). And (12) implies (13) because $s_H^0(f) \supseteq s_H^\varepsilon(f)$ for all $\varepsilon > 0$. If $s_H^0(f)$ is not contained in $s^{-1}(D)$, then there is $\gamma \in H$ with $s(\gamma) \notin D$ and $j(f)(\gamma) \neq 0$. Let $\varepsilon = \|j(f)(\gamma)\|/2$. Since $s(\gamma) \notin D$ and the bundle $\mathcal{A}$ is lower semicontinuous, $\|j(f)\|$ is lower semicontinuous in γ by Lemma 7.13. This gives an open neighbourhood U of γ with $\|j(f)(\eta)\| > \varepsilon$ for all $\eta \in U$. So $s_H^\varepsilon(f)$ has non-empty interior. This shows that (13) implies (10) and finishes the proof that (10)–(13) are equivalent.

The spaces X and $\widehat{A}$ are Baire spaces as well (see [17, Proposition 3.4.13]). The subset $D \subseteq X$ is meagre. Since ψ is open and continuous, this implies that $\psi^{-1}(D) \subseteq \widehat{A}$ is also meagre. By Lemma 7.15, the function $\|E_r(f^* * f)(x)\|$ is continuous in $X \backslash D$. We claim that the function $\|\tilde{E}_r(f^* * f)(\pi)\|$ is lower semicontinuous in $\psi^{-1}(X \backslash D)$. Indeed, if $x \in X \backslash D$, then there is a continuous section h of $\mathcal{A}|_X$ with $\lim_{y \to x} \|E_r(f^* * f)(y) - h(y)\| = 0$. Since $h \in A$, the function $\|\tilde{E}_r(h)(\pi)\| = \|\pi(h)\|$ is lower semicontinuous on $\widehat{A}$ (see [17, Proposition 3.3.2]). Since

$$\left| \|\tilde{E}_r(h)(\pi)\| - \|\tilde{E}_r(f^* * f)(\pi)\| \right| \leqslant \|E_r(f^* * f)(y) - h(y)\|$$

for $\pi \in \widehat{A_y} \subseteq \widehat{A}$, it follows that $\|\tilde{E}_r(f^* * f)(\pi)\|$ is lower semicontinuous in $\psi^{-1}(x)$.

Using the facts gathered in the previous paragraph, we may carry over the proof of the equivalence (10)–(13) to prove that (6)–(9) are equivalent and that (2)–(5) are equivalent. Finally, $E_r(f^* * f)(x) = 0$ is equivalent to $\|\pi(\tilde{E}_r(f^* * f))\| = 0$ for all $\pi \in \widehat{A_x}$, and to $j(f)|_{H_x} = 0$ by (7.1). Hence (6) is equivalent to (2) and (10). □

Remark 7.19. By definition, $C_r^*(H, \mathcal{A}) = C_{ess}^*(H, \mathcal{A})$ if and only if the only element $f \in C_r^*(H, \mathcal{A})$ that belongs to J_{sing} is the zero element. Thus Proposition 7.18 implies many equivalent characterisations for $C_r^*(H, \mathcal{A}) = C_{ess}^*(H, \mathcal{A})$.

Remark 7.20. An element f of $\mathrm{C}^*_\mathrm{r}(H)$ is called *singular* in [13] if $s^0_H(f) := \{\gamma \in H : j(f)(\gamma) \neq 0\}$ has empty interior. Proposition 7.18 shows that $f \in \mathrm{C}^*_\mathrm{r}(H)$ is singular in the notation of [13] if and only if it belongs to J_sing, provided H is covered by countably many bisections. This follows if H is countable at infinity.

