Wide Subcategories are Semistable

Toshiya Yurikusa[1]

Received: June 13, 2017
Revised: October 18, 2017

Communicated by Henning Krause

Abstract. For an arbitrary finite dimensional algebra Λ, we prove that any wide subcategory of $\mathsf{mod}\,\Lambda$ satisfying a certain finiteness condition is θ-semistable for some stability condition θ. More generally, we show that wide subcategories of $\mathsf{mod}\,\Lambda$ associated with two-term presilting complexes of Λ are semistable. This provides a complement for Ingalls-Thomas-type bijections for finite dimensional algebras.

2010 Mathematics Subject Classification: Primary 16G10; Secondary 18E30, 18E40, 19A13.
Keywords and Phrases: Representation theory of finite dimensional algebras, wide subcategories, semistable subcategories, τ-tilting theory

1 Introduction

The classification problem of subcategories is a well studied subject in representation theory, algebraic geometry and algebraic topology (e.g. [Hop, N, Th]). Among others, we refer to [Bru, Hov, IT, KS, MS, Ta] for recent developments on the classification of *wide subcategories*, which are full subcategories of an abelian category closed under kernels, cokernels and extensions.

Important examples of wide subcategories are given by geometric invariant theory for quiver representations [K]. Recall that a stability condition on $\mathsf{mod}\,\Lambda$ for a finite dimensional algebra Λ is a linear form θ on $K_0(\mathsf{mod}\,\Lambda)\otimes_{\mathbb{Z}}\mathbb{R}$, where $K_0(\mathsf{mod}\,\Lambda)$ is the Grothendieck group of $\mathsf{mod}\,\Lambda$. We say that $M \in \mathsf{mod}\,\Lambda$ is *θ-semistable* if $\theta(M) = 0$ and $\theta(L) \leq 0$ for any submodule L of M, or equivalently, $\theta(N) \geq 0$ for any factor module N of M. The full subcategory of θ-semistable Λ-modules is called the *θ-semistable subcategory* of $\mathsf{mod}\,\Lambda$. It is

[1]Supported by JSPS KAKENHI Grant Number JP17J04270.

basic that semistable subcategories of $\mathsf{mod}\,\Lambda$ are wide. They played important roles in representation theory and algebraic geometry (e.g. Igusa-Orr-Todorov-Weyman's work [IOTW] and Bridgeland's work [Bri]).
For quiver representations, Ingalls and Thomas [IT] gave bijections between wide/semistable subcategories and other important objects: For the path algebra kQ of a finite connected acyclic quiver Q over a field k, there are bijections (called *Ingalls-Thomas bijections*) between the following objects, where we refer to Subsection 3.1 for unexplained terminologies.

(1) Isomorphism classes of basic support tilting modules in $\mathsf{mod}(kQ)$.
(2) Functorially finite torsion classes in $\mathsf{mod}(kQ)$.
(3) Functorially finite wide subcategories of $\mathsf{mod}(kQ)$.
(4) Functorially finite semistable subcategories of $\mathsf{mod}(kQ)$.

They also proved that (1)-(4) above correspond bijectively with the clusters in the cluster algebra of Q and the isomorphism classes of basic cluster tilting objects in the cluster category of kQ.
Later, works of Adachi-Iyama-Reiten [AIR] and Marks-Stovicek [MS] gave the following Ingalls-Thomas-type bijections for an arbitrary finite dimensional k-algebra, where we refer to Subsection 3.1 for unexplained terminologies and explicit bijections.

Theorem 1.1. [AIR, Theorem 0.5][MS, Theorem 3.10] Let Λ be a finite dimensional algebra over a field k. There are bijections between the following objects:

(1) Isomorphism classes of basic support τ-tilting modules in $\mathsf{mod}\,\Lambda$.
(1′) Isomorphism classes of basic two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.
(2) Functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.
(2′) Functorially finite torsion free classes in $\mathsf{mod}\,\Lambda$.
(3) Left finite wide subcategories of $\mathsf{mod}\,\Lambda$.

Notice that the statement for semistable subcategories of $\mathsf{mod}\,\Lambda$ is missing in Theorem 1.1. The aim of this paper is to prove the following complement of Theorem 1.1.

Theorem 1.2. For a finite dimensional algebra Λ over a field k, the following objects are the same.

(3) Left finite wide subcategories of $\mathsf{mod}\,\Lambda$.
(4) Left finite semistable subcategories of $\mathsf{mod}\,\Lambda$.

Therefore, there are bijections between (1)-(4) in Theorem 1.1.

