The Hirzebruch-Mumford Volume for the Orthogonal Group and Applications

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Abstract. In this paper we derive an explicit formula for the Hirzebruch-Mumford volume of an indefinite lattice L of rank ≥ 3. If $\Gamma \subset \mathrm{O}(L)$ is an arithmetic subgroup and L has signature $(2,n)$, then an application of Hirzebruch-Mumford proportionality allows us to determine the leading term of the growth of the dimension of the spaces $S_k(\Gamma)$ of cusp forms of weight k, as k goes to infinity. We compute this in a number of examples, which are important for geometric applications.

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0 Introduction

In [Hi1] and [Hi2] Hirzebruch considered compact quotients of a homogeneous domain by an arithmetic group. He observed that the Chern numbers of such quotients are proportional to the Chern numbers of the compact duals of the homogeneous domains, and he also showed how the proportionality factor can be used to compute the dimension of spaces of automorphic forms. Later Mumford [Mum] extended Hirzebruch's approach to the case where the quotient is no longer compact, but only of finite volume. In this case the space of cusp forms of weight k with respect to some arithmetic group Γ grows asymptotically proportional to the dimension of the space of sections of the $(1-k)$-th power of the canonical bundle of the compact dual (for a precise formulation see Theorem 1.1). We call the proportionality constant the *Hirzebruch-Mumford volume*. Thus a computation of the Hirzebruch-Mumford volume for a given group Γ gives the leading term of the Hilbert polynomial of forms of weight k

with respect to Γ. Knowledge of this term is essential for many geometric applications, in particular when one considers the Kodaira dimension of modular varieties.

The subject starts with the seminal work of Siegel [Sie1] on the volume of the orthogonal group. Very many authors have taken up his theory and generalised it in many different directions, including Harder [Ha], Serre [Se], Prasad [Pr] and many others. Our specific interest lies in indefinite orthogonal groups (see the work by Shimura [Sh], Gross [Gr], Gan, Hanke and Yu [GHY], as well as Belolipetsky and Gan [BG], to name some important recent work in this direction). Motivated by possible applications (cf. [GHS1], [GHS2]) concerning moduli spaces of K3 surfaces and similar modular varieties we started to investigate the volume of certain arithmetic subgroups of orthogonal groups $\mathrm{O}(L)$ of even indefinite lattices of signature $(2,n)$. All our groups are defined over the rational numbers, but for the applications we have in mind we cannot restrict ourselves to unimodular or maximal lattices. To our knowledge there exist no results in the literature that allow an easy calculation of the Hirzebruch-Mumford volume for the groups we treat in this paper.

In order to compute these volumes we therefore decided to return to Siegel's work. Let L be an even indefinite lattice of signature $(2,n)$ and let $\mathrm{O}(L)$ be its group of isometries. The lattice L defines a domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C});\ (\mathbf{w},\mathbf{w})_L = 0,\ (\mathbf{w},\overline{\mathbf{w}})_L > 0\}.$$

This domain has two connected components $\mathcal{D}_L$ and $\mathcal{D}'_L$, which are interchanged by complex conjugation, where $\mathcal{D}_L = \mathrm{O}(2,n)/\mathrm{O}(2)\times\mathrm{O}(n)$. Let $\mathrm{O}^+(L)$ be the index 2 subgroup of $\mathrm{O}(L)$ which fixes $\mathcal{D}_L$. The fundamental problem of our paper is to determine the Hirzebruch-Mumford volume of this group. For this one has to compare the volume of the quotient $\mathrm{O}^+(L)\backslash\mathcal{D}_L$ to the volume of the compact dual $\mathcal{D}_L^{(c)} = \mathrm{O}(2+n)/\mathrm{O}(2)\times\mathrm{O}(n)$. To do so correctly, one has to choose volume forms on the domain $\mathcal{D}_L$ and the compact dual $\mathcal{D}_L^{(c)}$ that coincide at the common point of both domains given by a maximal compact subgroup. This is in fact a problem which does not depend on the complex structure of the domains, but can be considered in greater generality for indefinite lattices of signature (r,s). We use the volume form on $\mathcal{D}_L$ which was introduced by Siegel. It then turns out that this must be compared to the volume form on $\mathcal{D}_L^{(c)}$ which is given by $1/2$ of the volume form induced by the Killing form on the Lie algebra of the group $\mathrm{SO}(r+s)$. Comparing these two volumes gives us the main formula for the Hirzebruch-Mumford volume of $\mathrm{O}^+(L)$. This formula involves the Tamagawa (Haar) measure of the group $\mathrm{O}(L)$. However, again using a result of Siegel, the computation of the Tamagawa measure can be reduced to computing the local densities $\alpha_p(L)$ of the lattice L over the p-adic integers. Our main formula for any indefinite lattice L of rank $\rho \geq 3$ is

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{2}{g_{sp}^{+}}\,|\det L|^{(\rho+1)/2}\prod_{k=1}^{\rho}\pi^{-k/2}\Gamma(k/2)\prod_{p}\alpha_p(L)^{-1}$$

where g_{sp}^+ is the number of the proper spinor genera in the genus of L (see Theorem 2.1). Since everything is defined over the rationals, one can use Kitaoka's book [Ki] on quadratic forms to compute the local densities in question.

In order to illustrate our results, and particularly in view of applications, we compute the Hirzebruch-Mumford volume for several examples. The lattices and the groups which we consider are mostly related to moduli problems. We start with a series of even unimodular lattices, namely the lattices $II_{2,2m+8} = 2U \oplus mE_8(-1)$, where U denotes the hyperbolic plane and E_8 is the positive definite root lattice associated to E_8. The next series of examples consists of the lattices $L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d \rangle$, which are closely related to well known moduli problems. Let

$$\mathcal{F}_{2d}^{(m)} = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)}) \backslash \mathcal{D}_{L_{2d}^{(m)}}$$

where $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ is the subgroup of $\mathrm{O}^+(L_{2d}^{(m)})$ which acts trivially on the discriminant group. For $m = 0$ and d a prime number, $\mathcal{F}_{2d}^{(0)}$ is a moduli space of Kummer surfaces (see [GH]). The spaces $\mathcal{F}_{2d}^{(1)}$ parametrise certain lattice-polarised K3 surfaces and if $m = 2$, then $\mathcal{F}_{2d} = \mathcal{F}_{2d}^{(2)}$ is the moduli space of K3 surfaces of degree $2d$. We compute the Hirzebruch-Mumford volumes of the groups $\mathrm{O}^+(L_{2d}^{(m)})$ and $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ and obtain as a corollary the leading term controlling the growth behaviour of the dimension of the spaces of cusp forms for these groups. As a specialisation of this example we recover known formulae for the Siegel modular group in genus 2 and the paramodular group. Other series of examples considered in this paper, namely the even indefinite unimodular lattices (Section 3.3), their sublattices T (Section 3.4) and some lattices of signature $(2, 8m+2)$ (Section 3.6), are closely related to moduli of K3 surfaces and related quotients of homogeneous varieties of type IV. The volumes of these lattices determine the part of the obstruction for extending pluricanonical differential forms on $\mathcal{F}_{2d}^{(m)}$ to a smooth compactification of this variety which comes from the ramification divisor.

