The Motivic Cofiber of τ

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Abstract. Consider the Tate twist $\tau \in H^{0,1}(S^{0,0})$ in the mod 2 cohomology of the motivic sphere. After 2-completion, the motivic Adams spectral sequence realizes this element as a map $\tau \colon S^{0,-1} \longrightarrow S^{0,0}$, with cofiber $C\tau$. We show that this motivic 2-cell complex can be endowed with a unique E_∞ ring structure. Moreover, this promotes the known isomorphism $\pi_{*,*}C\tau \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ to an isomorphism of rings which also preserves higher products.

We then consider the closed symmetric monoidal category of $C\tau$-modules $({}_{C\tau}\mathrm{Mod}, - \wedge_{C\tau} -)$ which lives in the kernel of Betti realization. Given a motivic spectrum X, the $C\tau$-induced spectrum $X \wedge C\tau$ is usually better behaved and easier to understand than X itself. We specifically illustrate this concept in the examples of the mod 2 Eilenberg-Maclane spectrum $H\mathbb{F}_2$, the mod 2 Moore spectrum $S^{0,0}/2$ and the connective hermitian K-theory spectrum kq.

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1 INTRODUCTION

1.1 THE SETTING

The mod 2 cohomology of the motivic sphere spectrum $S^{0,0}$ over $\mathrm{Spec}\,\mathbb{C}$ was computed by Voevodsky in [Voe03a], and is given by

$$H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{where } |\tau| = (0,1).$$

Denote the mod 2 motivic Steenrod algebra of operations $[H\mathbb{F}_2, H\mathbb{F}_2]_{*,*}$ by $\mathcal{A}$. One can run the motivic Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2[\tau], \mathbb{F}_2[\tau]) \Longrightarrow \pi_{*,*}\left((S^{0,0})^\wedge_2\right),$$

as constructed in [Mor99], [DI10], [HKO]. Observe that the E_2-page contains a non-trivial element in Adams filtration 0, namely multiplication by τ on $\mathbb{F}_2[\tau]$. This is different from the topological Adams spectral sequence for S^0, where the only elements in Adams filtration 0 are the identity map and the zero map. It is easy to see that this element survives to the E_∞-page as it cannot be involved with any differential for degree reasons. Therefore, it detects a map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^\wedge_2,$$

whose Hurewicz image is $\tau \in H\mathbb{F}_{2*,*}((S^{0,0})^\wedge_2)$.
As we will explain in diagram (2.4), this map does not exist if we don't 2-complete the target. We will thus work 2-completed. Recall that 2-completion is given by the E-Bousfield localization at either the Moore spectrum $S^{0,0}/2$ or the Eilenberg-Maclane spectrum $H\mathbb{F}_2$. In particular, the 2-completed sphere $L_E S^{0,0}$ is also an E_∞ ring spectrum and admits a good category of (2-completed) modules. We will from now on work in the 2-completed category, i.e., in modules over the 2-completed sphere. We will denote the 2-completed sphere and the smash product in 2-completed spectra simply by $S^{0,0}$ and $-\wedge-$.

With this notation, the motivic Adams spectral sequence produces a non-trivial map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$.

Recall that the Betti realization functor $\mathrm{Re}_{\mathbb{C}}$ goes from (here 2-completed) motivic spectra $\mathrm{SPT}_{\mathbb{C}}$ over $\operatorname{Spec}\mathbb{C}$ to classical (2-completed) spectra SPT. This functor is for example constructed in [DI10, 2.6], [PPR09, Appendix A.7] or [Joa, Chapter 4], and is induced by taking $\mathbb{C}$-points of the involved $\mathbb{C}$-schemes. It is a left adjoint, with right adjoint usually denoted Sing, and admits the constant functor c as a section [Lev14]. The situation is summarized in the diagram

$$\begin{tikzcd}
\mathrm{SPT}_{\mathbb{C}} \arrow[r, shift left, "\mathrm{Re}_{\mathbb{C}}"] & \mathrm{SPT}. \arrow[l, shift left, "\mathrm{Sing}"] \arrow[l, bend right=60, dashed, "c"']
\end{tikzcd}$$

The Betti realization functor $\mathrm{Re}_{\mathbb{C}}$ therefore induces a split-surjection

$$\pi_{s,w}(S^{0,0}) \twoheadrightarrow \pi_s(S^0),$$

with section induced by the constant functor c. Moreover, it sends the map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$ to the identity $S^0 \xrightarrow{\mathrm{id}} S^0$, as shown in [DI10, Section 2.6]. Computationally, the Betti realization functor $\mathrm{Re}_{\mathbb{C}}$ can thus be interpreted as sending the element τ to 1. For example, on the homotopy of the mod 2 Eilenberg-Maclane spectrum it induces the quotient map

$$\pi_{*,*}(H\mathbb{F}_2) \cong \mathbb{M}_2 \twoheadrightarrow \pi_*(H\mathbb{F}_2) \cong \mathbb{F}_2,$$

which imposes the relation $\tau = 1$. Observe that there is another surjection $\mathbb{M}_2 \twoheadrightarrow \mathbb{F}_2$ with same source and target, namely the quotient map imposing the relation $\tau = 0$. One can thus ask if this map is also induced by a functor between $\mathrm{SPT}_{\mathbb{C}}$ and another homotopy theory. To answer this question, we are led to study the homotopy theoretical analogue of the algebraic operation of setting $\tau = 0$, which is to take the cofiber of the map τ. Consider the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1}, \tag{1.1}$$

where we denote the cofiber of the map τ by $C\tau$. This 2-cell complex already appeared in [Isa], where it is studied via its motivic Adams-Novikov spectral sequence. More precisely, it is proven that its Adams-Novikov spectral sequence collapses at the E_2-page with no possible hidden extensions. This provides a surprising isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \cong \pi_{*,*}(C\tau), \tag{1.2}$$

connecting two objects which are a priori unrelated. The left hand side is the cohomology of the classical (non-motivic) Hopf algebroid (BP_*, BP_*BP) and is very important in chromatic homotopy theory. In particular, it is the E_2-page of the Adams-Novikov spectral sequence for the topological sphere S^0. Notice that since it is the cohomology of a dga, namely the cobar complex associated to (BP_*, BP_*BP), it admits products and higher Massey products. All this algebraic structure gets transferred to the motivic homotopy groups $\pi_{*,*}(C\tau)$, formally endowing it with a (higher) ring structure. One can thus ask if this algebraic ring structure can be lifted to a topological ring structure on $C\tau$. The first goal of this paper is to answer this question, which we do in Section 3 by the following results.

THEOREM 1.1. *There exists a unique E_∞ ring structure on $C\tau$.*

We now explain what we mean by a motivic E_∞ ring spectrum, and refer to Section 3.1 for more details. Since $\mathrm{SPT}_{\mathbb{C}}$ is enriched over simplicial sets, one can talk about algebras over operads in simplicial sets. In fact, operads in simplical sets embedded in the motivic world are sometimes called constant operads. In this paper, we say that a motivic spectrum admits an E_∞ ring structure if it admits an algebra structure over a constant E_∞ operad, i.e., over any usual E_∞ operad in simplicial sets. We warn the reader that similarly to the equivariant case of [BH15], this notion of motivic E_∞ ring spectra is probably not the same as strictly commutative algebras in $\mathrm{SPT}_{\mathbb{C}}$.

There are two main tools involved in proving Theorem 1.1. We first use elementary techniques with triangulated categories to produce a unital, associative and commutative monoid in the homotopy category $\mathrm{Ho}(\mathrm{SPT}_{\mathbb{C}})$. We then rigidify this ring structure using Robinson's E_∞ obstruction theory [Rob03]. By tracing back to the origin of the isomorphism (1.2), we can now show that the algebraic structure on $\pi_{*,*}(C\tau)$ does come from $C\tau$.

PROPOSITION 1.2. *The isomorphism* (1.2)

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$$

is an isomorphism of rings which sends Toda brackets in $\pi_{*,*}$ to Massey products in Ext*, and vice-versa.*

Let's point out that the additive version of this theorem was already exploited by Isaksen in [Isa] to gain knowledge about the classical Adams-Novikov E_2-page. The idea is to compute $\pi_{*,*}(C\tau)$ in a range using its motivic Adams spectral sequence and the knowledge of $\pi_{*,*}(S^{0,0})$ in this range. Having a multiplicative structure available improves the correspondence in an obvious manner.

Having considered the cofiber of multiplication by τ, one can look at the less severe quotients $S^{0,0}/\tau^n =: C\tau^n$ of multiplication by τ^n. All together, these spectra sit in a tower of spectra

$$\begin{tikzcd}
C\tau & C\tau^2 \arrow[l] & C\tau^3 \arrow[l] & C\tau^4 \arrow[l] & \cdots \arrow[l] & (S^{0,0})^{\wedge}_{\tau}. \arrow[l] \\
 & \Sigma^{0,-1}C\tau \arrow[u] & \Sigma^{0,-2}C\tau \arrow[u] & \Sigma^{0,-3}C\tau \arrow[u] & &
\end{tikzcd} \tag{1.3}$$

Observe that the completion map

$$S^{0,0} \xrightarrow{\simeq} (S^{0,0})^{\wedge}_{\tau}$$

is a weak equivalence as it induces an isomorphism on homotopy groups. More precisely, its fiber is given by the formula $\operatorname{holim}_k S^{0,-k}$, whose homotopy groups can be computed using Milnor's $\lim^1$ exact sequence

$$0 \longrightarrow \lim_k{}^1 \pi_{s+1,w}(S^{0,-k}) \longrightarrow \pi_{s,w}(\operatorname{holim}_k S^{0,-k}) \longrightarrow \lim_k \pi_{s,w}(S^{0,-k}) \longrightarrow 0.$$

In any fixed bigrading $\pi_{s,w}$, both algebraic limits eventually become 0, as proven by Morel's vanishing result [Mor05], and can be seen explicitly in [GI17, Figure 1]. This shows that the middle group is also zero, and thus $S^{0,0} \longrightarrow (S^{0,0})^{\wedge}_{\tau}$ becomes a weak equivalence.

This show that the tower (1.3) reconstructs the sphere spectrum $S^{0,0}$, and that the spectrum $C\tau^n$ gets closer and closer to $S^{0,0}$ as n increases. This hints to the fact that every $C\tau^n$ should be an E_∞ ring spectrum and similarly to the natural reduction map $C\tau^n \longrightarrow C\tau^{n-1}$. Using similar techniques as in Theorem 1.2, one can show that every spectrum $C\tau^n$ is uniquely an A_∞ ring spectrum, and that it is homotopy commutative. Our method does not apply to show that $C\tau^n$ is E_∞ for any n, as the necessary obstruction groups do not vanish.

One can also consider the τ-Bockstein spectral sequence for $S^{0,0}$, which is the spectral sequence induced by applying $\pi_{*,*}$ to the tower (1.3). Surprisingly, this spectral sequence contains the same information as the motivic Adams-Novikov spectral sequence computing $\pi_{*,*}(S^{0,0})$. In fact, the E_1-page of the τ-Bockstein spectral sequence is isomorphic to the E_2-page of the motivic Adams-Novikov spectral sequence. Moreover, by [HKO11, Lemma 15], the motivic Adams-Novikov spectral sequence has only odd differentials, which are all of the form $d_{2r+1}(x) = \tau^r y$. Such a differential corresponds to a d_r differential of the τ-Bockstein spectral sequence, giving a one-to-one correspondence between the differentials of each spectral sequence. This implies that the E_{2r+2} page of the motivic Adams-Novikov spectral sequence is isomorphic to the E_{r+1} page of the τ-Bockstein spectral sequence. Following the referee's suggestion, observe that this situation is exactly analogous to the situation in [Lev15], since the τ-Bockstein spectral sequence is going twice as fast as the motivic Adams-Novikov spectral sequence. We strongly believe that this isomorphism of spectral sequence can be made more precise by using Levine's décalage's Theorem [Lev15, Proposition 4.3].

With an E_∞ ring structure in hand, any good model for motivic spectra produces a closed symmetric monoidal category of $C\tau$-modules with the relative smash product $- \wedge_{C\tau} -$, and a free-forget adjunction

$$\begin{tikzcd} \mathrm{Spt}_{\mathbb{C}} \arrow[r, shift left, "- \wedge C\tau"] \arrow[r, phantom, "\perp"] & {}_{C\tau}\mathrm{Mod}. \arrow[l, shift left] \end{tikzcd} \tag{1.4}$$

The remainder of this paper is devoted to the task of better understanding the category ${}_{C\tau}\mathrm{Mod}$. In Lemma 5.1 we show that the Betti realization of any $C\tau$-module is contractible, which means that the category of $C\tau$-modules lies in the kernel of Betti realization. This does not mean that the motivic spectrum $C\tau$ does not have topological applications, as there are other bridges between motivic and classical homotopy theory. Such a bridge is for example given by Proposition 1.2, relating the homotopy groups of the motivic spectrum $C\tau$ with the cohomology of the Hopf algebroid (BP_*, BP_*BP).

One strength of the category ${}_{C\tau}\mathrm{Mod}$ is that it is relatively easy to work with $C\tau$-modules. One first observes this phenomenon during the process of proving that $C\tau$ admits an E_∞ ring structure, with the many obstruction groups vanishing for degree reasons. We observe a similar phenomenon with related motivic spectra. For example, we can completely describe the ring spectra $C\tau \wedge C\tau$ and $\mathrm{End}(C\tau)$ by using elementary techniques. In joint work with Zhouli Xu and Guozhen Wang [GWX], we provide an equivalence between some category of (cellular) $C\tau$-modules and some category of derived BP_*BP-comodules. In particular, this implies that the homotopy category of cellular $C\tau$-modules is algebraic in the sense of [Sch10]. This is another reason why it feels easier to manipulate motivic spectra living in ${}_{C\tau}\mathrm{Mod}$, since algebraic categories are usually better behaved than topological categories. For example, algebraic categories admit a $\mathcal{D}(\mathbb{Z})$-enrichment which implies many pleasant properties. We refer the reader to Remark 5.10 for a concrete such example, and to [Sch10] for more details.

The category ${}_{C\tau}\mathrm{Mod}$ is the universal place in which the element τ has been killed. The strength of this benign statement lies in the fact that many motivic spectra naturally land in ${}_{C\tau}\mathrm{Mod}$, since at some point we were led to mod out by τ for one reason or another. For example, the relation $\tau\eta^4 = 0 \in \pi_{*,*}$ implies that the η-inverted sphere $S^{0,0}[\eta^{-1}]$, computed in [GI15] and [AM], lives in ${}_{C\tau}\mathrm{Mod}$. More generally, one can show that any element $x \in \pi_{s,w}$ with $s \neq 0$ admits a relation of the type $\tau^a x^b = 0$. In particular, inverting any such non-nilpotent element x when $a = 1$ yields a spectrum that naturally lives in ${}_{C\tau}\mathrm{Mod}$. In particular, this phenomenon applies to the exotic Morava K-theories $K(w_n)$ of [Ghe], detecting the motivic w_n-periodicity introduced in [And] by Michael Andrews and Haynes Miller.

We now describe the last Section of this paper, where we explicitly compute the homotopy of some specific $C\tau$-modules induced through the adjunction (1.4).

Given a spectrum X, we call the induced $C\tau$-module $X \wedge C\tau$ a $C\tau$-induced spectrum.

One of the first spectra to understand in ${}_{C\tau}\mathrm{Mod}$ is the $C\tau$-induced mod 2 Eilenberg-Maclane spectrum. This is the spectrum $H\mathbb{F}_2 \wedge C\tau$ that we denote by $\overline{H\mathbb{F}}_2$, which we will treat as a cohomology theory. Given any $C\tau$-module X one can consider the $C\tau$-linear mod 2 (co)homology of X defined by the homotopy of the spectra

$$F_{C\tau}(X, \overline{H\mathbb{F}}_2) \qquad \text{and} \qquad X \wedge_{C\tau} \overline{H\mathbb{F}}_2.$$

Here $F_{C\tau}(-,-)$ denotes the $C\tau$-linear function spectrum and $- \wedge_{C\tau} -$ denotes the relative smash product in ${}_{C\tau}\mathrm{Mod}$. Observe for example that since

$$X \wedge_{C\tau} \overline{H\mathbb{F}}_2 = X \wedge_{C\tau} (H\mathbb{F}_2 \wedge C\tau) \simeq X \wedge_{C\tau} C\tau \wedge H\mathbb{F}_2 \simeq X \wedge H\mathbb{F}_2,$$

the $C\tau$-linear $\overline{H\mathbb{F}}_2$-homology of X is isomorphic to the $H\mathbb{F}_2$-homology of the underlying spectrum of X. Consider the $C\tau$-linear (Steenrod) algebra of $\overline{H\mathbb{F}}_2$-(co-)operations, given by the homotopy of the spectra

$$F_{C\tau}(\overline{H\mathbb{F}}_2, \overline{H\mathbb{F}}_2) \qquad \text{and} \qquad \overline{H\mathbb{F}}_2 \wedge_{C\tau} \overline{H\mathbb{F}}_2.$$

These are the relevant (co-)operations acting on the $C\tau$-linear (co)homology of $C\tau$-modules, which we compute in Section 5.2. Recall that the dual mod 2 motivic Steenrod algebra over $\operatorname{Spec}\mathbb{C}$ is given by

$$\pi_{*,*}(H\mathbb{F}_2 \wedge H\mathbb{F}_2) \cong \mathbb{F}_2[\tau][\xi_1, \xi_2, \ldots, \tau_0, \tau_1, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}\,.$$

See Section 2.2 for more details. The following computation follows easily.

Proposition 1.3. *The Hopf algebra of $C\tau$-linear co-operations of $H\mathbb{F}_2$ is given by*

$$\mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots).$$

One can also as usual consider $\overline{H\mathbb{F}}_2$ as a (co)homology theory on $\mathrm{Spt}_{\mathbb{C}}$, and define the $\overline{H\mathbb{F}}_2$-homology and cohomology of any motivic spectrum X by the homotopy of the spectra

$$F(X, \overline{H\mathbb{F}}_2) \qquad \text{and} \qquad X \wedge \overline{H\mathbb{F}}_2.$$

The associated (co-)operations acting on the $\overline{H\mathbb{F}}_2$-(co-)homology of any spectrum is given by the homotopy of the spectra

$$F(\overline{H\mathbb{F}}_2, \overline{H\mathbb{F}}_2) \qquad \text{and} \qquad \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2.$$

We also compute these Hopf algebras in Section 5.2, in particular we get the following.