Since it is basic that semistable subcategories are wide subcategories, it suffices to show the converse. To construct a stability condition θ for a given left finite wide subcategory, we need the following preparation. Let T be a basic two-term silting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Then the corresponding support τ-tilting

Λ-module is given by $\mathrm{H}^0(T)$. In this paper, we mainly use T instead of $\mathrm{H}^0(T)$ since it is more convenient for our aim. There is a decomposition $T = T_\lambda \oplus T_\rho$ and a triangle

$$\Lambda \to T' \to T'' \to \Lambda[1] \tag{1.1}$$

in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, where $\mathsf{add}\,T' = \mathsf{add}\,T_\lambda$ and $\mathsf{add}\,T'' = \mathsf{add}\,T_\rho$ (see [AI, Lemma 2.25, Theorem 2.18]). Then T corresponds to the left finite wide subcategory

$$\mathcal{W}^T := \mathsf{Fac}\,\mathrm{H}^0(T_\lambda) \cap \mathrm{H}^0(T_\rho)^\perp \tag{1.2}$$

via the bijection between (1′) and (3) in Theorem 1.1 (see Subsection 3.1).
Our Theorem 1.2 is a consequence of the following result, where $\langle -, - \rangle$ is the Euler form (see (3.2)).

Theorem 1.3. *Let Λ be a finite dimensional algebra over a field k. Let T be a basic two-term silting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. We consider an $\mathbb{R}$-linear form θ defined by*

$$\sum_X a_X \langle X, - \rangle : K_0(\mathsf{mod}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

where X runs over all indecomposable direct summands of T_ρ, and a_X is an arbitrary positive real number for each X. Then $\mathcal{W}^T$ is the θ-semistable subcategory of $\mathsf{mod}\,\Lambda$.

We prove Theorem 1.3 in a more general setting. Any basic two-term presilting complex U in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ gives rise to a wide subcategory of $\mathsf{mod}\,\Lambda$ as follows: By [AIR, Proposition 2.9] (see also [BPP, Section 5]), there are two torsion pairs

$$(\mathcal{T}_U^+, \mathcal{F}_U^+) := ({}^\perp\mathrm{H}^{-1}(\nu U), \mathsf{Sub}\,\mathrm{H}^{-1}(\nu U)),$$
$$(\mathcal{T}_U^-, \mathcal{F}_U^-) := (\mathsf{Fac}\,\mathrm{H}^0(U), \mathrm{H}^0(U)^\perp)$$

in $\mathsf{mod}\,\Lambda$ such that $\mathcal{T}_U^+ \supseteq \mathcal{T}_U^-$ and $\mathcal{F}_U^+ \subseteq \mathcal{F}_U^-$. Then

$$\mathcal{W}_U := \mathcal{T}_U^+ \cap \mathcal{F}_U^-$$

is a wide subcategory of $\mathsf{mod}\,\Lambda$ (e.g. [DIRRT]), which is equivalent to $\mathsf{mod}\,C$ for some explicitly constructed finite dimensional algebra C (see [J, Theorem 1.4]).
Our Theorem 1.3 can be deduced from the following result since $\mathcal{W}^T = \mathcal{W}_{T_\rho}$ holds for any two-term silting complex T (see Lemma 3.5).

Theorem 1.4. *Let U be a basic two-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. We consider an $\mathbb{R}$-linear form θ defined by*

$$\sum_X a_X \langle X, - \rangle : K_0(\mathsf{mod}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

where X runs over all indecomposable direct summands of U, and a_X is an arbitrary positive real number for each X. Then $\mathcal{W}_U$ is the θ-semistable subcategory of $\mathsf{mod}\,\Lambda$.

Note that in the context of support τ-tilting modules, Theorem 1.4 was independently obtained by Brüstle-Smith-Treffinger [BST, Theorem 4.16] and Speyer-Thomas [ST].

Notations. Let Λ be a finite dimensional algebra over a field k, and $\mathsf{mod}\,\Lambda$ (resp., $\mathsf{proj}\,\Lambda$, $\mathsf{inj}\,\Lambda$) the category of finitely generated right Λ-modules (resp., projective right Λ-modules, injective right Λ-modules). For $M \in \mathsf{mod}\,\Lambda$, let $\mathsf{add}\,M$ (resp., $\mathsf{Fac}\,M$, $\mathsf{Sub}\,M$) be the category of all direct summands (resp., factor modules, submodules) of finite direct sums of copies of M. We denote by D the k-dual $\mathrm{Hom}_k(-,k)$.
For a full subcategory $\mathcal{S}$ of $\mathsf{mod}\,\Lambda$, let

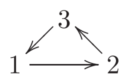
$$\mathcal{S}^{\perp} := \{M \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\Lambda}(\mathcal{S}, M) = 0\},$$

$$^{\perp}\mathcal{S} := \{M \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\Lambda}(M, \mathcal{S}) = 0\}.$$

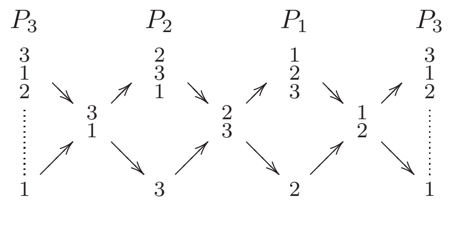
For an additive (resp., abelian) category $\mathcal{A}$, let $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ (resp., $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) be the homotopy (resp., derived) category of bounded complexes over $\mathcal{A}$. We denote by ν the Nakayama functor $D\Lambda \otimes_{\Lambda} - : \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda) \to \mathsf{K}^{\mathrm{b}}(\mathsf{inj}\,\Lambda)$.