In [GHS2] we use these results to obtain information about the Kodaira dimension of two series of modular varieties, including effective bounds on the degree d which guarantee that the varieties $\mathcal{F}_{2d}^{(m)}$ are of general type. The case of polarised K3 surfaces is considered in [GHS1].

The paper is organised as follows: in Section 1 we recall Hirzebruch-Mumford proportionality and the Hirzebruch-Mumford volume in the form in which we need it (see Theorem 1.1 and Corollary 1.2). In Section 2 we perform the necessary volume computations and derive the main formula (see Theorem 2.1). In Section 3 we treat in some detail several lattices which appear naturally in moduli problems.

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1 Hirzebruch-Mumford proportionality

In this section we consider an indefinite even lattice L of signature $(2, n)$. Let $\mathrm{O}(L)$ be its group of isometries. We denote by $(\ , \ )_L$ the form defined on L, extended bilinearly to $L \otimes \mathbb{R}$ and $L \otimes \mathbb{C}$. The domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C});\ (\mathbf{w}, \mathbf{w})_L = 0,\ (\mathbf{w}, \overline{\mathbf{w}})_L > 0\}$$

has two connected components, say $\Omega_L = \mathcal{D}_L \cup \mathcal{D}'_L$, which are interchanged by complex conjugation. By $\mathcal{D}_L^\bullet$ we denote the affine cone over $\mathcal{D}_L$ in $L \otimes \mathbb{C}$. Let $\Gamma \subset \mathrm{O}(L)$ be an arithmetic group which leaves the domain $\mathcal{D}_L$ invariant. A *modular form* of *weight* k with respect to the group Γ and with a (finite order) *character* $\chi : \Gamma \to \mathbb{C}^*$ is a holomorphic map

$$f : \mathcal{D}_L^\bullet \to \mathbb{C}$$

which has the two properties

$$\begin{array}{rcll} f(tz) & = & t^{-k} f(z) & \text{for } t \in \mathbb{C}^*, \\ f(g(z)) & = & \chi(g) f(z) & \text{for } g \in \Gamma. \end{array}$$

If $n \leq 2$ the function $f(z)$ must also be required to be holomorphic at infinity. A *cusp form* is a modular form which vanishes on the boundary.
We denote the spaces of modular forms and of cusp forms of weight k, with respect to the group Γ and character χ, by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively. These are finite dimensional vector spaces. Note that if $-\mathrm{id} \in \Gamma$ and $(-1)^k \neq \chi(-\mathrm{id})$ then obviously $M_k(\Gamma, \chi) = 0$.
Modular forms can be interpreted as sections of suitable line bundles. For this, we first assume that the group Γ is neat, in which case it acts freely on $\mathcal{D}_L$, and we also assume that the character χ is trivial. Then the transformation rules of modular forms of weight 1 define a line bundle $\mathcal{L}$ on the quotient $\Gamma \backslash \mathcal{D}_L$ and modular forms of weight k with trivial character become sections in $\mathcal{L}^{\otimes k}$. The line bundle $\mathcal{L}$, and its sections, extend to the Baily-Borel compactification $\overline{\Gamma \backslash \mathcal{D}_L}$. In fact, the Baily-Borel compactification is the normal projective variety associated to $\mathrm{Proj}\left(\bigoplus_k H^0(\mathcal{L}^{\otimes k})\right)$. In general, modular forms of weight k and with a character χ define sections of a line bundle $\mathcal{L}_{k,\chi}$ which differs from $\mathcal{L}^{\otimes k}$ only by torsion.

Every toroidal compactification $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$ has a morphism $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}} \to \overline{\Gamma\backslash\mathcal{D}}_L$ which is the identity on $\Gamma\backslash\mathcal{D}_L$. Via this morphism, we shall also consider $\mathcal{L}$ and $\mathcal{L}_{k,\chi}$ as line bundles on $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$, using the same symbol by abuse of notation. If Γ is not neat then the above remains true, as long as we consider $\mathcal{L}$ and $\mathcal{L}_{k,\chi}$ as $\mathbb{Q}$-line bundles or only consider weights k that are sufficiently divisible.
The connection with pluricanonical forms is as follows. There is an n-form dZ on $\mathcal{D}_L$ such that if f is a modular form of weight $n = \dim \mathcal{D}_L$ with character det, then $\omega = f dZ$ is a Γ-invariant n-form on $\mathcal{D}_L$. Hence, if the action of Γ on $\mathcal{D}_L$ is free, ω descends to an n-form on $\Gamma\backslash\mathcal{D}_L$. Similarly, modular forms of weight kn with character $\det^k$ define k-fold pluricanonical forms on $\Gamma\backslash\mathcal{D}_L$. If Γ does not act freely, then this is still true outside the ramification locus of the quotient map $\mathcal{D}_L \to \Gamma\backslash\mathcal{D}_L$. These forms will, in general, not extend to compactifications of $\Gamma\backslash\mathcal{D}_L$. If Γ is a neat group, then let $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$ be a smooth toroidal compactification (which always exists by [AMRT]). Let D be the boundary of such a toroidal compactification. If $\mathcal{L}_{n,\det}$ is the line bundle of modular forms of weight n and character det, then the canonical bundle is given by $\omega_{(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}} = \mathcal{L}_{n,\det} \otimes \mathcal{O}_{(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}}(-D)$. Hence, if f is a weight n form with character det, not vanishing at the boundary, then $f dZ$ defines an n-form on $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$ with poles along the boundary. However, if f is a cusp form, then $f dZ$ does define an n-form on $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$, and similarly forms of weight kn and character $\det^k$ that vanish along the boundary of order k define k-fold pluricanonical forms on $(\Gamma\backslash\mathcal{D}_L)^{\text{tor}}$. It should be pointed out that some authors define automorphic forms a priori as those functions that give rise to pluricanonical forms. In our context, this means a restriction to forms of weight kn. Moreover, the weight of these forms is sometimes defined as k. We shall refer to the latter as the *geometric* weight, in contrast to the *arithmetic* weight of our definition. This difference accounts for the fact that some of our formulae differ from corresponding formulae in the literature by powers of n.
The Hirzebruch-Mumford proportionality principle, which works very generally for quotients of a homogeneous domain $\mathcal{D}$ by an arithmetic group Γ, allows us to estimate the growth behaviour of spaces of cusp forms as a function of the weight k in terms of a suitably defined volume. This was first discovered by Hirzebruch [Hi1], [Hi2] in the case where the quotient $\Gamma\backslash\mathcal{D}$ is compact, and was generalised by Mumford [Mum] to the case where $\Gamma\backslash\mathcal{D}$ has finite volume. We denote the compact dual of $\mathcal{D}$ by $\mathcal{D}^{(c)}$. Let $\overline{X}$ be the Baily-Borel compactification of $X = \Gamma\backslash\mathcal{D}$ and let X^{tor} be some smooth toroidal compactification of X.

Theorem 1.1 *Let Γ be a neat arithmetic group which acts on a bounded symmetric domain $\mathcal{D}$. Let $S_k^{\text{geom}}(\Gamma) = S_{nk}(\Gamma, \det^k)$ be the space of cusp forms of geometric weight k with respect to Γ. Then*

$$\dim S_k^{\text{geom}}(\Gamma) = \text{vol}_{HM}(\Gamma) h^0(\omega_{\mathcal{D}^{(c)}}^{(1-k)}) + P_1(k)$$

where $P_1(k)$ is a polynomial whose degree is at most the dimension of $\overline{X}\backslash X$.