Proposition 1.4. *The Hopf algebra of co-operations of $\overline{H\mathbb{F}}_2$ is given by*

$$\mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots) \otimes E(\beta_\tau).$$

The extra co-operation β_τ is primitive in the coalgebra structure. It is induced by the τ-Bockstein corresponding to the composite

$$C\tau \xrightarrow{p} S^{1,-1} \xrightarrow{i} \Sigma^{1,-1}C\tau$$

of the projection of $C\tau$ on its top cell $S^{1,-1}$, followed by the inclusion as its bottom cell. It also appears in the Hopf algebra of operations, where it really deserves its name of τ-Bockstein. More precisely, the operation β_τ in cohomology is to τ as the usual Bockstein $\beta = \mathrm{Sq}^1$ is to the element 2. In particular, it allows to reconstruct $H\mathbb{F}_2$ via a motivic analogue of the Postnikov tower that runs in the weight direction. At each stage, the layer is a copy of $\overline{H\mathbb{F}_2}$, and the boundary map composed with the k-invariant is given by β_τ. However, if one is only interested in studying $C\tau$-modules internally to the category ${}_{C\tau}\mathrm{MOD}$, this is a noisy element and one should use $C\tau$-linear (co)homology.

Given a spectrum $X \in \mathrm{SPT}_{\mathbb{C}}$, the homotopy groups of the $C\tau$-induced spectrum $X \wedge C\tau$ are an extension of the τ-torsion and the residue mod τ of $\pi_{*,*}(X)$. Even though this proves to be very complicated in general, one principle that appears is the following. If all the obstructions to the spectrum $X \in \mathrm{SPT}_{\mathbb{C}}$ possessing some property or structure are τ-torsion, then the $C\tau$-induced spectrum $X \wedge C\tau$ posses the desired property or structure. Here are a few such examples that we study in Section 5.

Start with the 2-completed motivic mod 2 Moore spectrum $S^{0,0}/2$. The Toda bracket $\langle 2, \eta, 2\rangle \ni \tau\eta^2$ is the obstruction to both endowing it with a left unital multiplication, and to a v_1^1-self map. In Theorem 5.9, we will prove the following results about the $C\tau$-induced Moore spectrum, which we denote by $S/(2,\tau)$.

THEOREM 1.5. *The $C\tau$-induced motivic mod 2 Moore spectrum $S/(2,\tau)$ admits a unique structure of an E_∞ $C\tau$-algebra.*

PROPOSITION 1.6. *The $C\tau$-induced motivic mod 2 Moore spectrum $S/(2,\tau)$ admits a v_1^1-self map*

$$\Sigma^{2,1}S/(2,\tau) \xrightarrow{v_1} S/(2,\tau).$$

We also study the 2-completed connective[1] algebraic and hermitian K-theory spectra kgl and kq. Denote their $C\tau$-induced spectra by $\overline{kgl} := kgl \wedge C\tau$ and $\overline{kq} := kq \wedge C\tau$. The case of algebraic K-theory is very simple as both its (co)homology and homotopy are τ-free. For the case of the connective hermitian K-theory spectrum (constructed over $\operatorname{Spec}\mathbb{C}$ in [IS11], where it is denoted ko) we prove the following result.

PROPOSITION 1.7. *The $C\tau$-induced connective hermitian K-theory spectrum $\overline{kq}$ has homotopy groups*

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[v_1^2, \eta] \big/ 2\eta.$$

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[1] in the sense of [IS11, Definiton 4.9 and 4.11].

Recall that the homotopy of the motivic spectrum kq contains the 8-fold Bott periodicity element v_1^4, but does not contain v_1^2. One can compute the homotopy of kq and $\overline{kq}$ from their cohomology via the motivic May spectral sequence, followed by the motivic Adams spectral sequence. In the case of kq, there is a motivic May differential supported by v_1^2 and with τ-torsion target. In the case of $\overline{kq}$, the τ-torsion target of this differential gets shifted. We then resolve a hidden extension of this shifted element to show that it is in fact the periodicity element v_1^2. More precisely, we show that this element is a square root of the usual 8-fold Bott periodicity, making the $C\tau$-induced spectrum $\overline{kq}$ 4-fold periodic. In chromatic motivic language, up to the v_0-extensions this can be rewritten as $\mathbb{F}_2[v_0, v_1^2, w_0] \big/ v_0 w_0$. The relation $v_0 w_0 = 0$ is clear as it is already present in $\pi_{*,*}(S^{0,0})$, but this shows that v_1^2 and w_0 can coexist without any relation between them.

1.2 The Choice of Prime $p = 2$

This paper is written in a p-completed setting, where we chose the prime $p = 2$. However, the main results also apply to odd primes. In short, the $H\mathbb{F}_p$-based motivic Adams spectral sequence produces the map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^{\wedge}_p,$$

after p-completing the target for any prime p. Denote again its cofiber by $C\tau$, where the prime p does not appear in the notation. The isomorphism

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*) \cong \pi_{*,*}(C\tau)$$

still holds for any prime, producing the same vanishing regions in the homotopy of $C\tau$, and thus endowing $C\tau$ with an E_∞ ring structure.

For odd primes p, the motivic story is somehow easier since it is more closely related to the classical story. In particular, in the case of odd primes, the motivic Steenrod algebra (and its dual) are isomorphic as Hopf algebras to the classical Steenrod algebra (and its dual), with an extra primitive formal variable τ. This is not the case for $p = 2$, for example because of the relation $\tau_i^2 = \tau\xi_{i+1}$ in the dual motivic Steenrod algebra.

1.3 Organization

Section 2. This Section contains a brief summary of the motivic homotopy theory needed in order to define the spectrum $C\tau$. This contains a recall of the motivic category of spectra over $\operatorname{Spec}\mathbb{C}$, some functors relating $\mathrm{Spt}_{\mathbb{C}}$ with the topological category Spt, the mod 2 motivic cohomology, the structure of the mod 2 motivic Steenrod algebra and its dual, and the motivic Adams spectral sequence. After introducing the spectrum $C\tau$, we explain some vanishing regions both in its homotopy groups $\pi_{*,*}(C\tau)$ and in the homotopy classes of self-maps $[C\tau, C\tau]_{*,*}$. These results will be mostly used to endow $C\tau$ with an E_∞ ring structure.

SECTION 3. We first explain the notion of motivic A_∞ and E_∞ ring spectra that we will use in this paper, and adapt Robinson's obstruction theory [Rob03] to the motivic setting. We then apply this obstruction theory to endow the spectrum $C\tau$ with an E_∞ ring structure.

SECTION 4. In this Section we compute the homotopy types of the E_∞ ring spectrum $C\tau \wedge C\tau$ and of the A_∞ ring spectrum $\mathrm{End}(C\tau)$.

SECTION 5. This Section is about the symmetric monoidal category ${}_{C\tau}\mathrm{MOD}$. We start by showing some generalities on $C\tau$-modules. We then analyze more precisely a few specific $C\tau$-induced spectra:

(1) We compute the Steenrod algebra of operations and co-operations on the $C\tau$-induced mod 2 Eilenberg-Maclane spectrum $H\mathbb{F}_2 \wedge C\tau$.

(2) We show that the $C\tau$-induced mod 2 Moore spectrum $S/(2, \tau)$ admits a unique E_∞ structure as a $C\tau$-algebra, and that it admits a v_1^1-self map.

(3) We compute the homotopy groups of the $C\tau$-induced connective algebraic and hermitian K-theories $kgl \wedge C\tau$ and $kq \wedge C\tau$. In particular, a hidden extension shows that $kq \wedge C\tau$ contains a 4-fold periodicity by the element v_1^2, which is the square root of the usual 8-fold Bott periodicity observed in kq.

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2 NOTATION AND BACKGROUND ON $C\tau$

In this Section we give some brief background on motivic homotopy theory over $\mathrm{Spec}\,\mathbb{C}$ as well as properly introduce the spectrum $C\tau$. For a more detailed introduction to motivic homotopy theory we refer the reader to [Mor05], [MV99]. Most of our notation agrees with and is taken from [Isa].

2.1 MOTIVIC SPACES AND SPECTRA OVER $\mathrm{Spec}\,\mathbb{C}$

Denote by $\mathrm{SPC}_{\mathbb{C}}$ the category of *(pointed) motivic spaces* over $\mathrm{Spec}\,\mathbb{C}$ as defined in [MV99]. This category is endowed with a well-behaved $\mathbb{A}^1$-invariant homotopy theory, for example in the form of a closed symmetric monoidal, proper, simplicial and cellular model structure. The paper [Pel11, Chapter 2] is a good source for a careful construction of these model structures. There is a realization functor

$$\mathrm{Spc}_{\mathbb{C}} \xrightarrow{\mathrm{Re}_{\mathbb{C}}} \mathrm{Top},$$

from motivic spaces over $\operatorname{Spec}\mathbb{C}$ to topological spaces called *Betti realization*. This functor is for example constructed in [DI10, 2.6], [PPR09, Appendix A.7] or [Joa, Chapter 4], and is induced by taking $\mathbb{C}$-points of the involved $\mathbb{C}$-schemes. It is a strict symmetric monoidal left Quillen functor, whose right adjoint is usually denoted by Sing. In the same spirit as equivariant homotopy theory, motivic homotopy theory has two different types of spheres. We will denote the 1-dimensional simplicial sphere by $S^{1,0} \in \mathrm{Spt}_{\mathbb{C}}$ and the geometric sphere $\mathbb{G}_m$ by $S^{1,1} \in \mathrm{Spt}_{\mathbb{C}}$. The first coordinate m in the notation $S^{m,n}$ indicates the *topological dimension* of the sphere, and it is not hard to see that it Betti realizes to the topological sphere S^m. The second coordinate n indicates the *weight*, or the *Tate twist* of the sphere $S^{m,n}$. Over $\operatorname{Spec}\mathbb{C}$, the projective line $\mathbb{P}^1$ is a 2-dimensional topological sphere, whose homotopy type is described by the equation

$$\mathbb{P}^1 \simeq S^{1,0} \wedge S^{1,1} \simeq S^{2,1}. \tag{2.1}$$

The category of *motivic* ($\mathbb{P}^1$-)*spectra* $\mathrm{Spt}_{\mathbb{C}}$ over $\operatorname{Spec}\mathbb{C}$ is constructed by stabilizing with respect to the sphere $\mathbb{P}^1$, i.e., inverting the functor $- \wedge \mathbb{P}^1$. Observe that equation (2.1) implies that this is equivalent to inverting smashing with both fundamental spheres $- \wedge S^{1,0}$ and $- \wedge S^{1,1}$. This provides a bigraded suspension functor that we denote by $\Sigma^{m,n} = - \wedge S^{m,n}$. Smashing with the simplicial sphere $\Sigma = \Sigma^{1,0} = - \wedge S^{1,0}$ corresponds to the shift functor of the triangulated structure on the homotopy category. The category of motivic spectra $\mathrm{Spt}_{\mathbb{C}}$ also supports good model structures which are closed symmetric monoidal with respect to the smash product $- \wedge -$, proper, simplicial and cellular. The paper [Pel11, Chapter 2] constructs these models in details. Moreover, the realization and singular pair stabilizes to a Quillen adjunction[2]

$$\begin{tikzcd} \mathrm{Spt}_{\mathbb{C}} \arrow[r, shift left, "\mathrm{Re}_{\mathbb{C}}"] \arrow[r, phantom, "\perp"] & \mathrm{Spt}, \arrow[l, shift left, "\mathrm{Sing}"] \end{tikzcd}$$

where the Betti functor $\mathrm{Re}_{\mathbb{C}}$ is strict symmetric monoidal, see for example [PPR09, A.45].

Given two spectra $X, Y \in \mathrm{Spt}_{\mathbb{C}}$, the closed symmetric monoidal structure provides a *function motivic spectrum* that we denote by $F(X,Y) \in \mathrm{Spt}_{\mathbb{C}}$. When $X = Y$, we will usually write $\mathrm{End}(X) = F(X,X)$. As usual, we will denote the abelian group of homotopy classes of maps between X and Y by $[X,Y]$. When the source spectrum is a sphere $X = S^{s,w}$, the abelian group

$$\pi_{s,w}(Y) := [S^{s,w}, Y]$$

---

[2] Since the Betti realization of $\mathbb{P}^1$ is the topological sphere $\mathbb{P}^1(\mathbb{C}) \simeq S^2$, taking $\mathbb{C}$-points lands in the category of S^2-spectra, i.e., spectra with bonding maps $S^2 \wedge X_n \longrightarrow X_{n+1}$. This is also a model for stable homotopy theory, see [Joa, Section 4.1] for more details.

is called the homotopy group of Y in *stem* s and *weight* w. The relation between the two is given by the usual adjunction between the smash product and the function spectrum. After taking homotopy, this becomes the equation

$$\pi_{s,w}(F(X,Y)) \cong [\Sigma^{s,w}X, Y].$$

2.2 THE MOTIVIC ADAMS SPECTRAL SEQUENCE AND THE ELEMENT τ

Denote by $H\mathbb{Z}$ Voevodsky's motivic Eilenberg-Maclane spectrum representing integral motivic cohomology on schemes [Voe98, Section 6.1]. Denote by $H\mathbb{F}_2$ the cofiber of multiplication by 2 on $H\mathbb{Z}$, which sits in the cofiber sequence

$$H\mathbb{Z} \xrightarrow{\cdot 2} H\mathbb{Z} \longrightarrow H\mathbb{F}_2.$$

The spectrum $H\mathbb{F}_2$ represents mod 2 motivic cohomology on schemes. The coefficients of this spectrum were computed in [Voe03a] and are given by

$$H\mathbb{F}_{2*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{for } |\tau| = (0,-1).$$

Dually, the motivic cohomology of a point is

$$H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{for } |\tau| = (0,1),$$

where we abuse notation and use the same symbol τ to denote the Tate twist element in homology and its dual in cohomology. We use the same notation as in [Isa] for the coefficients

$$\mathbb{M}_2 := H\mathbb{F}_2{}^{*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau] \qquad \text{and} \qquad \mathbb{M}_2^\vee := H\mathbb{F}_{2*,*}(S^{0,0}) \cong \mathbb{F}_2[\tau].$$

We write $\mathcal{A}$ for the mod 2 motivic Steenrod algebra, i.e., the ring of stable cohomology operations on the motivic spectrum $H\mathbb{F}_2$. Its structure has been computed by Voevodsky in [Voe03b], [Voe10] : it is the bigraded Hopf algebra over $\mathbb{M}_2$ given by

$$\mathcal{A} \cong \mathbb{M}_2\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots\rangle \big/ \text{Adem relations}\cdot$$

Observe that as in topology, it is generated by the Steenrod squares Sq^n with the Adem relations between them. The Tate twist $\tau \in \mathbb{M}_2$ has bidegree $|\tau| = (0,1)$, and the Steenrod squares have bidegrees $|\mathrm{Sq}^{2n}| = (2n, n)$ and $|\mathrm{Sq}^{2n+1}| = (2n+1, n)$. Since we work at $p = 2$, the first square $\mathrm{Sq}^1 = \beta$ is again the usual Bockstein operation coming from the short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

The dual motivic Steenrod algebra

$$\mathcal{A}^\vee \cong \mathbb{M}_2^\vee[\xi_1, \xi_2, \ldots, \tau_0, \tau_1, \ldots] \Big/ \tau\xi_{i+1} = \tau_i^2, \tag{2.2}$$

was also computed by Voevodsky in [Voe03b]. Because we are now in homology, the Tate twist $\tau \in \mathbb{M}_2^\vee$ has bidegree $|\tau| = (0, -1)$. The ξ_i's and τ_i's have bidegrees $|\xi_i| = (2^{i+1} - 2, 2^i - 1)$ and $|\tau_i| = (2^{i+1} - 1, 2^i - 1)$. The coproduct is given by the formulas

$$\Delta(\xi_n) = \sum \xi_{n-k}^{2^k} \otimes \xi_k \qquad \text{and} \qquad \Delta(\tau_n) = \tau_n \otimes 1 + \sum \xi_{n-k}^{2^k} \otimes \tau_k.$$

One can now run the motivic Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2) \Longrightarrow \pi_{*,*}((S^{0,0})^{\wedge}_2)$$

constructed in [Mor99], [DI10], [HKO], that converges to the homotopy groups of the 2-completed motivic sphere $(S^{0,0})^{\wedge}_2$. The $\mathcal{A}$-module map

$$\mathbb{M}_2 \xrightarrow{\cdot\tau} \mathbb{M}_2$$

is an element in $\mathrm{Hom} = \mathrm{Ext}^0$ of Adams filtration 0 as τ is central in $\mathcal{A}$. This element survives to the E_∞-page as it cannot be involved with any differential for degree reasons. Therefore, it detects a map

$$S^{0,-1} \xrightarrow{\tau} (S^{0,0})^{\wedge}_2, \tag{2.3}$$

whose Hurewicz image is the element $\tau \in H\mathbb{F}_{2*,*}((S^{0,0})^{\wedge}_2)$.
Unfortunately, the map of equation (2.3) does not lift to a map before 2-completing the target. The situation can be summarized by the following commutative diagram

$$\begin{tikzcd}
S^{0,-1} \arrow[dr, "\tau"] \arrow[drr, dashed, bend left, "\exists\ \tau"] \arrow[ddr, dashed, bend right, "\nexists"'] & & \\
 & H\mathbb{F}_2 & (S^{0,0})^{\wedge}_2 \arrow[l] \\
 & H\mathbb{Z} \arrow[u] & S^{0,0}. \arrow[l] \arrow[u]
\end{tikzcd} \tag{2.4}$$

The top dotted arrow corresponds to the element (2.3) constructed by the motivic Adams spectral sequence, and the non-existence of the bottom dotted arrow shows that τ does not lift to a map $S^{0,-1} \longrightarrow S^{0,0}$. In fact, a map $S^{0,-1} \longrightarrow H\mathbb{Z}$ corresponds to a cohomology class in the group $H^{0,1}_{\mathrm{mot}}(\mathrm{Spec}\,\mathbb{C}; \mathbb{Z})$, which vanishes. Here is another argument kindly suggested by the referee, to explain the existence of the map (2.3). This element comes from the Tor spectral sequence for the 2-completed sphere, whenever there is an infinitely 2-divisible element in the Milnor-Witt K-theory $K_1^{\mathrm{MW}}(\mathrm{Spec}\,\mathbb{C})$, i.e., when the base field has all 2-power roots of unity.