2 Example

Before proving our results, we give an example.
Let Q be the quiver



and I be the two-sided ideal of the path algebra kQ generated by all paths of length three. Then $\Lambda := kQ/I$ is a finite dimensional k-algebra. The Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$ is the following



where P_i is the indecomposable projective Λ-module at vertex i. Table 1 gives a complete list of two-term silting complexes, support τ-tiling Λ-modules, functorially finite torsion classes and left finite wide subcategories in $\mathsf{mod}\,\Lambda$. The objects in each row correspond to each other under the bijections of Theorem 1.1. For $T \in$ 2-siltΛ, we write the class of indecomposable direct summands of T in $K_0(\mathsf{proj}\,\Lambda)$. Moreover, indecomposable direct summands X of T_ρ and $\mathrm{H}^0(X)$ are colored in blue.

Table 1: Example of Theorem 1.1

- 2-siltΛ : the set of isomorphism classes of basic two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.
- sτ-tiltΛ : the set of isomorphism classes of basic support τ-tilting modules in $\mathsf{mod}\,\Lambda$.
- f-torsΛ : the set of functorially finite torsion classes in $\mathsf{mod}\,\Lambda$.
- $\mathrm{f_L}$-wideΛ : the set of left finite wide subcategories of $\mathsf{mod}\,\Lambda$.

2-siltΛ	sτ-tiltΛ	f-torsΛ	$\mathrm{f_L}$-wideΛ
P_1, P_2, P_3	● ● ● ○ ○ ○ ○ ○ ○	● ● ● ● ● ● ● ● ●	● ● ● ● ● ● ● ● ●
$P_1, P_2,$ $\color{blue}{P_2 - P_3}$	○ ● ● ○ ○ ○ ○ ○ $\color{blue}\bullet$	○ ● ● ○ ● ● ● ○ ●	○ ● ● ○ ● ○ ● ○ ○
$P_1, P_3,$ $\color{blue}{P_1 - P_2}$	● ○ ● ○ ○ ○ $\color{blue}\bullet$ ○ ○	● ○ ● ● ○ ● ● ● ○	● ○ ● ○ ○ ● ○ ● ○
$P_2, P_3,$ $\color{blue}{P_3 - P_1}$	● ● ○ ○ ○ ○ ○ $\color{blue}\bullet$ ○	● ● ○ ● ● ○ ○ ● ●	● ● ○ ● ○ ○ ○ ○ ●
$P_1, \color{blue}{P_1 - P_3},$ $P_2 - P_3$	○ ○ ● ○ ○ $\color{blue}\bullet$ ○ ○ ●	○ ○ ● ○ ○ ● ● ○ ●	○ ○ ● ○ ○ ○ ○ ○ ●
$P_2, \color{blue}{P_2 - P_1},$ $\color{blue}{P_2 - P_3}$	○ ● ○ ○ $\color{blue}\bullet$ ○ ○ ○ $\color{blue}\bullet$	○ ● ○ ○ ● ○ ○ ○ ●	○ ● ○ ○ ○ ○ ○ ○ ○
$P_3, \color{blue}{P_3 - P_2},$ $P_1 - P_2$	● ○ ○ $\color{blue}\bullet$ ○ ○ ● ○ ○	● ○ ○ ● ○ ○ ● ● ○	● ○ ○ ○ ○ ○ ● ○ ○
$P_1, \color{blue}{P_1 - P_3},$ $\color{blue}{P_1 - P_2}$	○ ○ ● ○ ○ $\color{blue}\bullet$ $\color{blue}\bullet$ ○ ○	○ ○ ● ○ ○ ● ● ○ ○	○ ○ ● ○ ○ ○ ○ ○ ○
$P_2, \color{blue}{P_2 - P_1},$ $P_3 - P_1$	○ ● ○ ○ $\color{blue}\bullet$ ○ ○ ● ○	○ ● ○ ○ ● ○ ○ ● ●	○ ● ○ ○ ○ ○ ○ ● ○
$P_3, \color{blue}{P_3 - P_2},$ $\color{blue}{P_3 - P_1}$	● ○ ○ $\color{blue}\bullet$ ○ ○ ○ $\color{blue}\bullet$ ○	● ○ ○ ● ○ ○ ○ ● ○	● ○ ○ ○ ○ ○ ○ ○ ○