Proof. This is [Mum, Corollary 3.5]. □

Here $\mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D})$ denotes a suitably normalised volume of the quotient $\Gamma\backslash\mathcal{D}$, which we shall refer to as the *Hirzebruch-Mumford volume.* If Γ acts freely, then the Hirzebruch-Mumford volume is a quotient of Euler numbers

$$\mathrm{vol}_{HM}(\Gamma) = \mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D}) = \frac{e(\Gamma\backslash\mathcal{D})}{e(\mathcal{D}^{(c)})}.$$

If Γ does not act freely, then choose a normal subgroup $\Gamma' \lhd \Gamma$ of finite index which does act freely. Then

$$\mathrm{vol}_{HM}(\Gamma) = \frac{\mathrm{vol}_{HM}(\Gamma')}{[\mathrm{P}\Gamma : \Gamma']}$$

where $\mathrm{P}\Gamma$ is the image of Γ in $\mathrm{Aut}(\mathcal{D})$, i.e. the group Γ modulo its centre. This value is independent of the choice of the subgroup Γ'.
Hirzebruch [Hi1] first formulated his result in the case where the group is co-compact, i.e., where the quotient $X = \Gamma\backslash\mathcal{D}$ is compact. Since the Chern numbers of X and that of the compact dual are proportional and the factor of proportionality is given by the volume, one can use Riemann-Roch to compute the exact dimension of the space of modular foms (in this case it does not make sense to talk about cusp forms).
We shall now apply this to orthogonal groups.

Proposition 1.2 *Let L be an indefinite even lattice of signature $(2,n)$ and let Γ be an arithmetic subgroup which acts on the domain $\mathcal{D}_L$. Fix a positive integer k and a character χ. If $-\mathrm{id} \in \Gamma$, then we restrict to those k for which $(-1)^k = \chi(-\mathrm{id})$. Then the dimension of the space $S_k(\Gamma,\chi)$ of cusp forms of arithmetic weight k grows as*

$$\dim S_k(\Gamma,\chi) = \frac{2}{n!}\,\mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D}_L)k^n + O(k^{n-1}).$$

Proof. We shall first assume that Γ is neat (in which case automatically $-\mathrm{id} \notin \Gamma$) and that χ is trivial. We consider $\mathcal{L}$ as a line bundle on a smooth toroidal compactification X^{tor} of $X = \Gamma\backslash\mathcal{D}_L$. It follows from the definition of cusp forms that $H^0(X^{\mathrm{tor}}, \mathcal{L}^{\otimes k}(-D)) = S_k(\Gamma)$. Since $\mathcal{L}$ is big and nef and $K_{X^{\mathrm{tor}}} = \mathcal{L}^{\otimes n}(-D)$, it follows from Kawamata-Viehweg vanishing that $h^i(X^{\mathrm{tor}}, \mathcal{L}^{\otimes k}(-D)) = 0$ for $i \geq 1$ and $k \gg 0$ and hence $\chi(X^{\mathrm{tor}}, \mathcal{L}^{\otimes k}(-D)) = h^0(X^{\mathrm{tor}}, \mathcal{L}^{\otimes k}(-D))$ for $k \gg 0$. The leading term of the Riemann-Roch polynomial as a function of k is given by $c_1^n(\mathcal{L})/n!$. The same argument goes through for $\mathcal{L}_{k,\chi}$. Since $\mathcal{L}^{\otimes k}$ and $\mathcal{L}_{k,\chi}$ only differ by torsion they have the same leading coefficients.
In order to apply Theorem 1.1 we consider the line bundle $\mathcal{L}_{n,\det}$ of modular forms of weight n and character det. Note that $\mathcal{L}_{n,\det}^k = \mathcal{L}^{nk}$ for suitably divisible k. Also recall that in the orthogonal case the compact dual $\mathcal{D}^{(c)}$

is the complex n-dimensional quadric $Q_n \subset \mathbb{P}^{n+1}$ whose canonical bundle is $\omega_{Q_n} = \mathcal{O}_{Q_n}(-n)$ and it follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(n(k-1)-2) \to \mathcal{O}_{\mathbb{P}^{n+1}}(n(k-1)) \to \omega_{Q_n}^{(1-k)} \to 0$$

that the leading term of $h^0(\omega_{\mathcal{D}^{(c)}}^{(1-k)})$ is equal to $2n^n/n!$. It then follows from Hirzebruch-Mumford proportionality that

$$\frac{c_1^n(\mathcal{L}_{n,\det}^n)}{n!} = \frac{2n^n}{n!}\operatorname{vol}_{HM}(X)$$

and hence

$$\frac{c_1^n(\mathcal{L})}{n!} = \frac{2}{n!}\operatorname{vol}_{HM}(X)$$

which gives the claim in the case of a neat group.

We now consider a group Γ which is not necessarily neat and choose $\Gamma' \lhd \Gamma$ neat and of finite index. The group Γ acts on the total space of the line bundle $\mathcal{L}$, and if $-\operatorname{id} \in \Gamma$ then it follows from our assumptions on k that this element acts trivially. We can now apply the Lefschetz fixed point formula (cf. [T, Appendix to §2]), from which we obtain

$$\begin{aligned}
\dim S_k(\Gamma) &= \dim S_k(\Gamma')^{\Gamma} \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \cdot \sum_{\gamma \in \mathrm{P}\Gamma/\Gamma'} \operatorname{tr}\left(\gamma|_{S_k(\Gamma')}\right) \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \dim S_k(\Gamma') + O(k^{n-1}) \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \operatorname{vol}_{HM}(\Gamma' \backslash \mathcal{D}_L) \frac{2}{n!} k^n + O(k^{n-1}) \\
&= \frac{2}{n!} \operatorname{vol}_{HM}(\Gamma \backslash \mathcal{D}_L) k^n + O(k^{n-1}).
\end{aligned}$$

□

Note that the growth behaviour of the space of modular forms of weight k and that of the space of cusp forms are the same. This follows from the exact sequence

$$0 \to \mathcal{L}^{\otimes k}(-D) \to \mathcal{L}^{\otimes k} \to \mathcal{L}^{\otimes k}|_D \to 0.$$

2 Computation of volumes

In order to compute the leading coefficient that determines the growth of the dimension of spaces of cusp forms, we have to compare the volume of a fundamental domain of an arithmetic group Γ to the volume of the compact dual. For this, the complex structure is not important and we therefore consider, more generally, an indefinite integral lattice L of signature (r, s).