It is crucial for us that this element τ exists in the homotopy groups of the motivic sphere spectrum, and thus acts on the homotopy of any motivic spectrum. Recall that 2-completion is given by the E-Bousfield localization at either the Moore spectrum $S^{0,0}/2$ or the Eilenberg-Maclane spectrum $H\mathbb{F}_2$. In particular, the 2-completed sphere $(S^{0,0})^\wedge_2$ is also an E_∞ ring spectrum and admits a good category of (2-completed) modules. Denote temporarily its category of modules by $\widehat{\mathrm{Spt}_{\mathbb{C}}}$. The ring map $S^{0,0} \longrightarrow (S^{0,0})^\wedge_2$ induces a forgetful functor

$$\begin{tikzcd}\mathrm{Spt}_{\mathbb{C}} & \widehat{\mathrm{Spt}_{\mathbb{C}}} \arrow[l]\end{tikzcd}$$

from 2-completed motivic spectra to motivic spectra. As explained in [Pel11, Section 2.8], this forgetful functor creates a symmetric monoidal model structure on $\widehat{\mathrm{Spt}_{\mathbb{C}}}$. Moreover, as indicated in the diagram

$$\begin{tikzcd}\mathrm{Spt}_{\mathbb{C}} \arrow[r, shift left=2, "- \wedge (S^{0,0})^\wedge_2"] \arrow[r, shift right=2, "F\left((S^{0,0})^\wedge_2, -\right)"'] & \widehat{\mathrm{Spt}_{\mathbb{C}}}, \arrow[l]\end{tikzcd}$$

it is both a left and right Quillen functor via the usual adjunctions. It follows that the forgetful functor preserves all categorical constructions in $\widehat{\mathrm{Spt}_{\mathbb{C}}}$, i.e., the underlying spectrum of any (co)limit is computed in the underlying category of motivic spectra $\mathrm{Spt}_{\mathbb{C}}$. On finite spectra, 2-completing and smashing with the 2-completed sphere $(S^{0,0})^\wedge_2$ are equivalent functors. In this paper, we will only be concerned with finite spectra, and will from now on exclusively work in $\widehat{\mathrm{Spt}_{\mathbb{C}}}$ without further mention, and drop the completion symbol from the notation. For example, we will denote this category by $\mathrm{Spt}_{\mathbb{C}}$, the 2-completed motivic sphere spectrum by $S^{0,0}$, the smash product over the 2-completed sphere by $- \wedge -$, ...etc. With this notation, the motivic Adams spectral sequence produces a non-trivial map

$$S^{0,-1} \xrightarrow{\tau} S^{0,0},$$

which we can see as being an element in the homotopy groups $\pi_{0,-1}(S^{0,0})$.

2.3 The Spectrum $C\tau$ and its Homotopy

Recall that we work in a 2-completed setting. Define the 2-cell complex $C\tau$ by the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}, \tag{2.5}$$

where i denotes the inclusion of its bottom cell and p is the projection on its top cell. Recall from [DI10, Section 2.6] that the Betti realization functor $\mathrm{Spt}_{\mathbb{C}} \longrightarrow \mathrm{Spt}$ sends the map τ to the identity id, as shown in the diagram

$$\left(S^{0,-1} \xrightarrow{\tau} S^{0,0}\right) \longmapsto \left(S^{0} \xrightarrow{\mathrm{id}} S^{0}\right).$$

Moreover, it is a left Quillen functor and thus preserves cofiber sequences. This implies that it sends $C\tau$ to a contractible spectrum $* \in$ Top and thus that $C\tau$ is a purely motivic spectrum living in the kernel of Betti realization. Nonetheless, the motivic spectrum $C\tau$ has very tight connections to classical (non-motivic) homotopy theory. Surprisingly, a computation of Hu-Kriz-Ormsby in [HKO11], allows Isaksen in [Isa] to express the homotopy groups of this 2-cell complex $\pi_{*,*}(C\tau)$ in terms of the classical Adams-Novikov spectral sequence. Denote by $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*)$ the E_2-page of the classical (2-completed) Adams-Novikov spectral sequence for the topological sphere S^0, where as usual s is the Adams filtration and t is the internal degree.

Proposition 2.1 ([Isa, Proposition 6.2.5]). *The homotopy groups of $C\tau$ are given by*

$$\pi_{s,w}(C\tau) \cong \mathrm{Ext}^{2w-s,2w}_{BP_*BP}(BP_*, BP_*) \qquad \textit{for any } s, w \in \mathbb{Z}.$$

Remark 2.2. Proposition 2.1 is surprising as it is saying that the homotopy groups of a motivic 2-cell complex, which are in principle as complicated to compute as $\pi_{*,*}(S^{0,0})$, are completely algebraic. More precisely, they are given by the cohomology of the Hopf algebroid (BP_*, BP_*BP), which is a very important object in classical chromatic homotopy theory. This bridge allows computations to travel between the classical and the motivic world. See [Isa, Chapter 5 and 6] for examples where motivic computations of $\pi_{*,*}(C\tau)$ are used to deduce new information about the classical object $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$.

Remark 2.3. Since $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ admits a natural ring structure, the isomorphism of Proposition 2.1 induces an artificial ring structure on the motivic homotopy groups $\pi_{*,*}(C\tau)$. The starting point of this project was to ask if this induced ring structure of $\pi_{*,*}(C\tau)$ can be realized by a topological ring structure on the spectrum $C\tau$. Even further, the cohomology groups $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ admit higher structure (Massey products, algebraic squaring operations, ...) and one can hope that this is the shadow of a highly structured ring multiplication on $C\tau$. We will prove in Section 3 that $C\tau$ supports an E_∞ ring structure and that the isomorphism

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$$

preserves higher products (Toda brackets in homotopy and Massey products in algebra). In other words, the E_2-page of the classical Adams-Novikov spectral sequence can be realized with its higher structure as the homotopy of a motivic spectrum.

The ring structure mentioned in Remark 2.3 will be constructed by obstruction theory. To prepare the computations, we will now deduce some Corollaries of Proposition 2.1 about $\pi_{*,*}(C\tau)$ and $\pi_{*,*}(\mathrm{End}(C\tau))$.

Corollary 2.4 ([GI17]). *The group $\pi_{s,w}(C\tau)$ is zero when either $w > s$, or $w \leq \frac{1}{2}s$, or $s < 0$, except that $\pi_{0,0}(C\tau) \cong \hat{\mathbb{Z}}_2$. This is sketched in Figure 1.*

Proof. The vanishing regions in $\pi_{*,*}(C\tau)$ come from the vanishing regions of $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ via the isomorphism

$$\pi_{s,w}(C\tau) \cong \mathrm{Ext}^{2w-s,2w}_{BP_*BP}(BP_*, BP_*)$$

of Proposition 2.1. The region $w > s$ corresponds to the vanishing region above the line $t - s = s$ of slope 1 on the E_2-page of the Adams-Novikov spectral sequence, the region $w \leq \frac{1}{2}s$ corresponds to the E_2-page being 0 in negative Adams filtration $s \leq 0$, and finally $s < 0$ corresponds to E_2-page being zero in negative stems $t - s < 0$. ☐

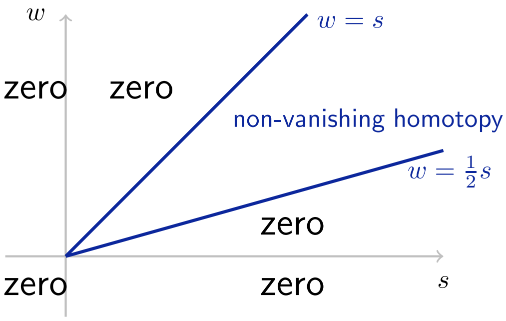

Figure 1: Vanishing regions of the homotopy groups $\pi_{s,w}(C\tau)$.

Corollary 2.5. *The group* $[\Sigma^{s,w}C\tau, C\tau]$ *is zero if either* $w > s+2$, *or* $w \leq \frac{1}{2}s$, *or* $s < -1$, *except that* $[C\tau, C\tau] \cong \hat{\mathbb{Z}}_2$ *in degree* $(0,0)$. *This is sketched in Figure 2.*

Proof. Using the cofiber sequence

$$S^{s,w} \xrightarrow{i} \Sigma^{s,w}C\tau \xrightarrow{p} S^{s+1,w-1},$$

we get a long exact sequence

$$\cdots \longleftarrow [S^{s,w}, C\tau] \xleftarrow{i^*} [\Sigma^{s,w}C\tau, C\tau] \xleftarrow{p^*} [S^{s+1,w-1}, C\tau] \longleftarrow \cdots,$$

after mapping into $C\tau$. The result follows by noticing that the hypothesis of this Corollary force both homotopy groups $\pi_{s,w}(C\tau)$ and $\pi_{s+1,w-1}(C\tau)$ to be 0 by the previous Corollary 2.4.

☐

Remark 2.6. This result is not sharp and one can slightly improve the non-vanishing region by being careful about choosing which of the 3 conditions of

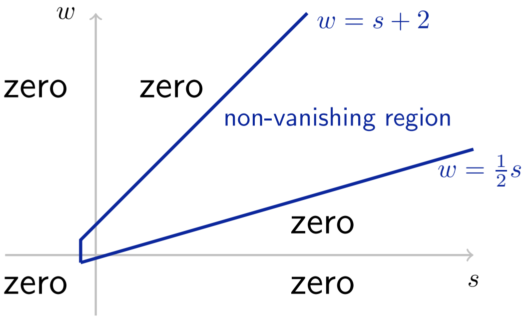

Figure 2: Vanishing regions of the abelian group $[\Sigma^{s,w}C\tau, C\tau]$.

Corollary 2.4 to use. For example, the group $\left[\Sigma^{-1,0}C\tau, C\tau\right]$ is zero as it sits in a long exact sequence

$$\cdots \longleftarrow \pi_{-1,0}(C\tau) \xleftarrow{i^*} \left[\Sigma^{-1,0}C\tau, C\tau\right] \xleftarrow{p^*} \pi_{0,-1}(C\tau) \longleftarrow \cdots,$$

and both homotopy groups surrounding it are zero. However, none of the 3 conditions of Corollary 2.5 are satisfied for the pair $(s, w) = (-1, 0)$ and thus we cannot use it to deduce that $\left[\Sigma^{-1,0}C\tau, C\tau\right]$ is zero.

The vanishing of the following groups of homotopy classes of maps will often be used in this document.

Corollary 2.7. *The following groups of homotopy classes of maps are zero*

(1) $\left[\Sigma^{0,-1}C\tau, C\tau\right] = 0$,

(2) $\left[\Sigma^{1,0}C\tau, C\tau\right] = 0$,

(3) $\left[\Sigma^{1,-1}C\tau, C\tau\right] = 0$,

(4) $[\Sigma^{n,-n}C\tau, C\tau] = 0$ *for any* $n \geq 1$.

3 The E_∞ Ring Structure on $C\tau$

In this Section we construct the E_∞ ring structure on the motivic spectrum $C\tau$. We start by endowing $C\tau$ with a homotopy unital, homotopy associative and homotopy commutative multiplication using elementary techniques with triangulated categories. The E_∞ coherences of such a multiplication cannot be constructed by hand via similar techniques and require some machinery. We will use a version of Robinson's obstruction theory from [Rob03], that we adapt to the motivic setting in Section 3.1.

3.1 MOTIVIC A_∞ AND E_∞ OPERADS AND OBSTRUCTION THEORY

Consider a simplicial symmetric monoidal model category presenting $\mathrm{SPT}_{\mathbb{C}}$, with smash product $- \wedge -$[3], and denote the simplicial mapping space by $\mathrm{Map}(X, Y)$. Given a motivic spectrum X, denote its *endomorphism operad* in simplicial sets by $\mathcal{E}\mathrm{nd}(X)$, where $\mathcal{E}\mathrm{nd}(X)_n$ is the simplicial set $\mathrm{Map}(X^{\wedge n}, X)$. If $F(-,-)$ denotes the internal (motivic) function spectrum, then we recover

$$\pi_n\left(\mathcal{E}\mathrm{nd}(X)_m\right) \cong \pi_{n,0}\left(F(X^{\wedge m}, X)\right), \tag{3.1}$$

only exploiting the weight zero homotopy groups of the function spectrum. Fix an A_∞ or E_∞ operad Θ in simplicial sets. A Θ-*algebra structure* on a motivic spectrum X is a map of operads

$$\Theta \longrightarrow \mathcal{E}\mathrm{nd}(X).$$

Equivalently, one can see Θ as an operad in motivic spaces via the constant functor and define a Θ-algebra via the motivic enrichment, which might seem more natural and internal to motivic homotopy theory. Because of this reason, classical (simplicial) operads transported into the motivic world are sometimes called *constant operads*.

In this paper, we will produce A_∞ and E_∞ structures by obstruction theory. The obstruction theory for A_∞ algebras is well-known, for example [Ang08, Theorem 3.1] (itself inspired by [Rob89]) exhibits an obstruction class in a certain abelian group. In all our cases, we will show that all the relevant abelian groups for the obstruction theory are zero. The obstruction theory for E_∞ algebras is less well-known. We will here briefly recap the work done in [Rob03] and adapt it to our motivic situation.

We will consider the simplicial E_∞ operad $\mathcal{T}$ defined in [Rob03, Section 5]. This operad is the product of a combinatorially defined cofibrant simplicial operad with the Barratt-Eccles E_∞ (simplicial) operad $E\Sigma_\bullet$. It inherits both properties and is thus a cofibrant E_∞ operad. The cofibrancy roughly means that the operadic composition maps

$$\mathcal{T}_n \times \mathcal{T}_m \xrightarrow{\circ_i} \mathcal{T}_{m+n-1} \tag{3.2}$$

are injective and that their images intersect in fairly small and regular subcomplexes. We refer to [RW02, Section 1.5] for more details. The injectivity of these maps is a key property that will be used for inductive arguments, since a map out of $\mathcal{T}_{m+n-1}$ is thus already determined on the image of all these composition maps. The bar filtration on the Barrat-Eccles operad induces a filtration on $\mathcal{T}$, where the n^{th}-filtration space of $\mathcal{T}_m$ is denoted by $\mathcal{T}_m^n \subseteq \mathcal{T}_m$. In particular $\mathcal{T}_m^n = \emptyset$ if $n < 0$. Consider now the *diagonal filtration* $\nabla^\bullet\mathcal{T}$ which is the sum of the bar filtration from the Barratt-Eccles operad and the filtration by operadic subspaces. More precisely, the n^{th}-graded piece $\nabla^n\mathcal{T} \subset \mathcal{T}$ has

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[3] For example Jardine's model of motivic symmetric spectra [Jar00].

m^{th}-space given by $\nabla^n \mathcal{T}_m = \mathcal{T}_m^{n-m}$. If $m > n$, then by definition we have $\nabla^n \mathcal{T}_m = \emptyset$. In particular, observe that $\nabla^n \mathcal{T}$ is not a suboperad as it does not contain m-ary operations for $m > n$.

Robinson defines an *n-stage for an E_∞ structure on X* to consist in a map $\nabla^n \mathcal{T} \longrightarrow \mathcal{E}\mathrm{nd}(X)$ satisfying some obvious coherences. More precisely, this is the data of Σ_m-equivariant maps

$$\mathcal{T}_m^{n-m} \longrightarrow \mathcal{E}\mathrm{nd}(X)_m$$

for $0 \leq m \leq n$, which on their restricted domain of definition satisfy the requirements for a morphism of operads. Since the operad $\mathcal{T}$ is non-unital and thus $\mathcal{T}_0 = \mathcal{T}_1 = \emptyset$, we only need to specify these maps for $2 \leq m \leq n$. From the definition of the diagonal filtration one can identify that

- a 2-stage is the data of a map $\mathcal{T}_2^0 \longrightarrow \mathcal{E}\mathrm{nd}(X)_2$, i.e., specifying a map $\mu\colon X \wedge X \longrightarrow X$,
- a 3-stage is the data of a 2-stage with the extra structure of an associative and commutative homotopy for the multiplication μ,
- a 4-stage is the data of a 3-stage with the extra structure of homotopies for the well-known pentagonal and hexagonal axioms [ML63], as well as a homotopy saying that the commutativity homotopy itself is homotopy commutative,
- an ∞-stage are the coherences of an E_∞ ring structure on X with multiplication μ.

An n-stage determines an $(n-1)$-stage by restriction, and an $(n-1)$-stage determines an n-stage on the boundary $\partial \nabla^n \mathcal{T}$ by injectivity of the composition maps of equation (3.2). We refer to [Rob03, Section 5.2] for more details. Therefore, given an $(n-1)$-stage, the data of an n-stage extending the underlying $(n-1)$-stage consists precisely in the data of extensions

$$\begin{tikzcd}
\partial \nabla^n \mathcal{T}_m \arrow[r] \arrow[dr] & \nabla^n \mathcal{T}_m \arrow[d, dashed] \\
 & \mathcal{E}\mathrm{nd}(X)_m
\end{tikzcd}$$

for every $0 \leq m \leq n$. The cofibrancy of the operad $\mathcal{T}$ is used again to show that for any m, the map

$$\partial \nabla^n \mathcal{T}_m \rightarrowtail \nabla^n \mathcal{T}_m$$

is a principal Σ_m-equivariant cofibration, whose cofiber is a wedge of spheres S^{n+2} indexed over a set with free Σ_m-action. This allows us to formulate the following result.