2-siltΛ	sτ-tiltΛ	f-torsΛ	$\mathrm{f_L}$-wideΛ
$P_1 - P_3,$ $P_2 - P_3, \color{blue}{-P_3}$	○ ○ ○ ○ ○ ● ○ ○ ●	○ ○ ○ ○ ○ ● ● ○ ●	○ ○ ○ ○ ○ ● ● ○ ●
$P_2 - P_1,$ $\color{blue}{P_2 - P_3, -P_1}$	○ ○ ○ ○ ● ○ ○ ○ $\color{blue}\bullet$	○ ○ ○ ○ ● ○ ○ ○ ●	○ ○ ○ ○ ● ○ ○ ○ ○
$P_3 - P_2,$ $P_1 - P_2, \color{blue}{-P_2}$	○ ○ ○ ● ○ ○ ● ○ ○	○ ○ ○ ● ○ ○ ● ● ○	○ ○ ○ ● ○ ○ ● ● ○
$P_1 - P_3,$ $\color{blue}{P_1 - P_2, -P_3}$	○ ○ ○ ○ ○ ● $\color{blue}\bullet$ ○ ○	○ ○ ○ ○ ○ ● ● ○ ○	○ ○ ○ ○ ○ ● ○ ○ ○
$P_2 - P_1,$ $P_3 - P_1, \color{blue}{-P_1}$	○ ○ ○ ○ ● ○ ○ ● ○	○ ○ ○ ○ ● ○ ○ ● ●	○ ○ ○ ○ ● ○ ○ ● ●
$P_3 - P_2,$ $\color{blue}{P_3 - P_1, -P_2}$	○ ○ ○ ● ○ ○ ○ $\color{blue}\bullet$ ○	○ ○ ○ ● ○ ○ ○ ● ○	○ ○ ○ ● ○ ○ ○ ○ ○
$P_2 - P_3,$ $\color{blue}{-P_1, -P_3}$	○ ○ ○ ○ ○ ○ ○ ○ ●	○ ○ ○ ○ ○ ○ ○ ○ ●	○ ○ ○ ○ ○ ○ ○ ○ ●
$P_1 - P_2,$ $\color{blue}{-P_2, -P_3}$	○ ○ ○ ○ ○ ○ ● ○ ○	○ ○ ○ ○ ○ ○ ● ○ ○	○ ○ ○ ○ ○ ○ ● ○ ○
$P_3 - P_1,$ $\color{blue}{-P_1, -P_2}$	○ ○ ○ ○ ○ ○ ○ ● ○	○ ○ ○ ○ ○ ○ ○ ● ○	○ ○ ○ ○ ○ ○ ○ ● ○
$\color{blue}{-P_1,}$ $\color{blue}{-P_2, -P_3}$	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○ ○ ○ ○

For a basic two-term presilting complex $U = U_1 \oplus \ldots \oplus U_m$ with indecomposable direct summands U_i in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, we consider the cone

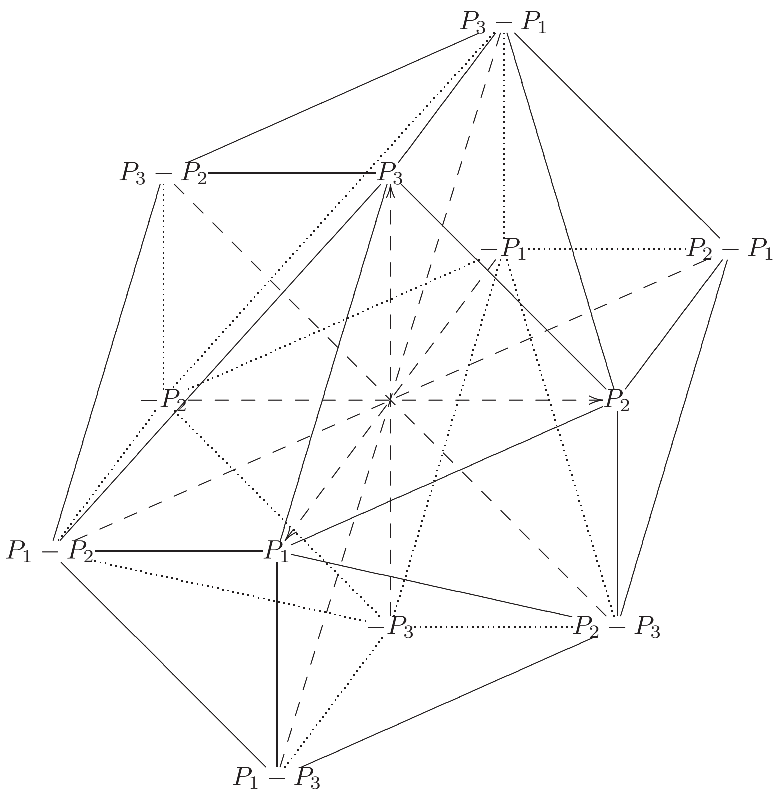
$$C(U) := \Big\{ \sum_{i=1}^{m} a_i [U_i] \mid a_i > 0 \ \ (1 \leq i \leq m) \Big\} \subseteq K_0(\mathsf{proj}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Since Λ is τ-tilting finite, we have a decomposition [DIJ]

$$(K_0(\mathsf{proj}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}) \setminus \{0\} = \bigsqcup_{U} C(U),$$

where U runs over isomorphism classes of basic two-term presilting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ (see Figure 1). By Theorem 1.4, any θ in the cone $C(U)$ gives rise to the wide subcategory $\mathcal{W}_U$ of $\mathsf{mod}\,\Lambda$.

Figure 1: The decomposition of $K_0(\mathsf{proj}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}$



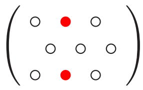
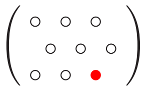
(1) We consider the case $U = U_1 \oplus U_2$, where

$$U_1 = (P_3 \to P_2), \ \ U_2 = (P_1 \to 0).$$

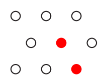
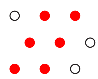
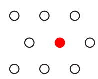
Applying the Nakayama functor,

$$\nu U = \nu U_1 \oplus \nu U_2 = (I_3 \to I_2) \oplus (I_1 \to 0)$$

holds. So we have

$$\mathrm{H}^{-1}(\nu U) = 3 \oplus \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}$$  and $\mathrm{H}^{0}(U) = 2$ .