Let $\mathcal{A}$ be a Fell line bundle. Remark 7.17 shows that $\mathcal{A}$ is a continuous field of Banach spaces over H. Exel and Pitts show that

$$\Gamma := \{f \in \mathrm{C}^*_\mathrm{r}(H, \mathcal{A}) : E_\mathrm{r}(f^*f)(x) = 0 \text{ for all } x \in X \text{ with } H(x) = \{x\}\}$$

is an ideal in $\mathrm{C}^*_\mathrm{r}(H, \mathcal{A})$, and they define the essential groupoid C*-algebra as the quotient $\mathrm{C}^*_\mathrm{r}(H, \mathcal{A})/\Gamma$. It is clear that $\Gamma \cap \mathrm{C}_0(X) = 0$ if and only if the set of $x \in H$ with $H(x) = \{x\}$ has empty interior, that is, H is topologically principal. Therefore, the map from $\mathrm{C}_0(X)$ to $\mathrm{C}^*_\mathrm{r}(H, \mathcal{A})/\Gamma$ is injective if and only if H is topologically principal. In contrast, we have defined $\mathrm{C}^*_\mathrm{ess}(H, \mathcal{A})$ so that the map $\mathrm{C}_0(X) \to \mathrm{C}^*_\mathrm{ess}(H, \mathcal{A})$ is always injective. Hence the essential twisted groupoid C*-algebras defined here and in [25] differ when H is not topologically principal. If, however, H is topologically principal, then the two definitions are equivalent:

Proposition 7.21. *Let H be topologically principal and let $\mathcal{A}$ be a Fell bundle over H. Assume either that $\mathcal{A}$ is a Fell line bundle or that $\mathcal{A}$ is continuous and H is covered by countably many bisections. Then*

$$J_\mathrm{sing} = \{f \in \mathrm{C}^*_\mathrm{r}(H, \mathcal{A}) : E_\mathrm{r}(f^* * f)(x) = 0 \text{ for all } x \in X \text{ with } H(x) = \{x\}\}.$$

Proof. Assume $E_\mathrm{r}(f^*f)(x) \neq 0$ for some $x \in X$ with $H(x) = \{x\}$. Then $E_\mathrm{r}(f^*f)$ is continuous at this point by Lemma 7.15. Hence there is an open neighbourhood of x on which $E_\mathrm{r}(f^*f)(y) \neq 0$. Thus $f \notin J_\mathrm{sing}$ by Proposition 7.18. Conversely, assume $E_\mathrm{r}(f^*f)(x) = 0$ for all $x \in X$ with $H(x) = \{x\}$. Since H is topologically principal, the set of $x \in X$ with $H(x) \neq \{x\}$ has empty interior. Hence the set of $x \in X$ with $E_\mathrm{r}(f^*f)(x) \neq 0$ has empty interior. Then $f \in J_\mathrm{sing}$ by Proposition 7.18. □

If $\mathcal{A}$ is a Fell line bundle, then the conditions above may also be related to supportive conditional expectations and the criterion in Lemma 5.23:

Theorem 7.22. *Let H be a topologically free étale groupoid with locally compact Hausdorff object space X. Let $\mathcal{L}$ be a Fell line bundle over H. The following are equivalent:*

(1) $\mathrm{C}^*_\mathrm{r}(H, \mathcal{L}) = \mathrm{C}^*_\mathrm{ess}(H, \mathcal{L})$;

(2) $\mathrm{C}_0(X)$ *supports* $\mathrm{C}^*_\mathrm{r}(H, \mathcal{L})$;

(3) $\mathrm{C}_0(X)$ *detects ideals in* $\mathrm{C}^*_\mathrm{r}(H, \mathcal{L})$;

(4) *for any $f \in \mathrm{C}^*_\mathrm{r}(H, \mathcal{L})^+\backslash\{0\}$, there is $a \in \mathrm{C}_0(X)^+\backslash\{0\}$ with $a \leqslant j(f)|_X$;*

(5) *the canonical weak expectation $\mathrm{C}^*_\mathrm{r}(H, \mathcal{L}) \to \mathrm{C}_0(X)''$ is supportive.*