As before, we denote the group of isometries of the lattice L by $\mathrm{O}(L)$. The lattice L defines a homogeneous domain $\mathcal{D}_{rs}$. In terms of groups the domain $\mathcal{D}_{rs}$ is the quotient of the orthogonal group $\mathrm{O}(L\otimes\mathbb{R})$ by a maximal compact subgroup, i.e.,

$$\mathcal{D}_{rs}=\mathcal{D}_L=\mathrm{O}(r,s)/\,\mathrm{O}(r)\times\mathrm{O}(s)=\mathrm{SO}(r,s)_0/\,\mathrm{SO}(r)\times\mathrm{SO}(s)$$

where all groups are real Lie groups and $\mathrm{SO}(r,s)_0$ is the connected component of the identity of $\mathrm{SO}(r,s)$.
The domain $\mathcal{D}_{rs}$ can be realised as a bounded domain in the form

$$\mathcal{D}_{rs}=\{X\in\mathrm{Mat}_{r\times s}(\mathbb{R})\,;\,I_r-X^tX>0\}$$

where $I_r\in\mathrm{Mat}_{r\times r}(\mathbb{R})$ is the identity matrix and the action of the orthogonal group is given in the usual form, namely by

$$M(X)=(AX+B)(CX+D)^{-1}$$

for

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{O}(r,s),\qquad A\in\mathrm{Mat}_{r\times r}(\mathbb{R}),\ D\in\mathrm{Mat}_{s\times s}(\mathbb{R}).$$

We consider the $\mathrm{O}(r,s)$-invariant metric given by

$$ds^2=\mathrm{tr}\big((I_r-X^tX)^{-1}\,dX\,(I_s-{}^tXX)^{-1}\,d^tX\big).$$

Since

$$\det((I_r-X^tX)^{-1})^s\cdot\det((I_s-{}^tXX)^{-1})^r=\det((I_r-X^tX)^{-1})^{r+s}$$

the corresponding volume form is given by

$$dV=(\det(I_r-X^tX)^{-1})^{\frac{r+s}{2}}\prod_{i,j}dx_{ij}.$$

Siegel computed the volume of $\mathcal{D}_{rs}$ with respect to this volume form in [Sie2] (see also [Sie3, Theorem 7, p. 155]). His result is

$$\mathrm{vol}_S(\mathrm{O}(L))=\mathrm{vol}_S(\mathrm{O}(L)\backslash\mathcal{D}_{rs})=2\alpha_\infty(L)|\det L|^{(r+s+1)/2}\,\gamma_r^{-1}\gamma_s^{-1}\,,\qquad(1)$$

where

$$\gamma_m=\prod_{k=1}^{m}\pi^{k/2}\,\Gamma(k/2)^{-1}\qquad(2)$$

and $\alpha_\infty(L)$ is the real Tamagawa (Haar) measure of the lattice L. Formula (1) is valid for any indefinite lattice L of rank ≥ 3. As indicated by the subscript, we shall refer to this volume as the *Siegel volume* of the group $\mathrm{O}(L)$.

We want to understand the Siegel metric in terms of Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the indefinite orthogonal group $\mathrm{O}(r,s)$ and its maximal compact subgroup $\mathrm{O}(r) \times \mathrm{O}(s)$ respectively. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. By [He, p. 239] this is isomorphic to

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & U \\ {}^tU & 0 \end{pmatrix}; \quad U \in \mathrm{Mat}_{r\times s}(\mathbb{R}) \right\}.$$

The space $\mathfrak{p}$ is isomorphic to the tangent space of $\mathcal{D}_{rs}$ at 0. A straightforward calculation shows that the $\mathrm{O}(r,s)$-invariant metric ds^2 is induced by the Killing functional $\mathrm{tr}(U_1{}^tU_2)$ on the tangent space at 0.

We now want to compare this to a suitable volume form on the compact dual. Recall that the general situation is as follows. Let H be a bounded homogeneous domain and $G = \mathrm{Aut}(H)_0$ be the connected component of the identity of the group of automorphisms of H. In particular, $H = G/K$ where $K = G_{z_0}$ is the stabiliser of some point z_0. There exists a unique compact real form G_u of the complex group $G_\mathbb{C}$ such that $G \cap G_u = K$ and the symmetric domain $H = G/K$ can be embedded into the compact manifold $\mathcal{D}^{(c)} = G_u/K$ as an open submanifold. In our situation

$$\mathcal{D}_{rs}^{(c)} = \mathrm{SO}(r+s)/\,\mathrm{SO}(r) \times \mathrm{SO}(s).$$

Again by [He, p.239] the tangent space of $\mathcal{D}_{rs}^{(c)}$ at the point I_{r+s} is given by the subspace

$$\mathfrak{p}' = \left\{ \begin{pmatrix} 0 & U \\ -{}^tU & 0 \end{pmatrix}; \quad U \in \mathrm{Mat}_{r\times s}(\mathbb{R}) \right\}$$

of the Lie algebra of $\mathrm{SO}(r+s)$. The Killing form $\mathrm{tr}(W_1{}^tW_2)$ of the Lie algebra of the compact group $\mathrm{SO}(r+s)$ induces the form $2\,\mathrm{tr}(U_1{}^tU_2)$ on the tangent space $\mathfrak{p}'$. In order to compare the volumes of $\mathcal{D}_{rs}$ and its compact dual $\mathcal{D}_{rs}^{(c)}$ we have to normalise this form in such a way that it coincides with the Siegel metric in the common base point $K \in \mathcal{D}_{rs} \subset \mathcal{D}_{rs}^{(c)}$, i.e. we have to use the form $\frac{1}{2}\,\mathrm{tr}(W_1{}^tW_2)$. Since the dimension of $\mathrm{SO}(n)$ is $\frac{1}{2}n(n-1)$, we get a factor $2^{-(r+s)(r+s-1)/4}$ in front of the volume of the compact group, calculated in terms of the volume form induced by the Killing functional on $\mathrm{SO}(r+s)$. The latter volume is computed in [Hua, §3.7]. Taking the above normalisation into account we find

$$\mathrm{vol}_S(\mathrm{SO}(m)) = 2^{m-1}\,\gamma_m \tag{3}$$

and we shall again refer to this volume as the Siegel volume. For the compact dual this gives

$$\mathrm{vol}_S(\mathcal{D}_{rs}^{(c)}) = 2\,\gamma_{r+s}\gamma_r^{-1}\gamma_s^{-1}. \tag{4}$$

Our aim is to compute the Hirzebruch-Mumford volume

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{\mathrm{vol}_S(\mathrm{O}(L)\backslash\mathcal{D}_{rs})}{\mathrm{vol}_S(\mathcal{D}_{rs}^{(c)})}. \tag{5}$$

To make the above equation effective, we have to determine the Tamagawa measure

$$\alpha_\infty(L) = \alpha_\infty(\mathrm{O}(L)\setminus \mathrm{O}(L\otimes\mathbb{R})) = \alpha_\infty(\mathrm{SO}(L)\setminus \mathrm{SO}(L\otimes\mathbb{R})).$$

The genus of the indefinite lattice L contains a finite number $g_{sp}^+(L)$ of (proper) spinor genera (for a definition see [Ki, §6.3]). (We consider only proper classes and proper spinor genera.) This number is always a power of two and can be calculated effectively. It is well known that the spinor genus of an indefinite lattice of rank ≥ 3 coincides with the class. As was proved by M. Kneser (see [Kn]) the weight of the representations of a given number m by a spinor genus is the same for all genera in the genus of L. The same arguments show that all spinor genera in the genus have the same mass. (We are grateful to R. Schulze-Pillot for drawing our attention to this fact.) It is easy to see this in adelic terms. A spinor genus corresponds to a double class $\mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,b\,\mathrm{SO}_{\mathbb{A}}(L)$ in the adelic group $\mathrm{SO}_{\mathbb{A}}(V)$, where $V = L\otimes\mathbb{Q}$ is the rational quadratic space and

$$\mathrm{SO}'_{\mathbb{A}}(V) = \ker \mathrm{sn}\colon\ \mathrm{SO}_{\mathbb{A}}(V) \to \mathbb{Q}_{\mathbb{A}}^\times/(\mathbb{Q}_{\mathbb{A}}^\times)^2$$

is the kernel of the spinor norm. We note that the genus of L is given by $\mathrm{SO}_{\mathbb{A}}(V)L$. It follows from the definition that the group $\mathrm{SO}'_{\mathbb{A}}(V)$ contains the commutator of $\mathrm{SO}_{\mathbb{A}}(V)$, therefore

$$\mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,b\,\mathrm{SO}_{\mathbb{A}}(L) = \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,\mathrm{SO}_{\mathbb{A}}(L)\,b.$$