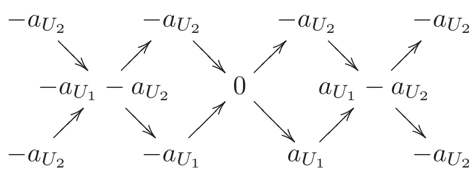
Then the cone $C(U) = \{a_1([P_2] - [P_3]) + a_2(-[P_1]) \mid a_1 > 0, a_2 > 0\}$ gives rise to the wide subcategory

$$\begin{matrix} \mathcal{T}_U^+ \\ \| \\ {}^{\perp}\mathrm{H}^{-1}(\nu U) \end{matrix}$$  $\cap$ $$\begin{matrix} \mathcal{F}_U^- \\ \| \\ \mathrm{H}^{0}(U)^{\perp} \end{matrix}$$  $= \mathcal{W}_U$  $= \mathsf{add}\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$

of $\mathsf{mod}\,\Lambda$. In fact, let

$$\theta = a_{U_1}\langle U_1, -\rangle + a_{U_2}\langle U_2, -\rangle : K_0(\mathsf{mod}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$$

be an $\mathbb{R}$-linear form, where $a_{U_1} > 0$ and $a_{U_2} > 0$. We can calculate the values of θ for indecomposable Λ-modules as follows:



Thus $\mathcal{W}_U = \mathsf{add}\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$ holds. In fact, an indecomposable Λ-module X satisfies $\theta(X) = 0$ if and only if $X = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ or ($X = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and $a_{U_1} = a_{U_2}$). Moreover, $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$ is θ-semistable since $\theta(2) > 0$, while $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ is never θ-semistable since $\theta(1) < 0$.

(2) Let $T = U \oplus U_3 \in$ 2-siltΛ, where

$$U_3 = (P_1 \to P_2).$$

Then there is a triangle

$$\Lambda \to U_3 \oplus U_3 \to U_1 \oplus U_2 \oplus U_2 \oplus U_2 \to \Lambda[1]$$

in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Thus $T_\lambda = U_3$ and $T_\rho = U_1 \oplus U_2 = U$. By Theorem 1.3, $\mathcal{W}^T$ is the θ-semistable subcategory of $\mathsf{mod}\,\Lambda$ for the above θ. In particular, we have $\mathcal{W}^T = \mathcal{W}_U = \mathsf{add}\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)$, as the second row of the right column in Table 1 shows.

3 Proofs of our results

3.1 Preliminaries

We recall unexplained terminologies and the bijections of Theorem 1.1 from [Ai, AI, AIR, ASS, KV].

Let $\mathcal{S}$ be a full subcategory of $\mathsf{mod}\,\Lambda$. We call $\mathcal{S}$ a *torsion class* (resp., *torsion free class*) if it is closed under extensions and quotients (resp., extensions and submodules) [ASS]. For subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathsf{mod}\,\Lambda$, a pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. Then $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class. Conversely, any torsion class (resp., torsion free class) gives rise to a torsion pair. We call $\mathcal{S}$ *functorially finite* if any Λ-module admits both a left and a right $\mathcal{S}$-approximation. More precisely, for any $M \in \mathsf{mod}\,\Lambda$, there are morphisms $g_1 : M \to S_1$ and $g_2 : S_2 \to M$ with $S_1, S_2 \in \mathcal{S}$ such that $\mathrm{Hom}_\Lambda(g_1, S)$ and $\mathrm{Hom}_\Lambda(S, g_2)$ are surjective for any $S \in \mathcal{S}$. Then g_1 is called a *left $\mathcal{S}$-approximation of M* and g_2 is called a *right $\mathcal{S}$-approximation of M*. We call $\mathcal{S}$ *left finite* if the minimal torsion class containing $\mathcal{S}$ is functorially finite (see [As]).

Let $M \in \mathsf{mod}\,\Lambda$. We call M *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, where τ is the Auslander-Reiten translation of $\mathsf{mod}\,\Lambda$ [AIR]. We call M *support τ-tilting* if M is τ-rigid and there is an idempotent e of Λ such that $Me = 0$ and $|M| = |\Lambda/\langle e\rangle|$, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M.