Proof. The dual groupoid of our action is H, and $C_0(X)$ is of Type I. So Theorem 6.13 shows that the inclusion $C_0(X) \hookrightarrow C^*(H, \mathcal{L}) \cong C_0(X) \rtimes S$ is aperiodic. Thus Theorem 6.5 shows that the essential crossed product $C^*_{\mathrm{ess}}(H, \mathcal{L})$ is the unique quotient of $C^*(H, \mathcal{L})$ in which $C_0(X)$ embeds and detects ideals, and also that $C_0(X)$ supports $C^*_{\mathrm{ess}}(H, \mathcal{L})$. So (1) and (3) are equivalent and they imply (2). And (2) implies (3) by Lemma 5.27. Hence the conditions (1)–(3) are equivalent. Assume (1) and let $f \in C^*_{\mathrm{r}}(H, \mathcal{L})^+ \backslash \{0\}$. Since $J_{\mathrm{sing}} = 0$, Proposition 7.18 implies that there are $\varepsilon > 0$ and an open subset $U \subseteq X$ with $j(f)|_X(x) = E_{\mathrm{r}}(f)(x) > \varepsilon$ for all $x \in U$. There is $a \in C_0(U)$ with $0 \leqslant a \leqslant \varepsilon$. It witnesses that f satisfies the condition in (4). So (1) implies (4). And (4) implies (5) by Lemma 5.23. Condition (5) implies (2) and (3) by Theorem 5.28. This shows that all conditions are equivalent. □

Remark 7.23. The existence of non-trivial singular elements in $C^*_{\mathrm{r}}(H, \mathcal{L})$ for a Fell line bundle $\mathcal{L}$ depends on the line bundle $\mathcal{L}$. Exel constructed in [21, Section 2] a non-Hausdorff, topologically principal, étale groupoid H such that $C_0(X)$ does not detect ideals in $C^*_{\mathrm{r}}(H)$. A line bundle $\mathcal{L}$ over the same groupoid such that $C_0(X)$ does detect ideals in $C^*_{\mathrm{r}}(H, \mathcal{L})$ is built in [25, Section 23].

7.5 Simplicity and pure infiniteness for groupoid Fell bundles

The following theorem carries our results for inverse semigroup crossed products over to the groupoid case:

THEOREM 7.24. *Let $\mathcal{A}$ be a Fell bundle over an étale groupoid H with locally compact Hausdorff object space X. Assume that $A := C_0(\mathcal{A}|_X)$ contains an essential ideal that is separable or of Type I. The inclusion $A \subseteq C^*(H, \mathcal{A})$ is aperiodic if and only if the dual groupoid $\widehat{A} \rtimes H$ is topologically free. This follows if $\mathrm{Prim}(A) \rtimes H$ is topologically free. If $\mathcal{A}$ is a continuous Fell bundle, then it follows also if H is topologically free.*

Proof. The first statement follows from Theorem 6.13 because $\widehat{A} \rtimes H$ is naturally isomorphic to the dual groupoid $\widehat{A} \rtimes \tilde{S}$ for the inverse semigroup action defined in Lemma 7.3. If $X \twoheadrightarrow Y$ is an H-equivariant, continuous and open map between two H-spaces, then $X \rtimes H$ inherits topological freeness from $Y \rtimes H$. Hence $\widehat{A} \rtimes H$ is topologically free if $\mathrm{Prim}(A) \rtimes H$ is topologically free. And $\mathrm{Prim}(A) \rtimes H$ is topologically free if $H = X \rtimes H$ is topologically free and the base map $\psi \colon \mathrm{Prim}(A) \to X$ is open. The base map is open if and only if A is a continuous $C_0(X)$-algebra if and only if $\mathcal{A}$ is a continuous Fell bundle (see Remark 7.17). □