The mass of a spinor genus

$$\tau(\mathrm{SO}(V)\setminus \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)b\,\mathrm{SO}_{\mathbb{A}}(L)) = \tau(\mathrm{SO}(V)\setminus \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,\mathrm{SO}_{\mathbb{A}}(L))$$

depends only on the genus, since the Tamagawa measure is invariant. The Tamagawa number of the orthogonal group is 2 (see [Sie1], [W], [Sh]), i.e., $\tau(\mathrm{SO}(V)\backslash\mathrm{SO}_{\mathbb{A}}(V)) = 2$. Then the Tamagawa measure $\alpha_\infty(L)$ can be computed via the local densities of the lattices $L\otimes\mathbb{Z}_p$ over the p-adic integers $\mathbb{Z}_p$ (the local Tamagawa measures). More precisely,

$$\alpha_\infty(L) = \alpha_\infty(\mathrm{SO}(L)\setminus \mathrm{SO}(L\otimes\mathbb{R})) = \frac{2}{g_{sp}^+(L)}\prod_p \alpha_p(L)^{-1}, \tag{6}$$

where p runs through all prime numbers and $g_{sp}^+(L)$ is the number of spinor genera in the genus of L. The local densities can be computed, at least for quadratic forms over $\mathbb{Q}$ and its quadratic extensions: see [Ki]. In order to find $\alpha_p(L)$ it is enough to know the Jordan decomposition of L over the p-adic integers.

We can now summarise our results as follows.

THEOREM 2.1 (MAIN FORMULA) *Let L be an indefinite lattice of rank $\rho \geq 3$. Then the Hirzebruch-Mumford volume of $\mathrm{O}(L)$ equals*

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{2}{g_{sp}^{+}(L)} \cdot |\det L|^{(\rho+1)/2} \prod_{k=1}^{\rho} \pi^{-k/2}\Gamma(k/2) \prod_{p} \alpha_p(L)^{-1} \qquad (7)$$

where the $\alpha_p(L)$ are the local densities of the lattice L and $g_{sp}^{+}(L)$ is the number of spinor genera in the genus of L.

Proof. This follows immediately from formulae (1), (2), (5) and (6). □

3 APPLICATIONS

In this section we want to apply the above results to compute the asymptotic behaviour of the dimension of spaces of cusp forms for a number of specific groups. The main applications have to do with locally symmetric varieties. In [GHS1] we prove general type results for the moduli spaces $\mathcal{F}_{2d}$ of K3 surfaces of degree $2d$, but in that special case we can use a different method. The results we have here are used in [GHS2] to prove similar results in greater generality.

3.1 GROUPS

We first have to clarify the various groups which will play a role. In this section, L will be an even indefinite lattice of signature $(2, n)$, containing at least one hyperbolic plane as a direct summand. By a classical result of Kneser we know that if the genus of an indefinite lattice L contains more than one class, then there is a prime p such that the quadratic form of L can be diagonalised over the p-adic numbers and the diagonal entries all involve distinct powers of p (see [CS, Chapter 15]). Therefore the genus of any indefinite lattice with one hyperbolic plane contains only one class.
As an immediate corollary of Theorem 2.1 we obtain

THEOREM 3.1 *Let L be a lattice of signature $(2, n)$ $(n \geq 1)$ containing at least one hyperbolic plane. Let Γ be an arithmetic subgroup of $\mathrm{O}(L)$. Then*

$$\mathrm{vol}_{HM}(\Gamma) = 2 \cdot [\mathrm{PO}(L) : \mathrm{P}\Gamma] |\det L|^{(n+3)/2} \prod_{k=1}^{n+2} \pi^{-k/2}\Gamma(k/2) \prod_{p} \alpha_p(L)^{-1}. \quad (8)$$

REMARK. In many interesting cases a subgroup Γ is given in terms of the orthogonal group of some sublattice L_1 of L. In this case one can use the volume in order to calculate the index (see Section 3.4 below).
We shall now discuss the various groups which are of importance to us and compute their indices in $\mathrm{O}(L)$. The group $\mathrm{O}(L)$ interchanges the two connected components of the domain Ω_L and we define $\mathrm{O}^{+}(L)$ as the index 2 subgroup which fixes each of these components (as sets). This group can also be described

using the (-1)-*spinor norm* on the group $\mathrm{O}(L\otimes\mathbb{R})$ which is defined as follows. Every element g can be represented as a product of reflections

$$g = \sigma_{v_1}\cdot\dots\cdot\sigma_{v_m}$$

and, following Brieskorn [Br], we define

$$\mathrm{sn}_{-1}(g) = \begin{cases} +1 & \text{if } (v_k, v_k) > 0 \ \text{ for an even number of } v_k \\ -1 & \text{otherwise.} \end{cases}$$

This is independent of the representation of g as a product of reflections. It is well known that

$$\mathrm{O}^+(L) = \mathrm{Ker}(\mathrm{sn}_{-1}) \cap \mathrm{O}(L).$$

To see this, note that any reflection with respect to a vector of negative square has (-1)-spinor norm equal to 1, and any reflection with respect to a vector of positive square has (-1)-spinor norm equal to -1 and interchanges the two components. The Hirzebruch–Mumford volume of $\mathrm{O}^+(L)$ is twice that of $\mathrm{O}(L)$. Let $L^\vee = \mathrm{Hom}(L,\mathbb{Z})$ be the dual lattice and $A_L = L^\vee/L$. The finite group A_L carries a discriminant quadratic form q_L with values in $\mathbb{Q}/2\mathbb{Z}$ [Ni, 1.3]. By $\mathrm{O}(q_L)$ we denote the corresponding group of isometries and the group $\widetilde{\mathrm{O}}(L)$, called the *stable orthogonal group*, is defined as the kernel of the natural homomorphism $\mathrm{O}(L)\to\mathrm{O}(q_L)$. Since L contains a hyperbolic plane, it follows from [Ni, Theorem 1.14.2] that this map is surjective. Set

$$\widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L)\cap\mathrm{O}^+(L).$$

Finally the groups $\mathrm{SO}^+(L)$ and $\widetilde{\mathrm{SO}}^+(L)$ are defined as the corresponding groups of isometries of determinant 1.