Let $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. We call P *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)}(P, P[i]) = 0$ for any $i > 0$ [Ai, AI, KV]. We call P *silting* if P is presilting and satisfies $\mathsf{thick}\,P = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, where $\mathsf{thick}\,P$ is the smallest subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ containing P which is closed under shifts, cones and direct summands. We say that $P = (P^i, d^i)$ is *two-term* if $P^i = 0$ for all $i \neq 0, -1$. We denote by 2-presiltΛ (resp., 2-siltΛ) the set of isomorphism classes of basic two-term presilting (resp., silting) complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

The bijections of Theorem 1.1 are given in the following way [AIR, Theorems 2.7, 3.2][MS, Lemma 3.8, Theorem 3.10][S, Theorem]:

$$\begin{aligned}
(1') \to (1) \quad &: \quad T \mapsto \mathrm{H}^0(T),\\
(1') \to (2) \quad &: \quad T \mapsto \mathcal{T}_T^- = \mathsf{Fac}\,\mathrm{H}^0(T),\\
(1') \to (2') \quad &: \quad T \mapsto \mathcal{F}_T^+ = \mathsf{Sub}\,\mathrm{H}^{-1}(\nu T),\\
(2) \to (2') \quad &: \quad \mathcal{T} \mapsto \mathcal{T}^{\perp}, \qquad (2) \leftarrow (2') \quad : \quad {}^{\perp}\mathcal{F} \leftarrow\!\shortmid \mathcal{F},\\
(1') \to (3) \quad &: \quad T \mapsto \mathcal{W}^T = \mathsf{Fac}\,\mathrm{H}^0(T_\lambda) \cap \mathrm{H}^0(T_\rho)^{\perp},
\end{aligned}$$

where $\mathrm{H}^i(T)$ is the i-th cohomology of T. Recall that we have

$$({}^{\perp}\mathrm{H}^{-1}(\nu T), \mathsf{Sub}\,\mathrm{H}^{-1}(\nu T)) = (\mathcal{T}_T^+, \mathcal{F}_T^+) = (\mathcal{T}_T^-, \mathcal{F}_T^-) = (\mathsf{Fac}\,\mathrm{H}^0(T), \mathrm{H}^0(T)^{\perp}) \tag{3.1}$$

for $T \in$ 2-siltΛ [AIR, Proposition 2.16].

3.2 Linear forms on Grothendieck groups

Let Λ be a finite dimensional algebra over a field k. Let $K_0(\mathsf{mod}\,\Lambda)$ and $K_0(\mathsf{proj}\,\Lambda)$ be the Grothendieck groups of the abelian category $\mathsf{mod}\,\Lambda$ and the exact category $\mathsf{proj}\,\Lambda$ with only split short exact sequences, respectively. Then we have natural isomorphisms $K_0(\mathsf{mod}\,\Lambda) \simeq K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda))$ and $K_0(\mathsf{proj}\,\Lambda) \simeq K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda))$. Moreover, $K_0(\mathsf{mod}\,\Lambda)$ has a basis consisting of the isomorphism classes S_i of simple Λ-modules, and $K_0(\mathsf{proj}\,\Lambda)$ has a basis consisting of the isomorphism classes P_i of indecomposable projective Λ-modules, where $\mathrm{top}\,P_i = S_i$.
The Euler form is a non-degenerate pairing between $K_0(\mathsf{proj}\,\Lambda)$ and $K_0(\mathsf{mod}\,\Lambda)$ given by

$$\langle P, X\rangle := \sum_{i\in\mathbb{Z}}(-1)^i \dim_k \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(P, X[i]) \tag{3.2}$$

for any $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$. Then $\{P_i\}$ and $\{S_i\}$ satisfies $\langle P_i, S_j\rangle = \delta_{ij}\dim_k \mathrm{End}_\Lambda(S_j)$ for any i and j, where δ_{ij} is the *Kronecker delta*. In particular, we have a $\mathbb{Z}$-linear form $\langle P, -\rangle : K_0(\mathsf{mod}\,\Lambda) \to \mathbb{Z}$ for $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.
Recall that there is a Serre duality, that is, a bifunctorial isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(P, X) \simeq D\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(X, \nu P) \tag{3.3}$$

for $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$. The following observation is basic.

Lemma 3.1. Let P be a two-term complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. For $M \in \mathsf{mod}\,\Lambda$, we have

$$\begin{aligned}\langle P, M\rangle &= \dim_k \mathrm{Hom}_\Lambda(\mathrm{H}^0(P), M) - \dim_k \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(P, M[1]) &(3.4)\\ &= \dim_k \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(M, \nu P) - \dim_k \mathrm{Hom}_\Lambda(M, \mathrm{H}^{-1}(\nu P)). &(3.5)\end{aligned}$$

Proof. Since P is two-term, $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(P, M[i]) = 0$ holds for any $i \neq 0, 1$. Moreover, we have $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(P, M) = \mathrm{Hom}_\Lambda(\mathrm{H}^0(P), M)$. Thus (3.4) holds. Similarly, (3.5) follows from (3.3). □

3.3 Proofs of Theorems 1.3 and 1.4

First, we make preparations to prove Theorem 1.4.

Lemma 3.2. For $U \in$ 2-presiltΛ, we have

$$\begin{aligned}\mathcal{T}_U^+ &= \{M \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(U, M[1]) = 0\}, &(3.6)\\ \mathcal{F}_U^- &= \{M \in \mathsf{mod}\,\Lambda \mid \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(M, \nu U) = 0\}. &(3.7)\end{aligned}$$

Proof. For $M \in \mathsf{mod}\,\Lambda$, by (3.3) we have

$$D\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(U, M[1]) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)}(M[1], \nu U) \simeq \mathrm{Hom}_\Lambda(M, \mathrm{H}^{-1}(\nu U)).$$

Thus (3.6) holds. Similarly, (3.7) holds. □

Using $\mathbb{R}$-linear forms on $K_0(\mathsf{mod}\,\Lambda)\otimes_{\mathbb{Z}}\mathbb{R}$, we have the following properties of torsion pairs $(\mathcal{T}_U^+,\mathcal{F}_U^+)$ and $(\mathcal{T}_U^-,\mathcal{F}_U^-)$ for $U\in$ 2-presiltΛ.