THEOREM 7.25. *Let $\mathcal{A}$ be a Fell bundle over an étale groupoid H with locally compact Hausdorff object space X. Assume that the inclusion $A := C_0(\mathcal{A}|_X) \subseteq C^*(H, \mathcal{A})$ is aperiodic. Then*

(1) *A supports $C^*_{\mathrm{ess}}(H, \mathcal{A})$;*

(2) A detects ideals in $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$, and $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ is the only quotient of $\mathrm{C}^*(H, \mathcal{A})$ with this property;

(3) if $J \in \mathbb{I}(\mathrm{C}^*(H, \mathcal{A}))$ and $J \cap A = 0$, then $J \subseteq \ker(\mathrm{C}^*(H, \mathcal{A}) \to \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A}))$;

(4) $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ is simple if and only if the dual groupoid $\widehat{A} \rtimes H$ is minimal;

(5) if $\widehat{A} \rtimes H$ is minimal, then $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ is simple and purely infinite if and only if every element of $A^+ \backslash \{0\}$ is infinite in $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$.

Proof. We may apply Theorems 6.5 and 6.6 and Corollary 6.7, replacing $A \rtimes S$ and $A \rtimes_{\mathrm{ess}} S$ by $\mathrm{C}^*(H, \mathcal{A})$ and $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$. This implies all the statements. $\square$

The simplicity criterion above strengthens the criterion of Renault [55] by removing the Hausdorffness assumption, replacing $\mathrm{Prim}(A) \rtimes H$ by $\widehat{A} \rtimes H$ in the topological freeness assumption, and weakening the separability assumptions. In the group case, Theorem 7.25 implies Kishimoto's theorem that purely outer group actions on simple C^*-algebras are simple, whereas Renault's criterion says nothing about this situation. Theorems 7.25 and 7.24 combined with the criteria for $\mathrm{C}^*_{\mathrm{r}}(H, \mathcal{A}) = \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$ also imply the results in [13] about the simplicity of $\mathrm{C}^*_{\mathrm{r}}(H)$.

THEOREM 7.26. *Let H be a minimal, topologically free, étale groupoid with locally compact Hausdorff object space X. Let $\mathcal{L}$ be a Fell line bundle over H. Then $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ is simple. And $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ is purely infinite if and only if every element of $\mathrm{C}_0(X)$ is infinite in $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$.*
In particular, $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ is purely infinite if, for every non-empty open subset $U \subseteq X$ there are $n \in \mathbb{N}$, a non-empty open subset $V \subseteq U$, and bisections $t_1, \ldots, t_n \in \mathrm{Bis}(H)$ on which $\mathcal{L}$ is trivial such that

$$ r(t_i) \cap r(t_j) = \varnothing \quad \text{for } 1 \leqslant i < j \leqslant n, \qquad V = \bigcup_{i=1}^{n} s(t_i), \qquad \overline{\bigcup_{i=1}^{n} r(t_i)} \subsetneq V. $$

Proof. Theorem 7.24 implies that the inclusion $\mathrm{C}_0(X) \subseteq \mathrm{C}^*(H, \mathcal{L})$ is aperiodic because $\widehat{A} = X$. Then the inclusion $\mathrm{C}_0(X) \subseteq \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ is aperiodic as well. Now Theorem 7.25 implies the first part of the statement. It remains to check that the criterion in the second paragraph implies that any $f \in \mathrm{C}_0(X)^+ \backslash \{0\}$ is infinite in $B := \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ (see Definition 5.25).
Let S be the set of bisections in $\mathrm{Bis}(H)$ that trivialise $\mathcal{L}$. This is a unital, wide inverse subsemigroup, and we may identify $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ with the essential crossed product $\mathrm{C}_0(X) \rtimes_{\mathrm{ess}} S$ by the inverse semigroup action $\mathcal{E} = (\mathcal{E}_t)_{t \in S}$ associated to the Fell line bundle $\mathcal{L}$. Let $U := \{x \in X : f(x) \neq 0\}$. Choose $\varnothing \neq V \subseteq U$ and $t_1, \ldots, t_n \in S$ as in the statement of the theorem. Let $a = b \in \mathrm{C}_0(X)^+ \backslash \{0\}$ be any non-zero positive function that is supported in the open subset $V \backslash \overline{\bigcup_{i=1}^{n} r(t_i)}$. Then $a \precsim f$. Since $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ is simple, f is infinite if a is infinite (see the proof of Proposition 5.26). Let $w_1, \ldots, w_n \in \mathrm{C}_0(X)$