PROPOSITION 3.1. *Let X be a motivic spectrum with a given $(n-1)$-stage for an E_∞ ring structure.*

(1) *If the homotopy groups $\pi_{n-3}(\mathcal{E}\mathrm{nd}(X)_m)$ are zero for every $2 \leq m \leq n$, the given $(n-1)$-stage lifts to an n-stage.*

(2) *If in addition the homotopy groups $\pi_{n-2}(\mathcal{E}\mathrm{nd}(X)_m)$ are zero for every $2 \leq m \leq n$, the extension is (essentially) unique.*

Proof. The fact that $\partial \nabla^n \mathcal{T}_m \rightarrowtail \nabla^n \mathcal{T}_m$ is a principal cofibration allows us to rotate it one step to the left, producing the unstable cofiber sequence of simplicial sets

$$\vee S^{n-3} \longrightarrow \partial \nabla^n \mathcal{T}_m \rightarrowtail \nabla^n \mathcal{T}_m \longrightarrow \vee S^{n-2}.$$

An $(n-1)$-stage produces a map $\partial \nabla^n \mathcal{T}_m \longrightarrow \mathcal{E}\mathrm{nd}(X)_m$, which extends as in the diagram

$$\begin{tikzcd}
\vee S^{n-3} \arrow[r] & \partial \nabla^n \mathcal{T}_m \arrow[r] \arrow[dr] & \nabla^n \mathcal{T}_m \arrow[r] \arrow[d, dashed] & \vee S^{n-2} \\
 & & \mathcal{E}\mathrm{nd}(X)_m &
\end{tikzcd}$$

if and only if the relevant composite is zero in the abelian group

$$\left[\vee S^{n-3}, \mathcal{E}\mathrm{nd}(X)_m\right] \cong \oplus \pi_{n-3}(\mathcal{E}\mathrm{nd}(X)_m).$$

Moreover, if $\left[S^{n-2}, \mathcal{E}\mathrm{nd}(X)_m\right] = 0$ then the extension is unique up to homotopy. □

By using equation (3.1) and the fact that a 3-stage is equivalent to a unital, associative and commutative monoid in the homotopy category, we get the following Corollary.

Corollary 3.2. *Let X be a motivic spectrum with a map $\mu\colon X \wedge X \longrightarrow X$ that is homotopy unital, homotopy associative and homotopy commutative.*

(1) *If the homotopy groups $\pi_{n-3,0}(F(X^{\wedge m}, X))$ are zero for every $n \geq 4$ and $2 \leq m \leq n$, then μ can be extended to an E_∞ ring structure on X.*

(2) *If in addition the homotopy groups $\pi_{n-2,0}(F(X^{\wedge m}, X))$ are zero for every $n \geq 4$ and $2 \leq m \leq n$, then μ can be extended to an E_∞ ring structure on X in essentially a unique way.*

Remark 3.3. These results are extracted from Robinson's work in [Rob03], even though they do not explicitly appear in this form in his paper. The reason is because this is not a powerful result when applied to the topological setting for the following reason. Fix a (topological) spectrum $X \in \mathrm{Spt}$. To apply this E_∞ obstruction theory to X, its endomorphism operad $\mathcal{E}\mathrm{nd}(X)$ has to satisfy the conditions of Proposition 3.1, which require the homotopy groups $\mathcal{E}\mathrm{nd}(X)_m$ to vanish for all $n \geq 4$ and $2 \leq m \leq n$. In particular, for any fixed m

the space $\mathcal{E}\mathrm{nd}(X)_m$ needs to have vanishing homotopy groups in degrees $n \geq m$. The paper [Rob03] proceeds to study what happens during an extension of an $(n-1)$-stage to an n-stage if one allows to perturb underlying stages. This reduces the size of the obstruction groups and gives a constraint between n and m, reducing the number of obstruction groups to check. In our motivic setting the obstructions live in the groups $\pi_{n-3,0}(\mathcal{E}\mathrm{nd}(X)_m)$, which are only a small fraction of all homotopy groups $\pi_{s,w}$. Corollary 3.2 will be sufficient to prove our result.

Remark 3.4. We should point out that, in analogy with the genuine G-equivariant E_∞ operads in [BH15] (called N_∞ operads), there ought to be a notion of motivic A_∞ and E_∞ operads. An algebra over such a motivic operad would have a lot more structure than an algebra over a constant operad, such as transfers upon changing the base scheme. In Jardine's category of motivic symmetric spectra [Jar00], a commutative algebra corresponds exactly to an algebra over the constant E_∞ operad as defined in this paper. For the purpose of this paper, constant A_∞ and E_∞ operads suffice. We will therefore drop the word "constant" and refer to those just as A_∞ and E_∞ operads.

3.2 The Homotopy Ring Structure on $C\tau$

In this Section we construct a ring structure on $C\tau$ up to homotopy. More precisely, we show that $C\tau$ is a unital, associative and commutative monoid in the homotopy category $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$. Recall that this is a 3-stage in Robinson's obstruction theory, which can be seen as the initial input to start the obstruction theory. In this Section, we will exclusively work in the stable triangulated category $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$, without further mentioning it.

Lemma 3.5. *There exists a unique left unital multiplication*

$$C\tau \wedge C\tau \xrightarrow{\mu} C\tau.$$

Proof. The equation (2.5) gives an exact triangle

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1},$$

where i denotes the inclusion of the bottom cell and p denotes the projection on the top cell. By smashing it with $- \wedge C\tau$, we get another triangle

$$S^{0,-1} \wedge C\tau \xrightarrow{\tau} S^{0,0} \wedge C\tau \xrightarrow{i_L} C\tau \wedge C\tau \xrightarrow{p_L} S^{1,-1} \wedge C\tau,$$

where i_L denotes a left unit and p_L the projection on the top cell of the left factor. Since the abelian group of maps $[\Sigma^{0,-1}C\tau, C\tau] = 0$ by Corollary 2.7, the map $\tau \in [\Sigma^{0,-1}C\tau, C\tau]$ is zero on $C\tau$. This produces a left unital multiplication

μ on $C\tau$ as shown in the diagram

$$\begin{tikzcd}
S^{0,-1}\wedge C\tau \arrow[r, "\tau"] & S^{0,0}\wedge C\tau \arrow[r, "i_L"] \arrow[dr, "\simeq"] & C\tau \wedge C\tau \arrow[r, "p_L"] \arrow[d, dashed, "\exists\ \mu"] & S^{1,-1}\wedge C\tau \\
& & C\tau. &
\end{tikzcd}$$

Moreover, since $\left[\Sigma^{1,-1}C\tau, C\tau\right] = 0$ by Corollary 2.7, there is no choice for such a map which is unique. ☐

Before studying the properties of this multiplication map μ, we show a fundamental equivalence that will be used throughout the document.

Lemma 3.6. *There is a canonical isomorphism*

$$C\tau \wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau.$$

Proof. Recall that since $[\Sigma^{0,-1}C\tau, C\tau] = 0$, the map τ is zero on $C\tau$. The exact triangle

$$S^{0,-1}\wedge C\tau \xrightarrow{\tau} S^{0,0}\wedge C\tau \xrightarrow{i_L} C\tau\wedge C\tau \xrightarrow{p_L} S^{1,-1}\wedge C\tau,$$

is thus split, giving both a retraction μ and a section s, as in the diagram

$$\begin{tikzcd}
S^{0,-1}\wedge C\tau \arrow[r, "\tau=0"] & S^{0,0}\wedge C\tau \arrow[r, "i_L"] & C\tau\wedge C\tau \arrow[r, "p_L"] \arrow[l, dotted, bend left, "\exists!\ \mu"] & S^{1,-1}\wedge C\tau \arrow[r, "\tau=0"] \arrow[l, dotted, bend left, "\exists!\ s"] & \cdots .
\end{tikzcd}$$

As it is the case for μ, the section s is unique since $[\Sigma^{1,-1}C\tau, C\tau] = 0$ by Corollary 2.7. Moreover, the relation $\mu\circ s \cong 0$ is forced since the composite lives in the zero group $[\Sigma^{1,-1}C\tau, C\tau] = 0$. This gives a canonical identification

$$C\tau\wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau,$$

via the inverse maps

$$C\tau\wedge C\tau \xrightarrow{(\mu,p_L)} C\tau\vee\Sigma^{1,-1}C\tau \qquad \text{and} \qquad C\tau\vee\Sigma^{1,-1}C\tau \xrightarrow{i_L+s} C\tau\wedge C\tau.$$

☐

Corollary 3.7. *For any $n \geq 2$, there is a canonical isomorphism*

$$C\tau^{\wedge n} \cong \bigvee_{i=0}^{n-1} \binom{n-1}{i} \Sigma^{i,-i}C\tau,$$

where we use $\binom{n-1}{i}\Sigma^{i,-i}C\tau$ to indicate a wedge sum of $\binom{n-1}{i}$ terms of the spectrum $\Sigma^{i,-i}C\tau$.

We will use the identification of Lemma 3.6 to show that μ endows $C\tau$ with a unital, associative and commutative monoid structure in $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$. We first compute the relevant maps on $C\tau \vee \Sigma^{1,-1}C\tau$ after composing with this identification.

Lemma 3.8. *After the canonical identification $C\tau \wedge C\tau \cong C\tau \vee \Sigma^{1,-1}C\tau$ of Lemma 3.6*

(1) the multiplication map $C\tau \wedge C\tau \xrightarrow{\mu} C\tau$ is given by the matrix

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{[\mathrm{id}\ 0]} C\tau,$$

i.e., by the canonical projection onto the first factor,

(2) the factor swap map $C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau$ is given by the matrix

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{\left[\begin{smallmatrix}\mathrm{id} & 0\\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]} C\tau \vee \Sigma^{1,-1}C\tau.$$

Proof.

(1) The composite

$$C\tau \vee \Sigma^{1,-1}C\tau \xrightarrow{i_L+s} C\tau \wedge C\tau \xrightarrow{\mu} C\tau$$

restricts to the identity on $C\tau$ since μ is a retraction of i_L, and to zero on $\Sigma^{1,-1}C\tau$ since $s \circ \mu = 0$ by Lemma 3.6.

(2) We claim that the following diagram

$$\begin{tikzcd}
C\tau \wedge C\tau \arrow[r, "\chi"] & C\tau \wedge C\tau \arrow[d, "{(\mu, p_L)}"] \\
C\tau \vee \Sigma^{1,-1}C\tau \arrow[u, "i_L + s"] \arrow[r, dashed, "\left[\begin{smallmatrix}\mathrm{id} & 0\\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]"] & C\tau \vee \Sigma^{1,-1}C\tau
\end{tikzcd}$$

commutes. First observe that the top right entry is forced to be zero since $\left[\Sigma^{1,-1}C\tau, C\tau\right] = 0$ by Corollary 2.7. The bottom left entry can be computed explicitly by a simple diagram chase. It is

$$S^{0,0} \wedge C\tau \xrightarrow{i\wedge\mathrm{id}} C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau \xrightarrow{p\wedge\mathrm{id}} S^{1,-1} \wedge C\tau,$$

which is homotopic to the composite

$$S^{0,0} \wedge C\tau \xrightarrow{\chi} C\tau \wedge S^{0,0} \xrightarrow{\mathrm{id}\wedge i} C\tau \wedge C\tau \xrightarrow{p\wedge\mathrm{id}} S^{1,-1} \wedge C\tau.$$

By commuting $\mathrm{id}\wedge i$ and $p\wedge\mathrm{id}$ and using the canonical equivalences $S^{0,0}\wedge C\tau = C\tau = C\tau\wedge S^{0,0}$ we can rewrite it as

$$C\tau \xrightarrow{p} S^{1,-1} \xrightarrow{i} \Sigma^{1,-1}C\tau.$$

For the diagonal entries, recall that $[C\tau, C\tau]\cong\hat{\mathbb{Z}}_2$ and that the matrix has to be an involution since χ is. This forces the diagonal entries to be $+\,\mathrm{id}$ and $-\,\mathrm{id}$. One could conclude by arguing that the top left entry arises by commuting $C\tau$ with $S^{0,0}$, and thus should be $+\,\mathrm{id}$, while the bottom right entry arises by commuting $C\tau$ with $S^{1,-1}$, and thus should be $-\,\mathrm{id}$. More precisely, consider the diagram

$$\begin{tikzcd}
S^{0,0}\wedge S^{0,0} \arrow[r, "i\wedge i"] \arrow[d, "\cong"'] & C\tau\wedge C\tau \arrow[d, "\mu"] \\
S^{0,0} \arrow[r, "i"] & C\tau.
\end{tikzcd}$$

By factoring the map $i\wedge i$ as $\mathrm{id}\wedge i$ followed by $i_L = i\wedge\mathrm{id}$, and using that $\mu\circ i_L = \mathrm{id}$, one sees that the diagram commutes up to the usual canonical equivalences of smashing with $S^{0,0}$. By factoring it the other way now, as $i\wedge\mathrm{id}$ followed by $\mathrm{id}\wedge i$, we get that $\mu\circ(\mathrm{id}\wedge i)=\mathrm{id}$. This shows that the top left entry of the matrix is id. The bottom right entry is thus forced to be $-\,\mathrm{id}$ since the matrix is an involution.

□

Proposition 3.9. *The unique left unital multiplication map $C\tau\wedge C\tau\xrightarrow{\mu}C\tau$ turns $C\tau$ into a unital, associative and commutative monoid in* $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & C\tau\wedge C\tau \arrow[r, "\chi"] \arrow[dd, "(\mu, p\wedge\mathrm{id})"] & C\tau\wedge C\tau \arrow[dr, "\mu"] & \\
C\tau \arrow[ur, "i_L"] \arrow[dr, dashed] & & & C\tau, \\
 & C\tau\vee\Sigma^{1,-1}C\tau \arrow[r, "\left[\begin{smallmatrix}\mathrm{id} & 0\\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]"] & C\tau\vee\Sigma^{1,-1}C\tau \arrow[uu, "i_L+s"'] \arrow[ur, "\left[\mathrm{id}\ 0\right]"'] &
\end{tikzcd} \tag{3.3}$$

which is commutative by Lemma 3.8. Since μ is left unital and since $p\circ i=0$, the dashed arrow is given by the canonical inclusion. It follows that the composite $\mu\circ\chi\circ i_L$ is simply given by the matrix multiplication

$$\left[\mathrm{id}\ 0\right]\cdot\left[\begin{smallmatrix}\mathrm{id} & 0\\ i\circ p & -\mathrm{id}\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}\mathrm{id}\\ 0\end{smallmatrix}\right]=\mathrm{id}\,.$$

Since the right unit is given by $\chi \circ i_L$, this shows that μ is right unital. To show that μ is commutative, we have to compute the composite

$$C\tau \wedge C\tau \xrightarrow{\chi} C\tau \wedge C\tau \xrightarrow{\mu} C\tau.$$

We can again read it from diagram (3.3), where it is given by the matrix multiplication

$$\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{id} & 0 \\ i \circ p & -\mathrm{id} \end{bmatrix} \cdot \begin{bmatrix} \mu \\ p \wedge \mathrm{id} \end{bmatrix} = \mu,$$

showing that μ is commutative. To see that μ is associative, we will show that the map

$$C\tau \wedge C\tau \wedge C\tau \xrightarrow{\mu \circ (1 \wedge \mu - \mu \wedge 1)} C\tau$$

is zero. By left and right unitatlity it restricts to zero on the subspectrum

$$\left(S^{0,0} \wedge C\tau \wedge C\tau\right) \vee \left(C\tau \wedge S^{0,0} \wedge C\tau\right) \vee \left(C\tau \wedge C\tau \wedge S^{0,0}\right) \hookrightarrow C\tau \wedge C\tau \wedge C\tau. \tag{3.4}$$

By [Str99, Lemma 3.6], there is a bijection between maps $C\tau \wedge C\tau \wedge C\tau \longrightarrow C\tau$ that restrict to zero on the subspectrum of equation (3.4), and maps

$$S^{3,-3} = S^{1,-1} \wedge S^{1,-1} \wedge S^{1,-1} \longrightarrow C\tau.$$

Here $S^{1,-1}$ appears because it is the cofiber of the unit map $S^{0,0} \longrightarrow C\tau$. By Corollary 2.4, we have that $\pi_{3,-3}(C\tau) = 0$, which shows that there is a unique such map. Since the zero map $C\tau \wedge C\tau \wedge C\tau \longrightarrow C\tau$ restricts to zero on the subspectrum of equation (3.4), it is the unique such map. This shows that $\mu \circ (1 \wedge \mu - \mu \wedge 1)$ is zero, i.e., that μ is associative. ☐

3.3 The E_∞ Ring Structure on $C\tau$

In this Section, we will use Robinson's obstruction theory from Section 3.1 to construct the E_∞ ring structure on $C\tau$. In the previous Section 3.2 we endowed $C\tau$ with a unital, associative and commutative monoid structure in the the homotopy category $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$. Recall that this to a 3-stage in Robinson's obstruction theory. We will now use Corollary 3.2 to rigidify this multiplication to an E_∞ ring structure in $\mathrm{Spt}_{\mathbb{C}}$. Although not needed for the E_∞ ring structure, as a warm-up, we first show in Proposition 3.10 that $C\tau$ admits a unique A_∞ ring structure.

Proposition 3.10. *The multiplication μ on $C\tau$ can be uniquely extended to an A_∞ multiplication.*

Proof. An A_2 structure corresponds to unital homotopies (left and right), and an A_3 structure adds an associative homotopy. We constructed both structures in Proposition 3.9. The A_∞ obstruction theory originated in [Rob89] exhibits obstruction classes to extend an A_{n-1} structure to an A_n structure. In more modern language, [Ang08, Theorem 3.1] exhibits the obstruction to go from A_{n-1} structure to an A_n structure as an element in the abelian group

$$\left[\Sigma^{n-3,0}S^{n,-n}, C\tau\right] \cong \left[S^{2n-3,-n}, C\tau\right] = \pi_{2n-3,-n}(C\tau). \tag{3.5}$$

Corollary 2.4 shows that these groups are zero for any n (we really just need $n \geq 4$), which shows that μ can be extended to an A_∞ structure. Furthermore, given that an A_{n-1} structure extends to A_n structure, the possible extensions are in bijection with the abelian group

$$\left[\Sigma^{n-2,0}S^{n,-n}, C\tau\right] \cong \pi_{2n-2,-n}(C\tau).$$

This group is also zero for any n, showing that μ can be uniquely extended to an A_∞ structure. ☐

Remark 3.11. For the case $n = 3$, i.e., to endow $C\tau$ with an A_3 structure, the obstruction group from equation (3.5) is $\pi_{3,-3}(C\tau)$. Observe that this is the exact same group that appears in Proposition 3.9, where we show with elementary techniques that $C\tau$ admits an A_3 structure.