Lemma 3.2 *Let $D = |\,\mathrm{O}(q_L)|$. Then we have the following diagram of groups with indices as indicated:*

$$\begin{array}{ccc} \widetilde{\mathrm{O}}(L) & \overset{D:1}{\subset} & \mathrm{O}(L) \\ \bigcup\ 2{:}1 & & \bigcup\ 2{:}1 \\ \widetilde{\mathrm{O}}^+(L) & \overset{D:1}{\subset} & \mathrm{O}^+(L) \\ \bigcup\ 2{:}1 & & \bigcup\ 2{:}1 \\ \widetilde{\mathrm{SO}}^+(L) & \overset{D:1}{\subset} & \mathrm{SO}^+(L). \end{array}$$

Proof. We shall first prove that the indices of the vertical inclusions are all 2. To do this, we choose a hyperbolic plane U in L, which exists by assumption. Let e_1, e_2 be a basis of U with $e_1^2 = e_2^2 = 0$ and $e_1.e_2 = 1$. If $u = e_1 - e_2$, $v = e_1 + e_2$, then $u^2 = -2$, $v^2 = 2$ and the two reflections σ_u and σ_v belong to $\widetilde{\mathrm{O}}(L)$, since they act trivially on the orthogonal complement of U. Moreover $\mathrm{sn}_{-1}(\sigma_v) = -1$ and $\mathrm{sn}_{-1}(\sigma_u) = 1$. Hence we can use σ_v to conclude that the

top two vertical inclusions are of index 2, whereas σ_u shows the same for the bottom two vertical inclusions.
We have already observed that the natural map $\mathrm{O}(L) \to \mathrm{O}(q_L)$ is surjective, which shows that the top horizontal inclusion has index D. Taking into account that the reflections σ_u and σ_v act trivially on the discriminant form, we obtain that

$$D = [\mathrm{O}(L) : \widetilde{\mathrm{O}}(L)] = [\mathrm{O}^+(L) : \widetilde{\mathrm{O}}^+(L)] = [\mathrm{SO}^+(L) : \widetilde{\mathrm{SO}}^+(L)].$$

□

Finally, we want to consider the projective groups $\mathrm{PO}(L)$, $\mathrm{PO}^+(L)$ and $\mathrm{P}\widetilde{\mathrm{O}}^+(L)$, i.e., the corresponding groups modulo their centres. It follows immediately from the above diagram that

$$[\mathrm{PO}(L) : \mathrm{P}\widetilde{\mathrm{O}}^+(L)] = \begin{cases} D & \text{if } -\mathrm{id} \notin \widetilde{\mathrm{O}}^+(L) \\ 2D & \text{if } -\mathrm{id} \in \widetilde{\mathrm{O}}^+(L). \end{cases} \qquad (9)$$

Note that $-\mathrm{id} \in \widetilde{\mathrm{O}}^+(L)$ if and only if A_L is a 2-group.

3.2 Local densities

Siegel's definition of local densities of a quadratic form over a number field K given by a matrix $S \in \mathrm{Mat}_{n\times n}(K)$ is

$$\alpha_p(S) = \frac{1}{2} \lim_{r\to\infty} p^{-\frac{rn(n-1)}{2}} |\left\{X \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \bmod p^r;\ {}^tXSX \equiv S \bmod p^r\right\}|.$$

The local densities can be calculated explicitly, at least in the cases where $K = \mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$ (see chapter 5 of the book [Ki] and references there). For the convenience of the reader we include the formulae over $\mathbb{Q}$ in the present paper. To calculate $\alpha_p(L)$ one should know the Jordan decomposition of the lattice L over the local ring $\mathbb{Z}_p$ of p-adic integers. The main difficulties arise for $p = 2$: see [Ki, Theorem 5.6.3].
Let us introduce some notation. Let L be a $\mathbb{Z}_p$-lattice in a regular (i.e. non-degenerate) quadratic space over $\mathbb{Q}_p$ of rank n, and let $(\mathbf{v}_i)$ be a basis of L. There are two invariants of L: the scale

$$\mathrm{scale}(L) = \{(\mathbf{x}, \mathbf{y})_L\,;\ \mathbf{x}, \mathbf{y} \in L\}$$

and the norm

$$\mathrm{norm}(L) = \{\textstyle\sum a_{\mathbf{x}}(\mathbf{x}, \mathbf{x})_L\,;\ \mathbf{x} \in L,\ a_{\mathbf{x}} \in \mathbb{Z}_p\}.$$

We have $2\,\mathrm{scale}(L) \subset \mathrm{norm}(L) \subset \mathrm{scale}(L)$. In fact, over $\mathbb{Z}_p$ ($p \neq 2$) we have $\mathrm{norm}(L) = \mathrm{scale}(L)$, whereas over $\mathbb{Z}_2$ we have either $\mathrm{norm}(L) = \mathrm{scale}(L)$ or $\mathrm{norm}(L) = 2\,\mathrm{scale}(L)$.

L is called p^r-modular, for $r \in \mathbb{Z}$, if the matrix $p^{-r}(\mathbf{v}_i, \mathbf{v}_j)_L$ belongs to $\mathrm{GL}_n(\mathbb{Z}_p)$. In this case we can write L as the scaling $N(p^r)$ of a unimodular lattice N. By a hyperbolic space we mean a (possibly empty) orthogonal sum of hyperbolic planes.
A regular lattice L decomposes as the orthogonal sum of lattices $\bigoplus_{j\in\mathbb{Z}} L_j$, where L_j is a p^j-modular lattice of rank $n_j \in \mathbb{Z}_{\geq 0}$. Put

$$w = \sum_j jn_j\big((n_j+1)/2 + \sum_{k>j} n_k\big)$$

and

$$P_p(n) = \prod_{i=1}^{n}(1-p^{-2i}).$$

For a regular quadratic space W over the finite field $\mathbb{Z}/p\mathbb{Z}$ one puts

$$\chi(W) = \begin{cases} 0 & \text{if } \dim W \text{ is odd,} \\ 1 & \text{if } W \text{ is a hyperbolic space,} \\ -1 & \text{otherwise.} \end{cases}$$

For a unimodular lattice N over $\mathbb{Z}_2$ with $\mathrm{norm}(N) = 2\,\mathrm{scale}(N)$ we define $\chi(N) = \chi(N/2N)$, where $N/2N$ is given the structure of a regular quadratic space over $\mathbb{Z}/2\mathbb{Z}$ via the quadratic form $Q(\mathbf{x}) = \frac{1}{2}(\mathbf{x},\mathbf{x})_N \bmod 2$.

For the local density $\alpha_p(L)$ for $p \neq 2$ we have the formula

$$\alpha_p(L) = 2^{s-1}p^w P_p(L) E_p(L) \tag{10}$$

where s is the number of non-zero p^j-modular terms L_j in the orthogonal decomposition of L, and

$$P_p(L) = \prod_j P_p([n_j/2]), \qquad E_p(L) = \prod_{j,\, L_j\neq 0} \big(1 + \chi(N_j)p^{-n_j/2}\big)^{-1}$$

where L_j is the p^j-scaling of the unimodular lattice N_j and $[n_j/2]$ denotes the integer part.