be a partition of unity subordinate to the open covering $V = \bigcup_{i=1}^{n} s(t_i)$. Let $a_i := a \cdot w_i^{1/2}$ for $i = 1, \ldots, n$. These functions vanish outside V, and a_i is supported in $s(t_i)$. Since $\mathcal{L}|_{t_i}$ is a trivial line bundle, a_i gives an element of $\mathcal{E}_{t_i}$, which we denote by $a_i \delta_{t_i}$. It belongs to $a \cdot \mathcal{E}_{t_i}$ by construction. The product $(a_i \delta_{t_i})^* a_j \delta_{t_j}$ is defined using the Fell bundle structure. It vanishes for $i \neq j$ because $r(t_i) \cap r(t_j) = \varnothing$. Similarly, $(a_i \delta_{t_i})^* a = 0$. And $\sum_{i=1}^{n} (a_i \delta_{t_i})^* a_i \delta_{t_i} = \sum_{i=1}^{n} a \cdot w_i = a$. Hence $x := \sum_{i=1}^{n} a_i \delta_{t_i}$ and $y := \sqrt{a}$ are elements of $a \cdot \mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$ such that $x^* x = a$, $y^* y = a \neq 0$ and $x^* y = 0$. Thus a is infinite in $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{L})$. In fact, the proof shows that a is properly infinite. $\square$

Remark 7.27. The condition in the second paragraph of Theorem 7.26 is satisfied, for instance, for the transformation groupoids of strongly boundary group actions (see [47]) and, more generally, for filling actions (see [30]). Hence the pure infiniteness results in [30, 47] are covered by Theorem 7.26.

For an étale, locally compact groupoid H, the condition of being *locally contracting* in [2] is the same as the condition in Theorem 7.26 with $n = 1$, without the trivialisation of the Fell line bundle. Thus $\mathrm{C}^*_{\mathrm{ess}}(H)$ is purely infinite and simple if H is a minimal, topologically free, étale, locally contracting groupoid with locally compact Hausdorff object space.

Now we specialise further to the trivial Fell bundle, which gives the groupoid C*-algebra $\mathrm{C}^*(H)$ without any twist. Then $\mathrm{C}^*(H) \cong \mathrm{C}_0(X) \rtimes S$ for any unital, wide inverse subsemigroup $S \subseteq \mathrm{Bis}(H)$, acting in the canonical way on $\mathrm{C}_0(X)$. We are going to show that $\mathrm{C}_0(X) \subseteq \mathrm{C}^*(H)$ has the generalised intersection property with essential quotient $\mathrm{C}^*_{\mathrm{ess}}(H)$ *if and only if* H is topologically free. One direction already follows from our general results. For the other direction, we construct the orbit representations of $\mathrm{C}^*(H)$. We construct them as covariant representations of the S-action on $\mathrm{C}_0(X)$. Let $x \in X$ and let $[x] := r(s^{-1}(x)) \subseteq X$ be the orbit of x. The *orbit representation of* $[x]$ takes place on the Hilbert space $\ell^2([x])$. Here $\mathrm{C}_0(X)$ acts by pointwise multiplication. If $U \in S$ is a bisection of H, then $f \in \mathrm{C}_0(U)$ acts on $\ell^2([x])$ by $\pi_U(f)\xi(r(\gamma)) := f(\gamma) \cdot \xi(s(\gamma))$ for all $\gamma \in U$ with $s(\gamma) \in [x]$, and $\pi_U(f)\xi(y) = 0$ for $y \in [x] \backslash r(U)$. Simple computations show that $\pi_U(f)^* = \pi_{U^*}(f^*)$, $\pi_U(f)\pi_V(g) = \pi_{UV}(f * g)$ for $U, V \in S$, $f \in \mathrm{C}_0(U)$, $g \in \mathrm{C}_0(V)$, and $\pi_U(f) = \pi_V(f)$ for $U \subseteq V$ and $f \in \mathrm{C}_0(U) \subseteq \mathrm{C}_0(V)$. Hence the maps π_U for $U \in S$ form a representation of the action of S on $\mathrm{C}_0(X)$. Then they induce a non-degenerate representation $\pi_{[x]}$ of $\mathrm{C}_0(X) \rtimes S$ on $\mathbb{B}(\ell^2[x])$. Let $\pi := \bigoplus_{x \in X} \pi_{[x]}$ be the direct sum of all these representations.