Remark 3.12. Mahowald conjectured that no non-trivial topological 2-cell complex posses an A_∞ structure. There are 2 trivial cases to exclude which are the cofiber of the zero map and the cofiber of the identity map, as shown in the cofiber sequences

$$S^0 \xrightarrow{0} S^0 \longrightarrow S^1 \vee S^0 \qquad \text{and} \qquad S^0 \xrightarrow{\text{id}} S^0 \longrightarrow *.$$

Since motivic spheres Betti realize to topological spheres, motivic 2-cell complexes Betti realize to topological 2-cell complexes. Moreover, since we are using simplicial (constant) operads, motivic algebras over A_n or E_n operads realize to classical algebras over the same A_n or E_n operads. However, the fact that $C\tau$ admits an A_∞ ring structure does not contradict Mahowald's conjecture, as the map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$ realizes to the identity map $S^0 \xrightarrow{\text{id}} S^0$.

Theorem 3.13. *The multiplication μ on $C\tau$ can be uniquely extended to an E_∞ multiplication.*

Proof. We showed in Proposition 3.9 that $C\tau$ is a unital, associative and commutative monoid in the homotopy category $\mathrm{Ho}(\mathrm{Spt}_{\mathbb{C}})$. This corresponds to a 3-stage in Robinson's obstruction theory. By Corollary 3.2, the obstructions of extending this 3-stage to an E_∞ ring structure live in

$$\pi_{n-3,0}(F(C\tau^{\wedge m}, C\tau)) \cong \left[\Sigma^{n-3,0}C\tau^{\wedge m}, C\tau\right]$$

for $n \geq 4$ and $2 \leq m \leq n$. Recall from Corollary 2.5 that $[\Sigma^{s,w}C\tau, C\tau]$ has in particular a vanishing region for $s \geq 0$ and $2w \leq s$. We now show that all obstruction groups live in this vanishing area. By the equivalence

$$[C\tau^{\wedge m}, C\tau] \cong \bigoplus_{i=0}^{m-1} \binom{m-1}{i} \left[\Sigma^{i,-i}C\tau, C\tau\right]$$

of Corollary 3.7, the homotopy groups $\pi_{n-3,0}(F(C\tau^{\wedge m}, C\tau))$ are given by

$$\left[\Sigma^{n-3,0}C\tau^{\wedge m}, C\tau\right] \cong \bigoplus_{i=0}^{m-1} \binom{m-1}{i} \left[\Sigma^{n-3+i,-i}C\tau, C\tau\right].$$

In particular, all the obstructions live in groups of the form $[\Sigma^{s,w}C\tau, C\tau]$ where the s-coordinate satisfies

$$s = n-3+i \geq 4-3+i \geq 1$$

while the w-coordinate satisfies both

$$w = -i \leq 0 \qquad \text{and} \qquad w = -i = n-s-3 \geq 1-s.$$

This corresponds to the region bounded by $s \geq 1$ and $1-s \leq w \leq s$, which lies entirely in the vanishing area described above. The situation is summarized in Figure 3. Similarly, recall from Corollary 3.2 that the obstructions for uniqueness of such an E_∞ ring structure live in groups of the form

$$\pi_{n-2,0}(F(C\tau^{\wedge m}, C\tau)) \cong \left[\Sigma^{n-2,0}C\tau^{\wedge m}, C\tau\right].$$

A similar analysis shows that all obstruction groups again live in the vanishing region, as described in Figure 3. This shows that $C\tau$ admits a unique E_∞ ring structure. ☐

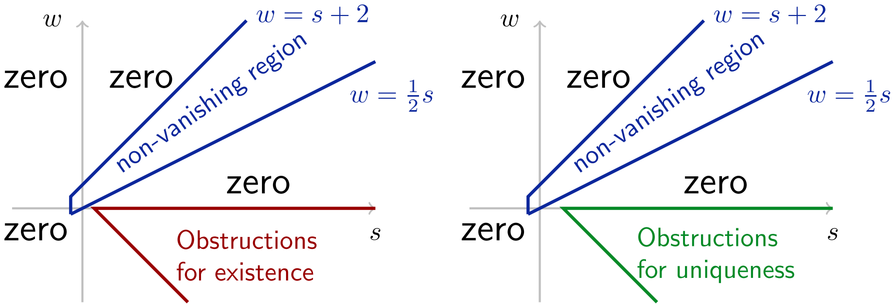

Figure 3: Chart of $[\Sigma^{s,w}C\tau, C\tau]$ where all obstruction groups live in the vanishing region.

COROLLARY 3.14. *There is an isomorphism of rings*

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*),$$

which sends Massey products in Ext *to Toda brackets in* $\pi_{*,*}$, *and vice-versa.*

Proof. Since $C\tau$ is an E_∞ ring spectrum, its motivic Adams-Novikov spectral sequence is multiplicative and converges to an associated graded of the ring $\pi_{*,*}(C\tau)$. Recall from Proposition 2.1 that the spectral sequence collapses at E_2 with no possible hidden extensions as a module over the spectral sequence for $S^{0,0}$. For the exact same reason, there are no possible hidden extensions as a multiplicative spectral sequence. By the Moss convergence theorem [Mos70], we get a highly structured bigraded isomorphism

$$\mathrm{Ext}_{BPGL_{*,*}BPGL}(BPGL_{*,*}, BPGL_{*,*}/\tau) \cong \pi_{*,*}(C\tau), \tag{3.6}$$

between the E_2-page and the output of the spectral sequence. More precisely, Massey products computed in Ext converge to Toda brackets computed in $\pi_{*,*}(C\tau)$.

Until the end of the proof, denote the motivic Brown-Peterson spectrum $BPGL$ by B. To finish the proof, we have to show that there is a highly structured ring isomorphism

$$\mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau) \cong \mathrm{Ext}_{B_{*,*}B}(B_{*,*}, B_{*,*}/\tau).$$

These are Ext-groups computed in comodules and since the first variable is projective (even free) over the base ring, both of those Ext terms can be computed from their cobar complex [Rav86, Corollary A1.2.12]. Moreover, since the cobar complex also controls the Massey products in the Ext-ring, this will give an isomorphism preserving this structure. The cobar complex of the left Ext-group is given by

$$\begin{aligned} B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}/\tau &\longrightarrow \\ B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \otimes_{B_{*,*}/\tau} B_{*,*}/\tau &\longrightarrow \cdots, \end{aligned}$$

while the cobar complex of the right term is given by

$$\begin{aligned} B_{*,*} \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}/\tau &\longrightarrow \\ B_{*,*} \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}B \otimes_{B_{*,*}} B_{*,*}/\tau &\longrightarrow \cdots. \end{aligned}$$

By iterating the ring isomorphism

$$B_{*,*}/\tau \otimes_{B_{*,*}/\tau} B_{*,*}B/\tau \cong B_{*,*} \otimes_{B_{*,*}} B_{*,*}/\tau,$$

these cobar complexes are isomorphic as dga's. By taking cohomology, we get an isomorphism

$$\mathrm{Ext}_{B_{*,*}B}(B_{*,*}, B_{*,*}/\tau) \cong \mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau) \tag{3.7}$$

that preserves Massey products. The trigraded $\mathrm{Ext}_{B_{*,*}B/\tau}(B_{*,*}/\tau, B_{*,*}/\tau)$ is really bigraded because of the relation $t = 2w$ between the internal degree t and the weight w. Therefore, when working mod τ, we can regrade everything in sight by keeping the internal degree and forgetting the weight. With this convention, the degree of $v_n \in B_*/\tau$ is the single number $2^{n+1} - 2$ and thus there is an isomorphism of Hopf algebroids $B_*B/\tau \cong BP_*BP$. This provides the (higher) ring isomorphism

$$\mathrm{Ext}_{B_*B/\tau}(B_*/\tau, B_*/\tau) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*). \tag{3.8}$$

By combining the isomorphisms of equation (3.6), (3.7) and (3.8), we get an isomorphism

$$\pi_{*,*}(C\tau) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$$

of higher rings, that sends Toda brackets to Massey products and vice-versa. $\square$

4 (CO-)OPERATIONS ON $C\tau$

In this Section we describe the homotopy types of $C\tau \wedge C\tau$ and $\mathrm{End}(C\tau)$ as ring spectra. Understanding their homotopy types is crucial for the computation of the Steenrod algebra of the spectrum $H\mathbb{F}_2 \wedge C\tau$ in Section 5.2. Most proofs are done by diagram chasing and identifying composites of maps.

4.1 THE SPECTRUM $C\tau \wedge C\tau$

The E_∞ ring structure on $C\tau$ induces an E_∞ ring structure on the smash product $C\tau \wedge C\tau$ via the multiplication

$$\mu_{C\tau\wedge C\tau}\colon (C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \xrightarrow{1\wedge\chi\wedge 1} C\tau \wedge C\tau \wedge C\tau \wedge C\tau \xrightarrow{\mu\wedge\mu} C\tau \wedge C\tau.$$

Here μ denotes the multiplication map on $C\tau$ and χ denotes the factor swap map. Recall from Lemma 3.6 that there is a canonical equivalence

$$C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1}C\tau,$$

describing the additive homotopy type of $C\tau \wedge C\tau$. The next lemma describes its ring structure.

LEMMA 4.1. *Under the canonical vertical identifications given by*

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "\mu_{C\tau\wedge C\tau}"] \arrow[d, "\simeq"'] & C\tau \wedge C\tau \arrow[d, "\simeq"] \\
(C\tau \vee \Sigma^{1,-1}C\tau) \wedge (C\tau \vee \Sigma^{1,-1}C\tau) \arrow[r] \arrow[d, "="'] & C\tau \vee \Sigma^{1,-1}C\tau \arrow[d, "="] \\
(C\tau \wedge C\tau) \vee (\Sigma^{1,-1}C\tau \wedge C\tau) \vee (C\tau \wedge \Sigma^{1,-1}C\tau) \vee (\Sigma^{1,-1}C\tau \wedge \Sigma^{1,-1}C\tau) \arrow[r, dashed] & C\tau \vee \Sigma^{1,-1}C\tau,
\end{tikzcd}$$

the multiplication on $C\tau \wedge C\tau$ is given by the maps

$$\begin{aligned} C\tau \wedge C\tau &\xrightarrow{(\mu,0)} C\tau \vee \Sigma^{1,-1}C\tau \\ \Sigma^{1,-1}C\tau \wedge C\tau &\xrightarrow{(0,\mu)} C\tau \vee \Sigma^{1,-1}C\tau \\ C\tau \wedge \Sigma^{1,-1}C\tau &\xrightarrow{(0,\mu)} C\tau \vee \Sigma^{1,-1}C\tau \\ \Sigma^{1,-1}C\tau \wedge \Sigma^{1,-1}C\tau &\xrightarrow{(0,0)} C\tau \vee \Sigma^{1,-1}C\tau. \end{aligned}$$

Proof. These four maps are given by a simple diagram chase, where we only have to be careful with the identifications. For simplicity, let's denote the sphere spectrum $S^{0,0}$ by S, and ignore or denote by 1 some identity maps id in the following diagrams. Recall the cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}$$

from equation (2.5). The first map $C\tau \wedge C\tau \longrightarrow C\tau \vee \Sigma^{1,-1}C\tau$ corresponds to the composite

$$(\mu, p \wedge 1) \circ (\mu \wedge \mu) \circ (1 \wedge \chi \wedge 1) \circ (i \wedge i),$$

which is embedded in the commutative diagram

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "1 \wedge \chi \wedge 1"] & C\tau \wedge C\tau \wedge C\tau \wedge C\tau \arrow[r, "\mu \wedge \mu"] & C\tau \wedge C\tau \arrow[r, "{(\mu, p \wedge 1)}"] & C\tau \vee \Sigma^{1,-1}C\tau \\
(S \wedge C\tau) \wedge (S \wedge C\tau) \arrow[u, "i \wedge i"] \arrow[r, dash, "\simeq"] & S \wedge S \wedge C\tau \wedge C\tau \arrow[u, "i \wedge i"] \arrow[r, dash, "\simeq"] & S \wedge C\tau \wedge C\tau. \arrow[u, "i \wedge \mu"] &
\end{tikzcd}$$

We can compute by the other path, where we use that the map

$$S \wedge C\tau \wedge C\tau \xrightarrow{i \wedge \mu} C\tau \wedge C\tau$$

decomposes as

$$S \wedge C\tau \wedge C\tau \xrightarrow{1 \wedge \mu} S \wedge C\tau \xrightarrow{i \wedge 1} C\tau \wedge C\tau,$$

and by using that $p \circ i = 0$ and $\mu \circ (i \wedge 1) = \mathrm{id}$. For the second map, the canonical splitting of Lemma 3.6 induces a splitting

$$\Sigma^{1,-1}C\tau \wedge C\tau \simeq \Sigma^{1,-1}C\tau \vee \Sigma^{2,-2}C\tau.$$

By Corollary 2.7 we have $\left[\Sigma^{1,-1}C\tau, C\tau\right] = \left[\Sigma^{2,-2}C\tau, C\tau\right] = 0$, and thus the second map

$$\Sigma^{1,-1}C\tau \wedge C\tau \longrightarrow C\tau \vee \Sigma^{1,-1}C\tau$$

corestricts to zero on $C\tau$. To compute the other part, recall first from Lemma 3.6 that the map $p \wedge 1$ admits a canonical section s, as shown in the cofiber sequence

$$\begin{tikzcd}
S^{0,-1} \wedge C\tau \arrow[r, "\tau = 0"] & S \wedge C\tau \arrow[r, "i_L"] & C\tau \wedge C\tau \arrow[r, "p_L"] \arrow[l, bend left, dotted, "\exists!\ \mu"] & S^{1,-1} \wedge C\tau \arrow[r, "\tau = 0"] \arrow[l, bend left, dotted, "\exists!\ s"] & \cdots .
\end{tikzcd}$$

The second map is the composite in the commutative diagram

$$\begin{tikzcd}
(C\tau \wedge C\tau) \wedge (C\tau \wedge C\tau) \arrow[r, "1 \wedge \chi \wedge 1"] & C\tau \wedge C\tau \wedge C\tau \wedge C\tau \arrow[r, "\mu \wedge \mu"] & C\tau \wedge C\tau \arrow[r, "p \wedge 1"] & \Sigma^{1,-1} C\tau \\
(S^{1,-1} \wedge C\tau) \wedge (S \wedge C\tau) \arrow[u, "s \wedge (i \wedge 1)"] \arrow[r, "s \wedge (1 \wedge 1)"'] & C\tau \wedge C\tau \wedge S \wedge C\tau \arrow[ul, dashed, "i"'] \arrow[r, dash, "\simeq"] & C\tau \wedge S \wedge C\tau \wedge C\tau. \arrow[ul, dashed, "i"'] \arrow[u, "1 \wedge \mu"'] &
\end{tikzcd}$$

We again compute it by following the other path

$$(p \wedge 1) \circ (1 \wedge \mu) \circ (s \wedge (1 \wedge 1)).$$

The result follows by noticing that the last two maps $p \wedge 1$ and $1 \wedge \mu$ commute with each other, together with the fact that s is a section of $p \wedge 1$. For the third map, we can either do a similar diagram chase, or use the fact that $C\tau \wedge C\tau$ is an E_∞ ring spectrum, and so the third map is homotopic to the second map we just computed. The last map is forced to be nullhomotopic since

$$\Sigma^{1,-1} C\tau \wedge \Sigma^{1,-1} C\tau \simeq \Sigma^{3,-3} C\tau \vee \Sigma^{2,-2} C\tau$$

and there are no non-trivial maps to both $C\tau$ and $\Sigma^{1,-1}C\tau$ by Corollary 2.5. ☐

The additive splitting $C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1}C\tau$ gives the isomorphism

$$\pi_{*,*}(C\tau \wedge C\tau) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau).$$

The class β_τ has degree $|\beta_\tau| = (1, -1)$, and is the unit element of the shifted copy given by the composite

$$S^{1,-1} \simeq S^{1,-1} \wedge S^{0,0} \xrightarrow{1 \wedge i} S^{1,-1} \wedge C\tau \xrightarrow{s} C\tau \wedge C\tau.$$

We call it β_τ because it induces a τ-Bockstein operations in $H\mathbb{F}_2 \wedge C\tau$-(co)homology, as we show in Propositions 5.5 and 5.6. Lemma 4.1 gives the following multiplicative description of the homotopy groups $\pi_{*,*}(C\tau \wedge C\tau)$.

COROLLARY 4.2. *The E_∞ ring spectrum $C\tau \wedge C\tau$ has homotopy ring*

$$\pi_{*,*}(C\tau \wedge C\tau) \cong \pi_{*,*}(C\tau)[\beta_\tau] \Big/ \beta_\tau^2 ,$$

where $|\beta_\tau| = (1, -1)$.

4.2 The Endomorphism Spectrum $\mathrm{End}(C\tau)$

In this Section we explicitly describe the homotopy type of $\mathrm{End}(C\tau)$ as a ring spectrum and give a presentation of its homotopy ring $\pi_{*,*}(\mathrm{End}(C\tau))$, in the same way that we did for $C\tau \wedge C\tau$. However, the endomorphism spectrum $\mathrm{End}(C\tau)$ is a little harder to understand than $C\tau \wedge C\tau$. First, it is only an associative A_∞ spectrum, whereas $C\tau \wedge C\tau$ is E_∞. Second, its multiplication comes from composition of morphisms and has nothing to do with the fact that $C\tau$ is a ring object, whereas the multiplication on $C\tau \wedge C\tau$ is easy to describe in terms of the multiplication of $C\tau$. Finally, it turns out that out of the eight maps that assemble together to give the multiplication on $\mathrm{End}(C\tau)$, only three are forced to be nullhomotopic for degree reasons, whereas five where forced to be nullhomotopic for $C\tau \wedge C\tau$.