Proposition 3.3. Let $U\in$ 2-presiltΛ and θ the corresponding $\mathbb{R}$-linear form on $K_0(\mathsf{mod}\,\Lambda)\otimes_{\mathbb{Z}}\mathbb{R}$ defined in Theorem 1.4. For $M\in\mathsf{mod}\,\Lambda$, the following assertions hold.

(a) If $M\in\mathcal{T}_U^+$, then $\theta(M)\geq 0$. Moreover, if $M\in\mathcal{T}_U^-$ is non-zero, then $\theta(M)>0$.
(b) If $M\in\mathcal{F}_U^-$, then $\theta(M)\leq 0$. Moreover, if $M\in\mathcal{F}_U^+$ is non-zero, then $\theta(M)<0$.

Proof. Let $M\in\mathcal{T}_U^+$. By (3.4) and (3.6), we have

$$\theta(M)=\sum_X a_X \dim_k \mathrm{Hom}_\Lambda(\mathrm{H}^0(X),M),$$

where X runs over all indecomposable direct summands of U. Since $a_X>0$ holds for any X, (a) holds. Similarly, (b) holds by (3.5) and (3.7). $\square$

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $M\in\mathcal{W}_U=\mathcal{T}_U^+\cap\mathcal{F}_U^-$. Then $\theta(M)=0$ holds by Proposition 3.3. Since $\mathcal{F}_U^-$ is a torsion free class, any submodule L of M is also belongs to $\mathcal{F}_U^-$. Thus $\theta(L)\leq 0$ holds by Proposition 3.3(b). Therefore, M is θ-semistable.
Conversely, assume that $M\in\mathsf{mod}\,\Lambda$ is θ-semistable. Since $(\mathcal{T}_U^-,\mathcal{F}_U^-)$ is a torsion pair, there is an exact sequence

$$0\to L\to M\to N\to 0,$$

where $L\in\mathcal{T}_U^-$ and $N\in\mathcal{F}_U^-$. Since $\theta(L)\leq 0$ holds, we have $L=0$ by Proposition 3.3(a). Thus $M=N\in\mathcal{F}_U^-$. Similarly, taking a canonical sequence of M with respect to the torsion pair $(\mathcal{T}_U^+,\mathcal{F}_U^+)$, we have $M\in\mathcal{T}_U^+$. Thus $M\in\mathcal{T}_U^+\cap\mathcal{F}_U^-=\mathcal{W}_U$ holds. $\square$

Next, we make preparations to prove Theorem 1.3. For $T\in$ 2-siltΛ, we have the following characterization of the corresponding torsion pairs $(\mathcal{T}_T^+,\mathcal{F}_T^+)$ and $(\mathcal{T}_T^-,\mathcal{F}_T^-)$ in $\mathsf{mod}\,\Lambda$.

Lemma 3.4. Let $T=T_\lambda\oplus T_\rho\in$ 2-siltΛ as in (1.1). The following equalities hold.

(a) $\mathcal{T}_T^+=\mathcal{T}_T^-=\mathsf{Fac}\,\mathrm{H}^0(T_\lambda)={}^\perp\mathrm{H}^{-1}(\nu T_\rho)$.
(b) $\mathcal{F}_T^+=\mathcal{F}_T^-=\mathrm{H}^0(T_\lambda)^\perp=\mathsf{Sub}\,\mathrm{H}^{-1}(\nu T_\rho)$.

Proof. By (3.1), we have $\mathcal{T}_T^+=\mathcal{T}_T^-$ and $\mathcal{F}_T^+=\mathcal{F}_T^-$.
Applying $\mathrm{H}^0(-)$ to the triangle $\Lambda\to T'\to T''\to\Lambda[1]$ in (1.1), we have an exact sequence

$$\Lambda\to\mathrm{H}^0(T')\to\mathrm{H}^0(T'')\to 0$$

in $\mathsf{mod}\,\Lambda$. Thus $\mathrm{H}^0(T_\rho) \in \mathsf{Fac}\,\mathrm{H}^0(T_\lambda)$ holds. Hence we have $\mathcal{T}_T^- = \mathsf{Fac}\,\mathrm{H}^0(T) = \mathsf{Fac}\,\mathrm{H}^0(T_\lambda)$ and $\mathcal{F}_T^- = \mathrm{H}^0(T)^\perp = \mathrm{H}^0(T_\lambda)^\perp$. Dually, the equations $\mathcal{T}_T^+ = {}^\perp\mathrm{H}^{-1}(\nu T_\rho)$ and $\mathcal{F}_T^+ = \mathsf{Sub}\,\mathrm{H}^{-1}(\nu T_\rho)$ hold. □

The following observation gives a connection between two constructions $\mathcal{W}^{(-)}$ and $\mathcal{W}_{(-)}$ of wide subcategories.