Lemma 7.28. *The representation π is faithful on $\mathrm{C}_0(X)$. If it factors through $\mathrm{C}^*_{\mathrm{ess}}(H, \mathcal{A})$, then H is topologically free.*

Proof. The algebra $\mathrm{C}_0(X)$ acts on $\ell^2([x])$ by pointwise multiplication. These representations are faithful when we sum over all $x \in X$. Assume that H is not topologically free. That is, there is a non-empty bisection $U \subseteq H \backslash \overline{X}$ with $r|_U = s|_U$. Since $U \cap \overline{X} = 0$, it follows that $EL(f) = 0$ for all $f \in \mathrm{C}_0(U)$. Choose any non-zero $f \in \mathrm{C}_c(U)$. Define $f_0 \in \mathrm{C}_c(s(U)) \subseteq \mathrm{C}_0(X)$ by $f_0(s(\gamma)) :=$

$f(\gamma)$ for all $\gamma \in U$. Since $r|_U = s|_U$, both $f \in C_0(U)$ and $f_0 \in C_0(X)$ act on $\ell^2([x])$ by pointwise multiplication with the same function f_0, for all $x \in X$. Hence $f - f_0 \in \ker \pi$. But $EL(f - f_0) = -f_0 \neq 0$. Hence $f - f_0 \notin \mathcal{N}_{EL}$. $\square$

THEOREM 7.29. *Let H be an étale groupoid with locally compact, Hausdorff unit space X. The inclusion $C_0(X) \subseteq C_0(X) \rtimes S = C^*(H)$ has the generalised intersection property with hidden ideal $\mathcal{N}_{EL}$ if and only if H is topologically free.*

Proof. The dual groupoid for the S-action on $C_0(X)$ is simply H, and $C_0(X)$ is of Type I. So Theorem 6.13 (or Theorem 7.24) shows that the action of S on $C_0(X)$ is aperiodic if and only if H is topologically free. Aperiodicity implies that $C_0(X) \subseteq C_0(X) \rtimes S = C^*(H)$ has the generalised intersection property with hidden ideal $\mathcal{N}_{EL}$. Conversely, if H is not topologically free, then Lemma 7.28 exhibits an ideal $\ker \pi$ with $\ker \pi \cap A = 0$, but $\ker \pi \nsubseteq \mathcal{N}_{EL}$. $\square$

The special case of Theorem 7.29 for transformation groups goes back to Kawamura–Tomiyama and Archbold–Spielberg (see [31, Theorem 4.1] and [4, Theorem 2]). The special case of Hausdorff groupoids with $C^*_r(H)$ instead of $C^*_{\text{ess}}(H)$ is similar to [5, Proposition 5.5]; replacing "topologically principal" by "topologically free" allows us to remove the second countability assumption, and replacing the reduced by the essential crossed product allows to remove the Hausdorffness assumption.

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