The local density $\alpha_2(L)$ is given by

$$\alpha_2(L) = 2^{n-1+w-q} P_2(L) E_2(L). \tag{11}$$

In this formula $q = \sum_j q_j$ where

$$q_j = \begin{cases} 0 & \text{if } N_j \text{ is even,} \\ n_j & \text{if } N_j \text{ is odd and } N_{j+1} \text{ is even,} \\ n_j + 1 & \text{if } N_j \text{ and } N_{j+1} \text{ are odd.} \end{cases}$$

A unimodular lattice N over $\mathbb{Z}_2$ is even if it is trivial or if $\operatorname{norm}(N) = 2\mathbb{Z}_2$, and odd otherwise. Any unimodular lattice can be represented as the orthogonal sum $N = N^{\text{even}} \oplus N^{\text{odd}}$ of even and odd sublattices such that $\operatorname{rank} N^{\text{odd}} \leq 2$. Then we put

$$P_2(L) = \prod_j P_2(\operatorname{rank} N_j^{\text{even}}/2).$$

The second factor is $E_2(L) = \prod_j E_j^{-1}$, where E_j is defined by

$$E_j = \frac{1}{2}(1 + \chi(N_j^{\text{even}})2^{-\operatorname{rank} N_j^{\text{even}}/2})$$

if both N_{j-1} and N_{j+1} are even, unless $N_j^{\text{odd}} \cong \langle\epsilon_1\rangle \oplus \langle\epsilon_2\rangle$ with $\epsilon_1 \equiv \epsilon_2 \mod 4$: in all other cases we put $E_j = 1/2$.

We note that E_j depends on N_{j-1}, N_j and N_{j+1} and $E_j = 1$ if all of them are trivial. Also q_j depends on N_j and N_{j+1} and $q_j = 0$ if N_j is trivial.

3.3 The even unimodular lattices $II_{2,8m+2}$

We start with the example

$$II_{2,8m+2} = 2U \oplus mE_8(-1), \qquad \text{where} \quad m \geq 0$$

which is a natural series of even unimodular lattices of signature $(2, 8m+2)$. Note that $II_{2,26} \cong 2U \oplus \Lambda$, where Λ is the Leech lattice.

The local densities are easy to calculate, since for every prime p the lattice $II_{2,8m+2} \otimes \mathbb{Z}_p$ over the p-adic integers is a direct sum of hyperbolic planes. Then using (10) and (11) we obtain

$$\alpha_p(II_{2,8m+2}) = 2^{\delta_{2,p}(8m+4)} P_p(4m+2)(1 + p^{-(4m+2)})^{-1}$$

where $\delta_{2,p}$ is the Kronecker delta. By our main formula (8) from Theorem 3.1 we obtain

$$\operatorname{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})) = 2^{-(8m+2)}\gamma_{8m+4}^{-1}\zeta(2)\zeta(4)\cdot\ldots\cdot\zeta(8m+2)\zeta(4m+2)$$

where γ_{8m+4} is as in formula (2). In order to simplify this expression we use the ζ-identity

$$\pi^{-\frac{1}{2}-2k}\Gamma(k)\Gamma\left(k + \frac{1}{2}\right)\zeta(2k) = (-1)^k\zeta(1-2k) = (-1)^{k+1}\frac{B_{2k}}{2k}. \tag{12}$$

Together with

$$\begin{aligned} &\pi^{-(4m+2)}\Gamma(4m+2)\zeta(4m+2) \\ &\quad = \quad 2^{4m+1}\pi^{-\frac{1}{2}-(4m+2)}\Gamma(2m+1)\Gamma(\frac{4m+3}{2})\zeta(4m+2) \\ &\quad = \quad 2^{4m+1}\frac{B_{4m+2}}{4m+2} \end{aligned}$$

where the first equality comes from the Legendre duplication formula of the Γ-function, and the second equality is again a consequence of the ζ-identity, we obtain

$$\mathrm{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})) = 2^{-(4m+1)}\frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{8m+2}}{(8m+2)!!} \cdot \frac{B_{4m+2}}{4m+2}.$$

Here $(2n)!! = 2 \cdot 4 \cdot \ldots \cdot 2n$. Since the discriminant group of the lattice $II_{2,8m+2}$ is trivial, we have the equality

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(II_{2,8m+2})) = \mathrm{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})).$$

In a similar way one can derive a formula for any indefinite unimodular lattice of signature (r,s). For example, for the odd unimodular lattice M defined by $x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$ we have to take into account that the even $(M \otimes \mathbb{Z}_2)^{\mathrm{even}}$ and odd $(M \otimes \mathbb{Z}_2)^{\mathrm{odd}}$ parts of the lattice M over 2-adic numbers depend on $r+s \mod 2$ and $r-s \mod 8$ (see [BG] for a different approach in this special case).
We can now use this to compute dimensions of cusp forms for this group and we obtain

$$\dim S_k(\widetilde{\mathrm{O}}^+(II_{2,8m+2}), \det{}^{\varepsilon}) = \frac{2^{-4m}}{(8m+2)!} \cdot \frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{8m+2}}{(8m+2)!!} \cdot \frac{B_{4m+2}}{4m+2} k^{8m+2} + O(k^{8m+1}).$$

Here $\varepsilon = \pm 1$ and we must assume that k is even, since otherwise there are no forms for trivial reasons.

3.4 The lattices $T_{2,8m+2}$

The orthogonal group of the lattice $II_{2,8m+2}$ for $m = 2$ defines an irreducible component of the branch divisor of the modular variety $\mathcal{F}_{2d}^{(m)}$. The same branch divisor contains another component defined by the lattice

$$T_{2,8m+2} = U \oplus U(2) \oplus mE_8(-1)$$

of discriminant 4. We note that this lattice is not maximal. For a prime number $p \neq 2$ the p-local densities of the lattices T and M coincide. Let us calculate $\alpha_2(T)$. Over the 2-adic ring we have $T_{2,8m+2} \otimes \mathbb{Z}_2 \cong (4m+1)U \oplus U(2)$. We have (see (11))

$$N_0 = N_0^{\mathrm{even}} = (4m+1)U, \quad N_1 = N_1^{\mathrm{even}} = U, \quad w = 3, \quad q = 0,$$

$$E_0 = \frac{1}{2}(1 + 2^{-(4m+1)}), \qquad E_1 = \frac{1}{2}(1 + 2^{-1}).$$

Thus

$$\alpha_2(T_{2,8m+2}) = 2^{8m+7}(1 - 2^{-2}) \cdot \cdots \cdot (1 - 2^{-8m})(1 - 2^{-(4m+1)}).$$

We note that $[\mathrm{PO}^+(T_{2,8m+2}) : \mathrm{P}\widetilde{\mathrm{O}}^+(T_{2,8m+2})] = 2$ since the finite orthogonal discriminant group of $T_{2,8m+2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. As a result we get

$$\begin{aligned}&\mathrm{vol}_{MH}(\widetilde{\mathrm{O}}^+(T_{2,8m+2})) = \\ &\qquad 2\,\gamma_{8m+4}^{-1}\zeta(2)\cdot\dots\cdot\zeta(8m+2)\zeta(4m+2)(1+2^{-(4m+1)})(1-2^{-(4m+2)}).\end{aligned}$$

Using the formula for the volume of $II_{2,8m+2}$ we see that

$$\frac{\mathrm{vol}_{MH}\,\widetilde{\mathrm{O}}^+(T_{2,8m+2})}{\mathrm{vol}_{MH}\,\widetilde{\mathrm{O}}^+(II_{2,8m+2})} = (2^{4m+1}+1)(2^{4m+2}-1). \tag{13}$$

If L_1 is a sublattice of finite index of a lattice L then $\widetilde{\mathrm{O}}^+(L)$ is a subgroup of $\widetilde{\mathrm{O}}^+(L_1)$. One can use the formula of Theorem 2.1 to calculate easily the index $[\widetilde{\mathrm{O}}^+(L_1) : \widetilde{\mathrm{O}}^+(L)]$. For example, formula (13) above gives the index of $\widetilde{\mathrm{O}}^+(II_{2,8m+2})$ in $\widetilde{\mathrm{O}}^+(T_{2,8m+2})$. This method is much shorter than the calculation in terms of finite geometry over $\mathbb{Z}/2\mathbb{Z}$.