LEMMA 3.5. Let $T = T_\lambda \oplus T_\rho \in$ 2-siltΛ. Then $\mathcal{W}^T = \mathcal{W}_{T_\rho}$ holds.

Proof. There are equalities

$$\mathcal{W}^T = \mathsf{Fac}\,\mathrm{H}^0(T_\lambda) \cap \mathrm{H}^0(T_\rho)^\perp = {}^\perp\mathrm{H}^{-1}(\nu T_\rho) \cap \mathrm{H}^0(T_\rho)^\perp = \mathcal{T}_{T_\rho}^+ \cap \mathcal{F}_{T_\rho}^- = \mathcal{W}_{T_\rho}$$

by (1.2) and Lemma 3.4(a). □

This result enables us to prove Theorem 1.3.

Proof of Theorem 1.3. The assertion immediately follows from Lemma 3.5 and Theorem 1.4. □

ACKNOWLEDGEMENTS. The author is a Research Fellow of Society for the Promotion of Science (JSPS). This work was supported by JSPS KAKENHI Grant Number JP17J04270.
The author would like to thank his supervisor Osamu Iyama for his guidance and helpful advice, and Laurent Demonet for helpful comments. He is also grateful to Thomas Brüstle and Kiyoshi Igusa for useful discussions related to their works [BST, I].

REFERENCES

[AIR] T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compos. Math. Vol. 150, 3 (2014) 415–452.

[Ai] T. Aihara, *Tilting-connected symmetric algebras*, Algebr. Represent. Theor. Vol. 16 (2013) 873–894.

[AI] T. Aihara and O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. Vol. 85 (2012) 633–668.

[As] S. Asai, *Semibricks*, arXiv:1610.05860.

[ASS] I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, Cambridge University Press, 2006.

[Bri] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. Math. Vol. 166, 2 (2007) 317–345.

[BPP] N. Broomhead, D. Pauksztello and D. Ploog, *Discrete derived categories II: the silting pairs CW complex and the stability manifold*, J. Lond. Math. Soc.(2) 93 (2016) 273–300.

[Bru] K. Brüning, *Thick subcategories of the derived category of hereditary algebra*, Homology Homotopy Appl. 9 (2) (2007) 165–176.

[BST] T. Brüstle, D. Smith and H. Treffinger, *Stability conditions, τ-tilting theory and maximal green sequences*, arXiv:1705.08227.

[DIJ] L. Demonet, O. Iyama and G. Jasso, *τ-tilting finite algebras, brick and g-vectors*, arXiv:1503.00285.

[DIRRT] L. Demonet, O. Iyama, N. Reading, I. Reiten and H. Thomas, *Lattice theory of torsion classes*, in preparation.

[Hop] M. Hopkins, *Global methods in homotopy theory*, Homotopy theory (Durham, 1985), 73–96, London Math. Soc. Lecture Note Ser. 117, Cambridge Univ. Press, Cambridge, 1987.

[Hov] M. Hovey, *Classifying subcategories of modules*, Trans. Amer. Math. Soc. 353 (8) (2001) 3181–3191.

[I] K. Igusa, *Linearity of stability conditions I*, in preparation, 2017.

[IOTW] K. Igusa, K. Orr, G. Todorov and J. Weyman, *Cluster complexes via semi-invariants*, Compos. Math. Vol. 145, 4 (2009) 1001–1034.

[IT] C. Ingalls and H. Thomas, *Noncrossing partitions and representations of quivers*, Compos. Math. Vol. 145, 6 (2009) 1533–1562.

[J] G. Jasso, *Reduction of τ-tilting modules and torsion pairs*, Int. Math. Res. Not. IMRN 16 (2015) 7190–7237.

[KV] B. Keller and D. Vossieck, *Aisles in derived categories*, Deuxieme Contact Franco-Belge en Algebre (Faulx-lesThombes, 1987). Bull. Soc. Math. Belg. 40 (1988) 239–253.

[K] A. King, *Moduli of representations of finite-dimensional algebras*, Q. J. Math. 45, 4 (1994) 515–530.

[KS] H. Krause and J. Stovicek, *The telescope conjecture for hereditary rings via Ext-orthogonal pairs*, Adv. Math. 225 (2010) 2341–2364.

[MS] F. Marks and J. Stovicek, *Torsion classes, wide subcategories and localisations*, arXiv:1503.04639.

[N] A. Neeman, *The chromatic tower for $D(R)$*, Topology 31 (1992) 519–532.

[S] S. O. Smalø, *Torsion theory and tilting modules*, Bull. Lond. Math. Soc. 16 (1984) 518–522.

[ST] D. Speyer and H. Thomas, in preparation.

[Ta] R. Takahashi, *Classifying subcategories of modules over a commutative Noetherian ring*, J. Lond. Math. Soc. (2) 78 (2008) 767–782.

[Th] R. W. Thomason, *The classification of triangulated subcategories*, Comp. Math. 105, no. 1 (1997) 1–27.

Toshiya Yurikusa
Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602, Japan
m15049q@math.nagoya-u.ac.jp