An important tool that we use is Spanier-Whitehead duality, adapted to the motivic setting from the categorical treatment in [LMSM86, Chapter 3]. We briefly recall some notation and elementary results from both [LMSM86, Chapter 3] and [Lur, Sections 4.6-7]. Consider two motivic spectra X and Y. If X is dualizable, its Spanier-Whitehead dual is defined to be the motivic spectrum

$$DX \coloneqq F(X, S^{0,0}).$$

In particular, finite cell complexes are dualizable. For spheres, there is a canonical identification

$$DS^{m,n} = F(S^{m,n}, S^{0,0}) \simeq F(S^{0,0}, S^{-m,-n}) \simeq S^{-m,-n}. \tag{4.1}$$

Given a map $f\colon X \longrightarrow Y$ between dualizable motivic spectra, denote its Spanier-Whitehead dual by

$$Df \coloneqq F(f, S^{0,0})\colon DY \longrightarrow DX.$$

If X is dualizable, the smashing morphism $F(X, S^{0,0}) \wedge X \xrightarrow{\wedge} F(X, S^{0,0} \wedge X)$ is an equivalence, giving the equivalence

$$DX \wedge X = F(X, S^{0,0}) \wedge X \xrightarrow{\simeq} F(X, S^{0,0} \wedge X) = \mathrm{End}(X). \tag{4.2}$$

Denote the evaluation map that is adjoint to the identity map on $F(X, S^{0,0})$ by

$$DX \wedge X = F(X, S^{0,0}) \wedge X \xrightarrow{\mathrm{ev}} S^{0,0}.$$

The endomorphism spectrum $\mathrm{End}(X)$ is always a motivic A_∞ ring spectrum with multiplication map given by the composite $\mu_{\mathrm{End}(X)}$ in the diagram

$$\begin{tikzcd}
\mathrm{End}(X) \wedge \mathrm{End}(X) \arrow[rr, dashed, "\mu_{\mathrm{End}(X)}"] \arrow[d, equal, "\mathrm{can.}"'] & & \mathrm{End}(X) \arrow[d, equal, "\mathrm{can.}"'] \\
DX \wedge X \wedge DX \wedge X \arrow[r, "1 \wedge \chi \wedge 1"] & DX \wedge DX \wedge X \wedge X \arrow[r, "1 \wedge \mathrm{ev} \wedge 1"] & DX \wedge S^{0,0} \wedge X.
\end{tikzcd} \tag{4.3}$$

The spectrum $C\tau$ is dualizable since it is a 2-cell complex. The A_∞ ring structure on $\mathrm{End}(C\tau)$ can thus be understood in terms of Spanier-Whitehead duality. For this, we have to compute the homotopy type of the Spanier-Whitehead dual $DC\tau$ and identify the evaluation map $DC\tau \wedge C\tau \xrightarrow{\mathrm{ev}} S^{0,0}$.

PROPOSITION 4.3. *We have the following identifications.*

1. *The Spanier-Whitehead dual of* $S^{0,-1} \xrightarrow{\tau} S^{0,0}$ *is* $D\tau \simeq \tau \colon S^{0,0} \longrightarrow S^{0,1}$.
2. *The Spanier-Whitehead dual of the cofiber sequence*

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \xrightarrow{i} C\tau \xrightarrow{p} S^{1,-1}$$

is the cofiber sequence

$$S^{0,1} \xleftarrow{\tau} S^{0,0} \xleftarrow{p} \Sigma^{-1,1}C\tau \xleftarrow{i} S^{-1,1}.$$

In particular we have $Di \simeq p$ and $Dp \simeq i$, and a canonical (up to homotopy) identification

$$DC\tau \simeq \Sigma^{-1,1}C\tau. \tag{4.4}$$

Proof.

1. Start with the map $S^{0,-1} \xrightarrow{\tau} S^{0,0}$. The functor $D = F(-, S^{0,0})$ and the canonical identification of equation (4.1) gives a map $S^{0,0} \xrightarrow{D\tau} S^{0,1}$, which by definition, sends 1 to τ on $\pi_{0,0}$. Since it lives in the group $\left[S^{0,0}, S^{0,1}\right] \cong \hat{\mathbb{Z}}_2$ generated by τ, we get that $D\tau \simeq \tau$.
2. Since the dualization functor D preserves cofiber sequences, we get the cofiber sequence

$$DS^{0,-1} \xleftarrow{D\tau} DS^{0,0} \xleftarrow{Di} DC\tau \xleftarrow{Dp} DS^{1,-1}.$$

To understand it, we use the canonical equivalences of equation (4.1) and embed it in the diagram

$$\begin{tikzcd}
DS^{0,-1} \arrow[d, equal, "\mathrm{can.}"'] & DS^{0,0} \arrow[l, "D\tau"'] \arrow[d, equal, "\mathrm{can.}"'] & DC\tau \arrow[l, "Di"'] & DS^{1,-1} \arrow[l, "Dp"'] \arrow[d, equal, "\mathrm{can.}"'] \\
S^{0,1} & S^{0,0} \arrow[l, "\tau"'] & \Sigma^{-1,1}C\tau \arrow[l, "p"'] \arrow[u, dashed] & S^{-1,1}. \arrow[l, "i"']
\end{tikzcd}$$

By the 5-lemma, the map $\Sigma^{-1,1}C\tau \longrightarrow DC\tau$ is an equivalence. Moreover, given two such equivalences, their difference would factor trough the map p and thus trough $S^{0,0}$. It follows that this equivalence is canonical up to homotopy, since by Corollary 2.4 we have

$$\pi_{0,0}(DC\tau) \cong \pi_{0,0}(\Sigma^{-1,1}C\tau) \cong \pi_{1,-1}(C\tau) = 0.$$

□

Lemma 4.4. *Up to a unit, the evaluation map* $DC\tau \wedge C\tau \xrightarrow{\text{ev}} S^{0,0}$ *is given by the commutative diagram*

$$\begin{tikzcd}
DC\tau \wedge C\tau \arrow[r, "ev"] \arrow[d, dash, "\simeq"', "\text{can.}"] & S^{0,0} \\
\Sigma^{-1,1}C\tau \wedge C\tau \arrow[r, "\mu"] & \Sigma^{-1,1}C\tau . \arrow[u, "p"']
\end{tikzcd}$$

Proof. We compute the relevant abelian group of homotopy classes of maps $\left[DC\tau \wedge C\tau, S^{0,0}\right]$. We have

$$\begin{aligned}
\left[DC\tau \wedge C\tau, S^{0,0}\right] &\cong \left[\Sigma^{-1,1}C\tau \wedge C\tau, S^{0,0}\right] && \text{by equation (4.4)} \\
&\cong \left[\Sigma^{-1,1}C\tau \vee C\tau, S^{0,0}\right] && \text{by Lemma 3.6} \\
&\cong \left[\Sigma^{-1,1}C\tau, S^{0,0}\right] \oplus \left[C\tau, S^{0,0}\right] && \\
&\cong \left[S^{0,0}, S^{0,0}\right] \oplus 0 && \text{via } \Sigma^{-1,1}C\tau \xrightarrow{p} S \\
&\cong \hat{\mathbb{Z}}_2
\end{aligned}$$

which is generated by the identity. This means that $\left[DC\tau \wedge C\tau, S^{0,0}\right]$ is generated by the composite

$$DC\tau \wedge C\tau \simeq \Sigma^{-1,1}C\tau \wedge C\tau \xrightarrow{\mu} \Sigma^{-1,1}C\tau \xrightarrow{p} S^{0,0}.$$

On the other side, by adjunction we have an isomorphism

$$[DC\tau, DC\tau] \cong \left[DC\tau \wedge C\tau, S^{0,0}\right],$$

which sends the identity map to the evaluation map (by definition of the evaluation map). This shows that ev is also one of the units in $\hat{\mathbb{Z}}_2$, finishing the proof. □

Lemma 4.5. *Under the vertical identifications given by*

$$\begin{tikzcd}
\mathrm{End}(C\tau) \wedge \mathrm{End}(C\tau) \arrow[rr, "\mu_{\mathrm{End}(C\tau)}"] \arrow[d, dash, "\simeq"'] & & \mathrm{End}(C\tau) \arrow[d, dash, "\simeq"] \\
(\Sigma^{-1,1}C\tau \vee C\tau) \wedge (\Sigma^{-1,1}C\tau \vee C\tau) \arrow[rr] \arrow[d, dash, "="'] & & \Sigma^{-1,1}C\tau \vee C\tau \arrow[d, dash, "="] \\
(\Sigma^{-1,1}C\tau \wedge \Sigma^{-1,1}C\tau) \vee (\Sigma^{-1,1}C\tau \wedge C\tau) \vee (C\tau \wedge \Sigma^{-1,1}C\tau) \vee (C\tau \wedge C\tau) \arrow[rr, dashed] & & \Sigma^{-1,1}C\tau \vee C\tau ,
\end{tikzcd}$$

the multiplication on End$(C\tau)$ *is given (up to a unit) by the maps*

$$\Sigma^{-1,1}C\tau \wedge \Sigma^{-1,1}C\tau \xrightarrow{(p\wedge 1,0)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$\Sigma^{-1,1}C\tau \wedge C\tau \xrightarrow{(\mu,0)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$C\tau \wedge \Sigma^{-1,1}C\tau \xrightarrow{(\mu,p\wedge 1)} \Sigma^{-1,1}C\tau \vee C\tau$$

$$C\tau \wedge C\tau \xrightarrow{(0,\mu)} \Sigma^{-1,1}C\tau \vee C\tau.$$

Sketch of proof. This proof is by tedious diagram chases, and is in the spirit as the proof of Lemma 4.1. We will now briefly sketch the steps in the proof. The first part is to break End$(C\tau) \wedge$ End$(C\tau)$ in more manageable summands via Spanier-Whitehead duality, and the necessary identifications are done in Proposition 4.3. We then use the definition of the multiplication map on End$(C\tau)$ from diagram (4.3), as a composite of the factor swap map and the evaluation map. The evaluation map was explicitly computed in Lemma 4.4. The remainder of the proof consists on carefully identifying composites. ☐

The additive splitting End$(C\tau) \simeq C\tau \vee \Sigma^{-1,1}C\tau$ gives the isomorphism

$$\pi_{*,*}(\mathrm{End}(C\tau)) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau).$$

The class β_τ has degree $|\beta_\tau| = (-1,1)$, and is the unit element of the shifted copy given by the composite given by the composite

$$C\tau \xrightarrow{p} S^{1,-1} \xrightarrow{\Sigma i} \Sigma^{1,-1}C\tau.$$

Lemma 4.5 gives the following multiplicative description of the homotopy groups $\pi_{*,*}(\mathrm{End}(C\tau))$.

COROLLARY 4.6. *The A_∞ ring spectrum* End$(C\tau)$ *has homotopy ring (up to a unit) given by*

$$\pi_{*,*}\left(\mathrm{End}(C\tau)\right) \cong \pi_{*,*}\left(C\tau\right)\langle\beta_\tau\rangle \Big/ \begin{array}{l} \alpha\beta_\tau - (-1)^{|\alpha|}\beta_\tau\alpha = i\circ p(\alpha) \\ \beta_\tau^2 = 0 \end{array}$$

where β_τ is a non-commutative variable and α span the elements of $\pi_{*,*}(C\tau)$.

REMARK 4.7. The canonical inclusion $C\tau \longrightarrow$ End$(C\tau)$ is a map of A_∞ ring spectra and on homotopy is the inclusion of $\pi_{*,*}(C\tau)$ onto the non-shifted factor. We can also think of the ring $\pi_{*,*}(\mathrm{End}(C\tau))$ as being the abelian group

$$\pi_{*,*}(\mathrm{End}(C\tau)) \cong \pi_{*,*}(C\tau) \oplus \beta_\tau \cdot \pi_{*,*}(C\tau)$$

with ring structure given by the following multiplication table (up to a unit)

$$\begin{aligned} \alpha\circ\alpha' &= \alpha\alpha' \\ \alpha\circ\beta_\tau\alpha' &= (-1)^{|\alpha|}\beta_\tau\alpha\alpha' + (i\circ p(\alpha))\alpha' \\ \beta_\tau\alpha\circ\alpha' &= \beta_\tau\alpha\alpha' \\ \beta_\tau\alpha\circ\beta_\tau\alpha' &= \beta_\tau(i\circ p(\alpha))\alpha', \end{aligned}$$

where $\alpha, \alpha' \in \pi_{*,*}(C\tau)$ and $\beta_\tau\alpha, \beta_\tau\alpha' \in \beta_\tau \cdot \pi_{*,*}(\Sigma^{-1,1}C\tau)$.

Remark 4.8. Since $S^{0,0} \xrightarrow{i} C\tau$ is the ring map which induces the $\pi_{*,*}(S^{0,0})$-module structure on $\pi_{*,*}(C\tau)$, we have the compatibility formula

$$i(\alpha)\alpha' = \alpha\alpha' \qquad \text{for } \alpha \in \pi_{*,*}(S^{0,0}), \alpha' \in \pi_{*,*}(C\tau).$$

The first multiplication uses the ring structure of $C\tau$ while the second uses the $S^{0,0}$-module structure on $C\tau$. This simplifies some of the formulas of Corollary 4.6, for example by $\beta_\tau\alpha \circ \beta_\tau\alpha' = \beta_\tau p(\alpha)\alpha'$ since $p(\alpha)$ is in the homotopy groups of the motivic sphere.

5 Examples of $C\tau$-modules

Since the 2-cell complex $C\tau$ is a (cofibrant) commutative ring spectrum, we can use [Pel11, Section 2.8] to endow the category ${}_{C\tau}\mathrm{Mod}$ with a closed symmetric monoidal model structure. The closed monoidal structure is given by the the relative smash product $- \wedge_{C\tau} -$ and the internal function spectrum $F_{C\tau}(-,-)$. Moreover, the model structure is created by the forgetful functor, and is thus part of the Quillen adjunction

$$\mathrm{Spt}_{\mathbb{C}} = {}_{S^{0,0}}\mathrm{Mod} \begin{tikzcd} {} \arrow[r, shift left, "-\wedge C\tau"] \arrow[r, phantom, "\perp"] & {} \arrow[l, shift left, "U"] \end{tikzcd} {}_{C\tau}\mathrm{Mod}. \tag{5.1}$$

In this Section we will first give some elementary lemmas about the category ${}_{C\tau}\mathrm{Mod}$, and then study some important spectra that are induced up from $S^{0,0}$-modules by smashing with $- \wedge C\tau$. We call such a spectrum a *$C\tau$-induced spectrum.*

We start with the $C\tau$-induced Eilenberg-Maclane spectrum $H\mathbb{F}_2 \wedge C\tau$ which has homotopy groups $\pi_{*,*}(H\mathbb{F}_2 \wedge C\tau) \cong \mathbb{F}_2$ in degree $(0,0)$. We will compute its Steenrod algebra of operations (and its dual) as a Hopf algebra, both in $\mathrm{Spt}_{\mathbb{C}}$ and ${}_{C\tau}\mathrm{Mod}$. This computation is used in future work [Ghe] to construct Morava K-theories for the motivic w_i periodic operators. The first operator w_1 was introduced in [And]. We then show that the $C\tau$-induced Moore spectrum $S/(2,\tau)$ admits a unique structure of an E_∞ algebra over $C\tau$. We also observe that it admits a v_1^1-self map, whereas $S^{0,0}/2$ only admits a v_1^4-self map. Finally, we compute the homology and homotopy of the $C\tau$-induced connective algebraic and hermitian K-theory spectra kgl and kq. Here again an interesting phenomenon arises in hermitian K-theory: an obstruction is killed and we can see the element v_1^2 in the homotopy of $kq \wedge C\tau$, whereas we only see its square v_1^4 in kq.

5.1 Elementary Results on $C\tau$-Modules

Let X be a (left) $C\tau$-module with action map $\phi_X\colon C\tau \wedge X \longrightarrow X$. The left unitality condition says that the triangle in the diagram

$$\begin{tikzcd}
S^{0,-1} \wedge X \arrow[r, "\tau"] & S^{0,0} \wedge X \arrow[r, "i"] \arrow[dr, "\simeq"] & C\tau \wedge X \arrow[r, "p"] \arrow[d, "\phi_X"] & S^{1,-1} \wedge X \\
 & & X &
\end{tikzcd}$$

commutes, i.e., that ϕ_X is a retraction of the unit. This produces a splitting

$$C\tau \wedge X \xrightarrow{(\phi_X,p)} X \vee \Sigma^{1,-1}X \tag{5.2}$$

up to homotopy, whose inverse map requires a choice of section of p. There is however a canonical choice of section given by the composite

$$S^{1,-1} \wedge S^{0,0} \wedge X \xrightarrow{\mathrm{id}\wedge i \wedge \mathrm{id}} S^{1,-1} \wedge C\tau \wedge X \xrightarrow{s\wedge \mathrm{id}} C\tau \wedge C\tau \wedge X \xrightarrow{\mathrm{id}\wedge \phi_X} C\tau \wedge X,$$

by using the canonical section $s\colon \Sigma^{1,-1}C\tau \longrightarrow C\tau \wedge C\tau$ from Lemma 3.6. The Betti realization functor $\mathrm{SPT}_{\mathbb{C}} \longrightarrow \mathrm{SPT}$ naturally extends to ${}_{C\tau}\mathrm{MOD}$ by composing with the forget functor

$${}_{C\tau}\mathrm{MOD} \longrightarrow \mathrm{SPT}_{\mathbb{C}} \xrightarrow{\mathrm{Re}_{\mathbb{C}}} \mathrm{SPT}.$$

Lemma 5.1. *Every $C\tau$-module realizes to a contractible spectrum in* Top.

Proof. Consider a spectrum $X \in \mathrm{SPT}_{\mathbb{C}}$ endowed with a structure of $C\tau$-module. Since the Betti realization functor is (strict) symmetric monoidal and sends $C\tau$ to a contractible spectrum, we have

$$\mathrm{Re}_{\mathbb{C}}(C\tau \wedge X) \simeq \mathrm{Re}_{\mathbb{C}}(C\tau) \wedge \mathrm{Re}_{\mathbb{C}}(X) \simeq *.$$

It follows that $\mathrm{Re}_{\mathbb{C}}(X) \simeq *$ as X is a retract of $C\tau \wedge X$ by equation (5.2). □

The next two elementary lemmas will often be used for studying $C\tau$-induced spectra.

Lemma 5.2. *Let X be a spectrum with τ-free homotopy (resp. homology) groups, i.e., multiplication by τ is injective on $\pi_{*,*}(X)$ (resp. on $H\mathbb{F}_{2*,*}(X)$). Then the homotopy (resp. homology) groups of the $C\tau$-induced spectrum $X \wedge C\tau$ are given by*

$$\pi_{*,*}(X \wedge C\tau) \cong \pi_{*,*}(X)\,/_{\tau} \qquad \textit{(resp. } H\mathbb{F}_{2*,*}(X \wedge C\tau) \cong H\mathbb{F}_{2*,*}(X)\,/_{\tau}\ \textit{).}$$

Moreover if X is an E_∞ ring spectrum, then this isomorphism is a ring isomorphism.

Proof. This follows by the long exact sequence induced from the cofiber sequence

$$\Sigma^{0,-1}X \xrightarrow{\tau} X \xrightarrow{i} C\tau \wedge X$$

since multiplication by τ is injective. Moreover, if X is an E_∞ ring spectrum, then the map

$$S^{0,0} \wedge X \xrightarrow{i \wedge \mathrm{id}} C\tau \wedge X$$

is a map of E_∞ ring spectra as well. □

Lemma 5.3. *Let X be a spectrum with τ-free $H\mathbb{F}_2$-cohomology groups, i.e., multiplication by τ is injective on $H\mathbb{F}_2{}^{*,*}(X)$. Then the cohomology groups of the $C\tau$-induced spectrum $X \wedge C\tau$ are given by*

$$H\mathbb{F}_2{}^{*,*}(X \wedge C\tau) \cong {H\mathbb{F}_2{}^{*,*}(\Sigma^{1,-1}X)}/_{\tau}.$$

Proof. Similarly to the proof of Lemma 5.2, this just follows by the long exact sequence induced from the cofiber sequence

$$C\tau \wedge X \longrightarrow \Sigma^{1,-1}X \xrightarrow{\tau} \Sigma^{1,0}X$$

since multiplication by τ is injective. □

5.2 The $C\tau$-Induced Eilenberg-Maclane Spectrum

Consider the $C\tau$-induced Eilenberg-Maclane spectrum

$$\overline{H\mathbb{F}_2} := H\mathbb{F}_2 \wedge C\tau,$$

which has homotopy $\pi_{*,*}(\overline{H\mathbb{F}_2}) \cong \mathbb{F}_2$ concentrated in degree $(0,0)$ by Lemma 5.2. Unlike $H\mathbb{F}_2$, this spectrum detects both cells of $C\tau$ since

$$\overline{H\mathbb{F}_2}^{*,*}(C\tau) \cong \begin{cases} \mathbb{F}_2 & \text{if } (*,*) = (0,0) \\ \mathbb{F}_2 & \text{if } (*,*) = (1,-1) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.4. As pointed out by the referee, an important observation about the spectrum $\overline{H\mathbb{F}_2}$ is that it lives in the heart of Morel's homotopy t-structure [Mor04]. It is possible that this observation leads to a universal property of $\overline{H\mathbb{F}_2}$, which could also hold over other bases than $\operatorname{Spec}\mathbb{C}$, giving a way to generalize the results of this paper over other bases.

Denote the $\overline{H\mathbb{F}_2}$-Steenrod algebra of operations in $\overline{H\mathbb{F}_2}$-cohomology by

$$\overline{\mathcal{A}} \cong \pi_{-*,-*}\left(F(\overline{H\mathbb{F}_2}, \overline{H\mathbb{F}_2})\right),$$

and its dual algebra of co-operations in $\overline{H\mathbb{F}_2}$-homology by

$$\overline{\mathcal{A}}^\vee \cong \pi_{*,*}\left(\overline{H\mathbb{F}_2} \wedge \overline{H\mathbb{F}_2}\right).$$

The two main ingredients for these computations are our previous knowledge of the $H\mathbb{F}_2$-Steenrod algebra $\mathcal{A}$, which we recalled in Section 2.2, and the descriptions of $C\tau \wedge C\tau$ and $\mathrm{End}(C\tau)$ from Section 4. Since $\tau \in \mathbb{M}_2$ is an element of the base ring, there is an induced Hopf algebra structure over $\mathbb{M}_2/\tau \cong \mathbb{F}_2$ on the quotients ${\mathcal{A}}/_{\tau}$ and ${\mathcal{A}^\vee}/_{\tau}$.

Proposition 5.5. *The dual $\overline{H\mathbb{F}}_2$-Steenrod algebra $\overline{\mathcal{A}}^\vee$ has the following Hopf algebra structure*

$$\overline{\mathcal{A}}^\vee \cong \mathcal{A}^\vee/_\tau \otimes E(\beta_\tau) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots) \otimes E(\beta_\tau)$$

where β_τ is a τ-Bockstein in degree $(1, -1)$ which is primitive in the coalgebra structure.

Proof. The dual $\overline{H\mathbb{F}}_2$-Steenrod algebra is given by the homotopy groups of the E_∞ ring spectrum

$$\overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2 = H\mathbb{F}_2 \wedge C\tau \wedge H\mathbb{F}_2 \wedge C\tau \simeq H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau \wedge C\tau.$$

Since $\pi_{*,*}(\overline{H\mathbb{F}}_2) \cong \mathbb{F}_2$, the left and right units of the Hopf algebroid $\pi_{*,*}\left(\overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2\right)$ are flat maps and they agree, turning it into a Hopf algebra. If we smash the canonical equivalence $C\tau \wedge C\tau \simeq C\tau \vee \Sigma^{1,-1}C\tau$ of Lemma 3.6 with $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we get an additive splitting

$$\overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2 \simeq (H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau) \vee \left(\Sigma^{1,-1}H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau\right),$$

into two wedge summands that we can understand individually. Since the dual Steenrod algebra $\mathcal{A}^\vee$ is τ-free, Lemma 5.2 gives a ring description of the homotopy

$$\pi_{*,*}(H\mathbb{F}_2 \wedge H\mathbb{F}_2 \wedge C\tau) \cong \mathcal{A}^\vee/_\tau,$$

and thus the dual $\overline{H\mathbb{F}}_2$-Steenrod algebra is a free module of rank 2 over $\mathcal{A}^\vee$. The first generator in degree $(0, 0)$ is the unit given by the ring map

$$S^{0,0} \xrightarrow{i} \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2.$$

The second generator in degree $(1, -1)$ that we call β_τ is given by the map

$$\beta_\tau \colon S^{1,-1} \xrightarrow{i} \Sigma^{1,-1}C\tau \xrightarrow{s} C\tau \wedge C\tau \xrightarrow{i\wedge i} \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2,$$

where i denotes the inclusion of the bottom cell and s denotes the canonical section of μ, as in Lemma 3.6. We choose the name β_τ because its dual element in the $\overline{H\mathbb{F}}_2$-Steenrod algebra does behave like a τ-Bockstein in cohomology, as we explain in Proposition 5.6. To finish the description of the ring structure of $\overline{\mathcal{A}}^\vee$, we have to compute the product $\beta_\tau \cdot \beta_\tau$ which lands in degree $(2, -2)$. This product is the homotopy class of the composite

$$\beta_\tau \cdot \beta_\tau \colon S^{1,-1} \wedge S^{1,-1} \xrightarrow{\beta_\tau \wedge \beta_\tau} \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2 \xrightarrow{\mu} \overline{H\mathbb{F}}_2 \wedge \overline{H\mathbb{F}}_2$$

which is nullhomotopic since $\mu_{C\tau} \circ s \simeq 0$. This gives the ring structure as the tensor products

$$\overline{\mathcal{A}}^\vee \cong \mathcal{A}^\vee/\tau \otimes E(\beta_\tau) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots) \otimes E(\beta_\tau).$$

For the coalgebra structure, the counit is forced as there is only a copy of $\mathbb{F}_2$ in degree $(0,0)$. It thus only remains to compute the coproduct. The ring map

$$H\mathbb{F}_2 \xrightarrow{i} \overline{H\mathbb{F}_2}$$

induces the following map of Hopf algebroids

$$\mathcal{A}^\vee \xrightarrow{\psi} \overline{\mathcal{A}}^\vee \cong \mathcal{A}^\vee/\tau \otimes E(\beta_\tau) \colon a \longmapsto a \otimes 1,$$

which can be factored as reduction modulo τ and then inclusion into the $- \otimes 1$ factor. It follows that the coproduct $\Delta(a \otimes 1)$ can be computed by choosing a pre-image a of $a \otimes 1$, computing the coproduct in $\mathcal{A}^\vee$, and then pushing it back via ψ. Since the coproduct formula on the ξ_i's and τ_i's in $\mathcal{A}^\vee$ does not involve any τ-multiples, the exact same formula holds for the coproduct of elements of the form $a \otimes 1 \in \overline{\mathcal{A}}^\vee$. It only remains to compute the diagonal on the element $1 \otimes \beta_\tau$. We show in the next Proposition 5.6 that its dual is exterior in the algebra structure of $\overline{\mathcal{A}}$, implying that $1 \otimes \beta_\tau$ is primitive. ☐

Proposition 5.6. *The* $\overline{H\mathbb{F}_2}$*-Steenrod algebra* $\overline{\mathcal{A}}$ *has the following Hopf algebra structure*

$$\overline{\mathcal{A}} \cong \mathcal{A}/_{\tau} \otimes E(\beta_\tau)$$

where β_τ *is a* τ*-Bockstein in degree* $(1,-1)$ *which is primitive in the coalgebra structure.*

Proof. Since $C\tau$ is dualizable we can rewrite

$$F(\overline{H\mathbb{F}_2}, \overline{H\mathbb{F}_2}) = F(H\mathbb{F}_2 \wedge C\tau, H\mathbb{F}_2 \wedge C\tau) \simeq F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau \wedge DC\tau.$$

By the identification of Section 4.2 we further have

$$F(\overline{H\mathbb{F}_2}, \overline{H\mathbb{F}_2}) \simeq (F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau) \vee \left(\Sigma^{-1,1} F(H\mathbb{F}_2, H\mathbb{F}_2) \wedge C\tau\right).$$

By Lemma 5.2 we get that $\overline{\mathcal{A}}$ is a free $\mathcal{A}/\tau$-module of rank 2 with generators given by the operations

$$\mathrm{id} \colon \overline{H\mathbb{F}_2} \longrightarrow \overline{H\mathbb{F}_2} \qquad \text{and} \qquad \beta_\tau \colon \overline{H\mathbb{F}_2} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H\mathbb{F}_2},$$

where p denotes the projection of $C\tau$ on its top cell, while i denotes the inclusion of it bottom cell. The definition of β_τ explains why we call it a τ-Bockstein. Since the Steenrod algebra is defined as negative homotopy groups of the endomorphism spectrum, the τ-Bockstein β_τ is in degree $(1,-1)$. This settles the additive structure of $\overline{\mathcal{A}}$, and it remains to understand its Hopf algebra structure. Since $\overline{\mathcal{A}}^\vee$ is a Hopf algebra of finite type, we can dualize its structure from Proposition 5.5 to get the desired Hopf algebra structure of $\overline{\mathcal{A}}$. Recall that we did not yet finish the proof of Proposition 5.5, as we still have to show that $\beta_\tau \in \overline{\mathcal{A}}^\vee$ is primitive. This is equivalent to $\beta_\tau \in \overline{\mathcal{A}}$ being exterior, which is clear since it is the composite

$$\beta_\tau \circ \beta_\tau \colon \overline{H\mathbb{F}_2} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H\mathbb{F}_2} \xrightarrow{p} \Sigma^{1,-1} H\mathbb{F}_2 \xrightarrow{i} \Sigma^{1,-1}\overline{H\mathbb{F}_2},$$

which is nullhomotopic as $p \circ i \simeq 0$. □

REMARK 5.7 ($C\tau$-linear $\overline{H\mathbb{F}_2}$-homology and cohomology). We can define the $C\tau$-linear homology and cohomology of a $C\tau$-module X to be

$$\overline{H\mathbb{F}_2}^{C\tau}_{*,*}(X) \coloneqq \pi_{*,*}(\overline{H\mathbb{F}_2} \wedge_{C\tau} X) \quad \text{and} \quad \overline{H\mathbb{F}_2}^{*,*}_{C\tau}(X) \coloneqq \pi_{-*,-*}\left(F_{C\tau}(X, \overline{H\mathbb{F}_2})\right).$$

The relevant $\overline{H\mathbb{F}_2}$-Steenrod algebra of $C\tau$-linear operations and co-operations are then

$$\pi_{-*,-*}\left(F_{C\tau}(\overline{H\mathbb{F}_2}, \overline{H\mathbb{F}_2})\right) \qquad \text{and} \qquad \overline{\mathcal{A}}^\vee \cong \pi_{*,*}\left(\overline{H\mathbb{F}_2} \wedge_{C\tau} \overline{H\mathbb{F}_2}\right).$$

Their computation follows from Lemmas 5.2 and 5.3, and the result is the usual motivic Steenrod algebra and its dual, modulo τ. The only difference with the computations of Propositions 5.5 and 5.6 is that the $C\tau$-linear Steenrod algebras do not contain the τ-Bockstein element β_τ. In particular, the dual $C\tau$-linear $\overline{H\mathbb{F}_2}$-Steenrod algebra enjoys the nice formula

$$\mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_0, \tau_1, \ldots)$$

that is very reminiscent of the odd-primary classical Steenrod algebra.

5.3 THE $C\tau$-INDUCED MOORE SPECTRUM

Denote by $S^0/2$ the mod 2 Moore spectrum in the usual category of topological spectra SPT. Recall that the classical Toda bracket $\langle 2, \eta, 2\rangle = \eta^2$ implies that $\pi_2(S^0/2) \cong \mathbb{Z}/4$. This shows that multiplication by 2 is not a nullhomotopic map on $S^0/2$, and thus that there is no possible filler in the diagram

$$\begin{tikzcd}
S^0 \wedge S^0/2 \arrow[r, "2"] & S^0 \wedge S^0/2 \arrow[r] \arrow[dr, "\simeq"] & S^0/2 \wedge S^0/2 \arrow[r] \arrow[d, dashed, "\nexists\, \mu"] & \Sigma^1 S^0/2 \\
 & & S^0/2. &
\end{tikzcd}$$

This shows that there exists no left unital multiplication on $S^0/2$.

Denote now the motivic mod 2 Moore spectrum by $S^{0,0}/2$. Similarly, we can compute the motivic homotopy group $\pi_{2,0}(S^{0,0}/2) \cong \mathbb{Z}/4$ via the same argument. More precisely, the analoguous Toda bracket is $\langle 2, \tau\eta, 2\rangle = \tau^2\eta^2$, where $\eta \in \pi_{1,1}(S^{0,0})$ and thus $\tau\eta \in \pi_{1,0}(S^{0,0})$. This again implies that there is no left unital multiplication on the Moore spectrum $S^{0,0}/2$. Observe that this could

also have been noticed by the fact that a left unital multiplication on $S^{0,0}/2$ would induce one on $S^0/2$ by Betti realization.
Denote the cofiber of multiplication by τ on $S^{0,0}/2$ by $S/(2,\tau)$. This spectrum does admit a left unital multiplication since

$$\langle 2, \eta, 2\rangle = \tau\eta^2 \equiv 0 \qquad \text{modulo } \tau.$$

This does not imply that there is a ring structure on $S/(2,\tau)$ as this bracket is just one possible obstruction (the obstruction to left unitality). In Theorem 5.9 we show that all obstructions are of this type and that $S/(2,\tau)$ admits the structure of an E_∞ algebra over $C\tau$.
Since cofibers in $C\tau$-modules can be computed in the underlying category of motivic spectra, it follows that the cofiber of 2 on $C\tau$ has underlying spectrum $S/(2,\tau)$. Consider now $S/(2,\tau)$ as a $C\tau$-module, for example as constructed in the category ${}_{C\tau}\text{MOD}$ by the cofiber sequence

$$C\tau \xrightarrow{2} C\tau \xrightarrow{i} S/(2,\tau) \xrightarrow{p} \Sigma^{1,0}C\tau. \tag{5.3}$$

To equip $S/(2,\tau)$ with an E_∞ $C\tau$-algebra structure, we will proceed very similarly as in Section 3, which we refer to for more details.

PROPOSITION 5.8. *There is a unique homotopy unital and homotopy commutative $C\tau$-algebra structure on $S/(2,\tau)$.*

Proof. The computation of $[S/(2,\tau), S/(2,\tau)]_{C\tau} \cong \mathbb{Z}/2$ generated by the identity map shows that $\cdot 2$ is nullhomotopic on $S/(2,\tau)$, providing a left unital multiplication μ from diagram

$$\begin{tikzcd}
C\tau \wedge_{C\tau} S/(2,\tau) \arrow[r, "2"] & C\tau \wedge_{C\tau} S/(2,\tau) \arrow[r, "i_L"] \arrow[dr, "\simeq"] & S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[r, "p_L"] \arrow[d, dashed, "\exists\, \mu"] & \Sigma^{1,0} C\tau \wedge_{C\tau} S/(2,\tau) \\
 & & S/(2,\tau). &
\end{tikzcd}$$

The computation $\left[\Sigma^{1,0}C\tau \wedge_{C\tau} S/(2,\tau), S/(2,\tau)\right]_{C\tau} = 0$ shows that there is a unique left unital multiplication up to homotopy on $S/(2,\tau)$. As in Lemma 3.6, it also implies that there is a unique section s of p_L, giving a canonical additive splitting

$$S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \simeq S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau). \tag{5.4}$$

The induced multiplication $\widetilde{\mu}$ after this identification is again just projection onto the first factor, and the factor swap map χ is given by the following diagram

$$\begin{tikzcd}
S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[r, "\chi"] & S/(2,\tau) \wedge_{C\tau} S/(2,\tau) \arrow[d, "{(\mu, p_L)}"] \\
S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau) \arrow[u, "i_L + s"] \arrow[r, dashed, "{\left[\begin{smallmatrix} 1 & 0 \\ i\circ p & 1 \end{smallmatrix}\right]}"] & S/(2,\tau) \vee \Sigma^{1,0} S/(2,\tau).
\end{tikzcd}$$

The matrix can be completely determined since $[S/(2,\tau),S/(2,\tau)]_{C\tau}\cong\mathbb{Z}/2$. By an easy matrix multiplication as in Proposition 3.9, this shows that μ is right unital and homotopy commutative. □

The next step is to show that this (unique) multiplication map μ on $S/(2,\tau)$ can be extended to an E_∞ multiplication. We proceed in the exact same way as we did in Proposition 3.10 and Theorem 3.13.

Theorem 5.9. *The $C\tau$-algebra structure on $S/(2,\tau)$ can be uniquely extended to an E_∞ structure.*

Proof. We first extend it to an A_∞ structure as in Proposition 3.10, with obstructions living in the abelian group

$$\left[\Sigma^{n-3,0}(\Sigma^{1,0}C\tau)^{\wedge n},S/(2,\tau)\right]_{C\tau}\cong\left[\Sigma^{2n-3,0}C\tau^{\wedge n},S/(2,\tau)\right]_{C\tau}$$

for $n\geq 3$. Here we used $\Sigma^{1,0}C\tau$ since it is the cofiber of the unit map $C\tau\xrightarrow{i}S/(2,\tau)$. By using the decomposition formula for $C\tau^{\wedge n}$ from Corollary (3.7), the obstructions live in the group

$$\bigoplus_{i=0}^{n}\binom{n}{i}\left[\Sigma^{2n-3+i,-i}C\tau,S/(2,\tau)\right]_{C\tau}.$$

By the free-forget adjunction these groups are

$$\pi_{2n-3+i,-i}(S/(2,\tau)).$$

For $n\geq 3$ and for any $0\leq i\leq n$ this homotopy group is zero, making the obstruction group zero and allowing μ to extend to an A_∞ structure. Similarly the obstructions for uniqueness live in zero groups, showing that $S/(2,\tau)$ admits a unique A_∞ algebra structure over $C\tau$.
The A_3 structure gives an associative homotopy, and thus we now have a unital, associative and commutative monoid in the homotopy category. This is a 3-stage in Robinsin's obstruction theory, so we can apply Corollary 3.2 to extend it to an E_∞ ring structure. The obstructions live in

$$\left[\Sigma^{n-3,0}S/(2,\tau)^{\wedge m},S/(2,\tau)\right]_{C\tau}$$

for $n\geq 4$ and $2\leq m\leq n$, where the smash product is over $C\tau$. As in the proof of Theorem 3.13, we first break the source in smaller pieces by recursively using equation (5.4). It is then easy to show that all of those groups are zero by using cofiber sequences in the first variable to reduce it to homotopy groups of $S/(2,\tau)$. Similarly, the obstructions for uniqueness live in

$$\left[\Sigma^{n-2,0}S/(2,\tau)^{\wedge m},S/(2,\tau)\right]_{C\tau}$$

for $n\geq 4$ and $2\leq m\leq n$. We show by the exact same method that all those groups are zero, finishing the proof. □

REMARK 5.10. The fact that multiplication by 2 is nullhomotopic on $S/(2,\tau) \simeq C\tau/2$ is not so surprising, as $C\tau$ is somehow of algebraic nature. In fact, multiplication by n on X/n is always nullhomotopic in such algebraic categories, as explained in [Sch10, Proposition 1].

REMARK 5.11. The Toda bracket $\langle 2, \eta, 2\rangle = \eta^2$ is also responsible for the non-existence of a v_1^1-self map on the topological Moore spectrum $S^0/2$. This is illustrated in the diagram

$$\begin{tikzcd}
S^2/2 \arrow[d, dashed, "\nexists"'] & S^2 \arrow[l, "i"'] \arrow[d, "\eta"] \arrow[dl, dashed, "\exists\, \widetilde{\eta}"'] & S^2 \arrow[l, "2"'] \\
S^0/2 \arrow[r, "p"] & S^1 \arrow[r, "2"] & S^1.
\end{tikzcd}$$

The map $\widetilde{\eta}$ exists since $2\eta = 0$, but there is no v_1^1-self map as $2 \cdot \widetilde{\eta} \neq 0$. Motivically, the same diagram has the same problem because of the non-vanishing of the bracket $\langle 2, \eta, 2\rangle = \tau\eta^2$. However, in $C\tau$-modules this bracket vanishes and the $C\tau$-induced Moore spectrum admits a v_1^1-self map. The diagram

$$\begin{tikzcd}
\Sigma^{2,1}S/(2,\tau) \arrow[d, dashed, "\exists\, v_1"'] & \Sigma^{2,1}C\tau \arrow[l, "i"'] \arrow[d, "\eta"] \arrow[dl, dashed, "\exists\, \widetilde{\eta}"'] & \Sigma^{2,1}C\tau \arrow[l, "2"'] \\
S/(2,\tau) \arrow[r, "p"] & \Sigma^{1,0}C\tau \arrow[r, "2"] & \Sigma^{1,0}C\tau
\end{tikzcd}$$

exhibits this v_1-self map

$$\Sigma^{2,1}S/(2,\tau) \xrightarrow{v_1} S/(2,\tau).$$

More precisely, this follows since the computation $\left[\Sigma^{2,1}C\tau, S/(2,\tau)\right] \cong \mathbb{Z}/2$ forces the relation $2 \cdot \widetilde{\eta} \simeq 0$.

5.4 THE $C\tau$-INDUCED ALGEBRAIC AND HERMITIAN K-THEORY SPECTRA

Consider the motivic algebraic K-theory spectrum KGL constructed in [Voe98], which represents algebraic K-theory on smooth schemes. More precisely, given any smooth scheme X, the KGL-cohomology of its stabilization $\Sigma_+^\infty X$ computes the algebraic K-theory of the scheme X. Voevodsky constructed in [Voe02] its 0-effective cover $kgl := f_0(KGL)$. In the 2-completed category, a model for kgl is also given by the connective cover kgl as described in [IS11] over Spec $\mathbb{C}$ and in [NSØ15] over more general basis. It is shown in [NSØ15] that both KGL and kgl admit a unique E_∞ ring structure. Recall that we work in the 2-completed category, and we use kgl to denote the 2-completed connective algebraic K-theory spectrum. Its coefficients and mod 2 homology of kgl over $\mathrm{Spec}\,\mathbb{C}$ are computed in [IS11] and given by

$$\begin{aligned} \pi_{*,*}(kgl) &\cong \hat{\mathbb{Z}}_2[\tau, v_1] \\ H\mathbb{F}_{2*,*}(kgl) &\cong \mathbb{F}_2[\tau][\xi_1, \xi_2, \ldots][\tau_2, \tau_3, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}\,, \end{aligned}$$

where the element v_1 is in degree $(2,1)$ and corresponds to the usual Bott periodicity. Its homology is written as a subalgebra of the mod 2 homology of $H\mathbb{F}_2$ recalled in equation (2.2).

Consider now the hermitian K-theory spectrum KQ defined in [Hor05] and studied in [RØ16]. The paper [IS11] defines its connective cover kq over $\mathrm{Spec}\,\mathbb{C}$, by taking appropriate C_2-fixed points (although it is denoted by ko in that paper). It also computes its coefficients and mod 2 homology

$$\begin{aligned} \pi_{*,*}(kq) &\cong \hat{\mathbb{Z}}_2[\tau, \eta, a, b] \Big/ 2\eta, \tau\eta^3, a\eta, a^2 = 4b \\ H\mathbb{F}_{2*,*}(kq) &\cong \mathbb{F}_2[\tau][\xi_1^2, \xi_2, \ldots][\tau_2, \tau_3, \ldots] \Big/ \tau_i^2 = \tau\xi_{i+1}\,. \end{aligned}$$

To explain the homotopy ring $\pi_{*,*}(kq)$, Figure 4 displays the E_∞-page of the motivic Adams spectral sequence computing $\pi_{*,*}(kq)$. The horizontal axis represents the stem, i.e., the s in $\pi_{s,w}(kq)$, while the vertical axis represents the Adams filtration. As it is usually done with motivic charts, the weight w in $\pi_{s,w}(kq)$ is suppressed from the chart and one can imagine it on a third axis perpendicular to the page.

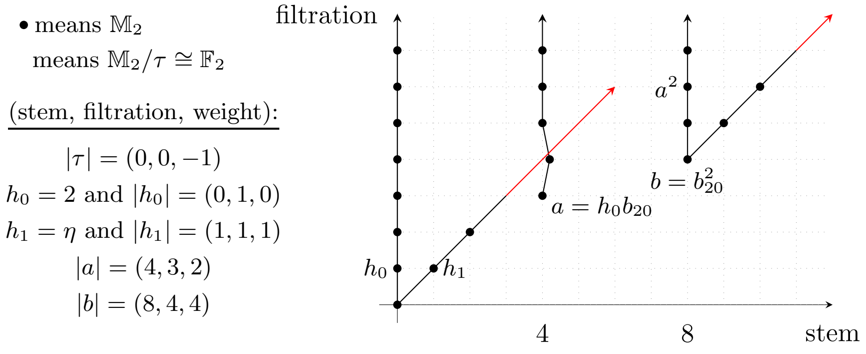

Figure 4: The E_∞-page of the Adams spectral sequence computing $\pi_{*,*}(kq)$.

In this Section we consider the $C\tau$-induced spectra that we denote by

$$\overline{kgl} := kgl \wedge C\tau \qquad \text{and} \qquad \overline{kq} := kq \wedge C\tau.$$

Both of them are $C\tau$-algebras, where $\overline{kgl}$ is an E_∞ algebra as it is the smash product of two E_∞ rings.

The case of algebraic K-theory $\overline{kgl}$

The fact that both its homotopy and homology are τ-free makes the description of $\overline{kgl}$ straightforward. Indeed, by Lemma 5.2 we immediately get

$$\pi_{*,*}(\overline{kgl}) \cong \hat{\mathbb{Z}}_2[v_1] \qquad \text{and} \qquad H\mathbb{F}_{2*,*}(\overline{kgl}) \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots] \otimes E(\tau_2, \tau_3, \ldots).$$

The case of hermitian K-theory $\overline{kq}$

Its homology is τ-free and so again we immediately get

$$H\mathbb{F}_{2*,*}(\overline{kq}) \cong \mathbb{F}_2[\xi_1^2, \xi_2, \ldots] \otimes E(\tau_2, \tau_3, \ldots).$$

Its homotopy is more interesting as it is not τ-free, and we will get contributions both from the cokernel and kernel of multiplication by τ. Moreover, a surprising fact occurs as there is a hidden extension which makes $\overline{kq}$ contain the periodicity element v_1^2 in its homotopy.

Proposition 5.12. *The homotopy ring* $\pi_{*,*}(\overline{kq})$ *has the presentation*

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, v_1^2] \,/_{2\eta}.$$

Proof. The usual cofiber sequence (2.5) for $C\tau$, smashed with kq gives the cofiber sequence

$$\Sigma^{0,-1}kq \xrightarrow{\tau} kq \xrightarrow{i} \overline{kq} \xrightarrow{p} \Sigma^{1,-1}kq.$$

Since the homology $H\mathbb{F}_{2*,*}\left(\Sigma^{0,-1}kq\right)$ is τ-free, we ge the short exact sequence

$$0 \longrightarrow H\mathbb{F}_{2*,*}\left(\Sigma^{0,-1}kq\right) \xrightarrow{\tau} H\mathbb{F}_{2*,*}\left(kq\right) \xrightarrow{i} H\mathbb{F}_{2*,*}\left(\overline{kq}\right) \longrightarrow 0$$

in homology. For any motivic spectrum X, denote by $\mathrm{Ext}^*(X)$ the trigraded term

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^\vee\text{-comod}}(H\mathbb{F}_{2*,*}(S^{0,0}), H\mathbb{F}_{2*,*}(X))$$

that represents the E_2 page of the motivic Adams spectral sequence for X. We use the indicated grading in $\mathrm{Ext}^*(X)$ to denote the homological degree in Ext, i.e., the Adams filtration on the E_2 page. From the above short exact sequence, we get a long exact sequence in Ext-groups

$$\cdots \xrightarrow{\tau} \mathrm{Ext}^*(kq) \xrightarrow{i_*} \mathrm{Ext}^*(\overline{kq}) \xrightarrow{p_*} \mathrm{Ext}^{*+1}(\Sigma^{0,-1}kq) \xrightarrow{\tau} \cdots,$$

i.e., a long exact sequence in E_2 pages. This gives short exact sequences

$$0 \longrightarrow \mathrm{Ext}^*(kq)\,/_{\tau} \xrightarrow{i_*} \mathrm{Ext}^*(\overline{kq}) \xrightarrow{p_*} {}_{\tau}\,\mathrm{Ext}^{*+1}(\Sigma^{0,-1}kq) \longrightarrow 0,$$

where the left term is the cokernel of τ while the right term is the τ-torsion. Since i is a ring map, the term $\mathrm{Ext}(kq)/\tau$ includes as a subring of $\mathrm{Ext}(\overline{kq})$. However, this cokernel can act non-trivially on the τ-torsion part, giving potential extension problems to solve. Since the motivic Adams spectral sequence for kq collapses at the E_2 page with no hidden extensions, the term $\mathrm{Ext}(kq)$ is given by the Figure 4 on page 1121. These two pieces assemble to give the additive description of the E_2 page of the motivic Adams spectral sequence for $\overline{kq}$ as described in Figure 5. It still remains to solve the possible extension problems and possible Adams differentials. The only possible extension is whether or not $2 \cdot \widetilde{h_1^3} = \overline{a}$, as indicated in Figure 5. Consider the Toda bracket $\langle \tau, \eta^3, 2\rangle$ as in the diagram

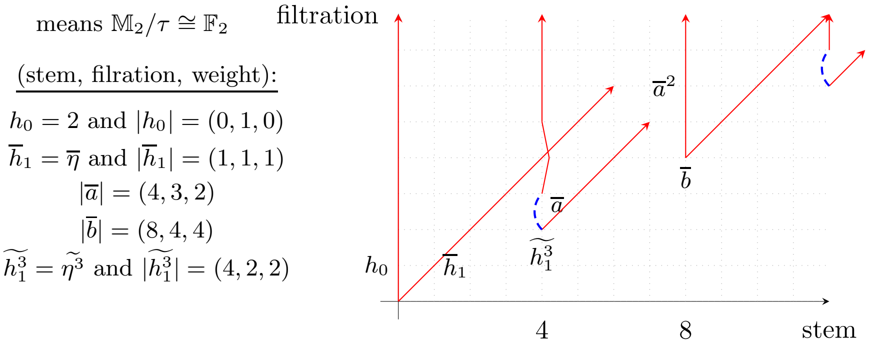

Figure 5: The E_2 page of the motivic Adams spectral sequence for $\overline{kq}$ as an $\mathbb{F}_2$-vector space.

$$\begin{tikzcd}
S^{3,2} \arrow[r, "2"] \arrow[rrdd, dashed] & S^{3,2} \arrow[r, "\eta^3"] \arrow[rd, dashed, "\widetilde{\eta^3}"'] & \Sigma^{0,-1}kq \arrow[r, "\tau"] & kq \\
 & & \Sigma^{-1,0}\overline{kq} \arrow[u, "p"'] & \\
 & & \Sigma^{-1,0}kq, \arrow[u, "i"'] &
\end{tikzcd}$$

where we have that $2 \cdot \widetilde{\eta^3} \in i_*\langle \tau, \eta^3, 2\rangle$ by [Isa, Section 3.1.1]. We can compute this bracket in the motivic May spectral sequence using May's convergence Theorem. See [May69] for the original reference, and [Isa, Theorem 2.2.3] for an exposition of the motivic version. More precisely, we can compute it on the motivic May E_3-page via the differential $d_3(b_{20}) = \tau h_1^3$ (since $h_0 h_1$ is already zero). This bracket has no indeterminacy giving

$$\langle \tau, h_1^3, h_0 \rangle = \{b_{20}h_0\}\,.$$

Recall from Figure 4 that $a = b_{20}h_0$ giving that indeed, in $\pi_{*,*}(\overline{kq})$, there is an extension $2 \cdot \widetilde{h_1^3} = \overline{a}$. This h_0-extension appears as the round dotted line on Figure 5. We now spell out the ring structure of this E_2 page. First observe that

$$4\left(\widetilde{h_1^3}\right)^2 = \left(2\widetilde{h_1^3}\right)^2 = \overline{a}^2 = 4\overline{b}^2,$$

and because there are no possible extensions in that column, we get that $\left(\widetilde{h_1^3}\right)^2 = \overline{b}$. The E_2 page of the motivic Adams spectral sequence for $\overline{kq}$ has therefore the ring presentation

$$E_2 \cong \mathbb{F}\left[h_0, \overline{h}_1, \widetilde{h_1^3}\right] \big/ h_0\overline{h}_1\,.$$

There are no possible Adams differentials on these 3 generators, and thus Figure 5 also represents the E_∞ page of the Adams spectral sequence for $\overline{kq}$. Except the h_0-towers, there are no possible hidden extensions, giving the multiplicative description

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, \widetilde{h_1^3}] \big/ 2\eta.$$

Finally, we show that $\widetilde{h_1^3}$ detects the element v_1^2. We can smash the cofiber sequence

$$\Sigma^{1,1}kq \xrightarrow{\eta} kq \xrightarrow{i} kgl$$

with $C\tau$ to obtain the cofiber sequence

$$\Sigma^{1,1}\overline{kq} \xrightarrow{\eta} \overline{kq} \xrightarrow{\overline{i}} \overline{kgl}.$$

Since i is a ring map, then so is the induced map $\overline{i}$. The ring map $\overline{i}$ sends the 8-fold Bott periodicity element $\overline{b} = \left(\widetilde{h_1^3}\right)^2$ to the 8-fold Bott periodicity element v_1^4, which forces $\widetilde{h_1^3}$ to be sent to v_1^2. The E_2 page of $\overline{kq}$ has therefore the ring presentation

$$\pi_{*,*}(\overline{kq}) \cong \hat{\mathbb{Z}}_2[\eta, v_1^2] \big/ 2\eta. \qquad \square$$

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