3.5 The lattices $L_{2d}^{(m)}$

We consider the lattice

$$L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d\rangle$$

of signature $(2, 8m+3)$. The lattice $L_{2d}^{(m)}$ is not maximal if d is not square free. This lattice is of particular interest, as the lattice $L_{2d}^{(2)}$ is closely related to the moduli space of polarised K3 surfaces of degree $2d$. More precisely, the quotient space

$$\mathcal{F}_{2d} = \widetilde{\mathrm{O}}^+(L_{2d}^{(2)})\backslash\mathcal{D}_{L_{2d}^{(2)}}$$

is the moduli space of K3 sufaces of degree $2d$. As we shall see, there is also a relation to Siegel modular forms for both the group $\mathrm{Sp}(2,\mathbb{Z})$ and the paramodular group.

Again, the lattices over the p-adic integers are easy to understand, since $E_8(-1)\otimes\mathbb{Z}_p$ is the direct sum of four copies of a hyperbolic plane. By (10) and (11) we find

$$\begin{aligned}
&\alpha_p(L_{2d}^{(m)}) = P_p(4m+2) && \text{if } p \nmid 2d\\
&\alpha_p(L_{2d}^{(m)}) = 2p^s P_p(4m+2)(1+p^{-(4m+2)})^{-1} && \text{if } p \text{ is odd, } p^s\|d\\
&\alpha_2(L_{2d}^{(m)}) = 2^{8m+6}P_2(4m+2) && \text{if } d \text{ is odd}\\
&\alpha_2(L_{2d}^{(m)}) = 2^{8m+7+s}P_2(4m+2)(1+2^{-(4m+2)})^{-1} && \text{if } d \text{ is even, } 2^s\|d
\end{aligned}$$

where the expression $p^s \| d$ means that p^s is the highest power of p which divides d. Therefore

$$\prod_p \alpha_p(L_{2d}^{(m)})^{-1} = \zeta(2)\zeta(4)\dots\zeta(8m+4)\,(2d)^{-1}2^{-\rho(d)-8m-5}\prod_{p|d}(1+p^{-(4m+2)})$$

where $\rho(d)$ denotes the number of prime divisors of d.
We shall need the following.

Lemma 3.3 *Let $R = \langle -2d\rangle$. Then the order of the discriminant group* $\mathrm{O}(q_R)$ *is* $2^{\rho(d)}$.

Proof. Let g be the standard generator of $A_R = \mathbb{Z}/2d\mathbb{Z}$, given by the equivalence class of 1. Then $q_R(g) = -1/2d \mod 2\mathbb{Z}$. If $\varphi \in \mathrm{O}(q_R)$, then $\varphi(g) = xg$ for some x with $(x, 2d) = 1$. Hence φ is orthogonal if and only if

$$-\frac{x^2}{2d} \equiv -\frac{1}{2d} \mod 2\mathbb{Z},$$

or equivalently

$$x^2 \equiv 1 \mod 4d\mathbb{Z}.$$

This equation has $2^{\rho(d)+1}$ solutions modulo $4d\mathbb{Z}$, and hence $2^{\rho(d)}$ solutions modulo $2d\mathbb{Z}$. □

From this it follows also that the discriminant group of the lattice $L_{2d}^{(m)}$ also has order $2^{\rho(d)}$.
From (9) it follows that

$$[\mathrm{PO}(L_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})] = 2^{\rho(d)} \quad \text{if} \quad d > 1$$

and 2 if $d = 1$. We first assume that $d > 1$. We put $n = 8m + 3$, which is the dimension of the homogeneous domain. It follows from Corollary 3.1 that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = 2^{\rho(d)+1}(2d)^{\frac{n+3}{2}}\gamma_{n+2}^{-1}\prod_p \alpha_p(L_{2d}^{(m)})^{-1}.$$

If $d = 1$ we have to multiply the right hand side by a factor 2. Using the ζ identity, a straightforward calculation gives (again for $d > 1$ and $n = 8m + 3$)

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = \left(\frac{d}{2}\right)^{\frac{n+1}{2}} \prod_{p|d}(1+p^{-\frac{n+1}{2}}) \cdot \frac{|B_2 \cdot B_4 \cdot \dots \cdot B_{n+1}|}{(n+1)!!}.$$

We want to apply this to the moduli space of K3 surfaces of degree $2d$. This is the case $m = 2$: the dimension of the domain is $n = 19$. Using Hirzebruch-Mumford proportionality and specialising the above volume computation to this case, we compute the dimension of the spaces of cusp forms:

$$\dim S_k(\widetilde{\mathrm{O}}^+(L_{2d}^{(2)}), \det{}^{\varepsilon}) = \frac{2^{-9}}{19!} d^{10} \cdot \prod_{p|d}(1+p^{-10}) \frac{|B_2 \cdot B_4 \cdot \ldots \cdot B_{20}|}{20!!} \cdot k^{19} + O(k^{18})$$

which holds for $d > 1$, with an additional factor 2 for $d = 1$. In the latter case we must assume that k and ε have the same parity. For $d > 1$ there is no restriction since $-\operatorname{id} \notin \widetilde{\mathrm{O}}^+(L_{2d}^{(2)})$. This should be compared to Kondo's formula [Ko] where, however, the Hirzebruch-Mumford volume has not been computed explicitly. It should also be noted that Kondo uses the geometric, rather than the arithmetic, weight.

3.5.1 Siegel modular forms

The case $m = 0$ gives applications to Siegel modular forms. We shall first consider the case $d = 1$. Recall that

$$\widetilde{\mathrm{SO}}^+(L_2^{(0)}) \cong \widetilde{\mathrm{O}}^+(L_2^{(0)})/\{\pm \operatorname{id}\} \cong \mathrm{Sp}(2,\mathbb{Z})/\{\pm \operatorname{id}\}.$$

From our previous computation we obtain that

$$\operatorname{vol}_{HM}(\widetilde{\mathrm{SO}}^+(L_2^{(0)})) = \operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_2^{(0)})) = 2^{-4}|B_2B_4|$$

and by Hirzebruch-Mumford proportionality this gives

$$\dim S_k(\mathrm{Sp}(2,\mathbb{Z})) = 2^{-4}3^{-1}|B_2B_4|k^3 + O(k^2).$$

Note that this coincides with [T, p. 428], taking into account that Tai's formula refers to modular forms of weight $3k$. Tai uses Siegel's computation of the volume of the group $\mathrm{Sp}(2,\mathbb{Z})$, rather than the orthogonal group.

3.5.2 The paramodular group

Finally, we consider the case $m = 0$ and $d > 1$. This is closely related to the so-called *paramodular* group $\Gamma_d^{(\mathrm{Sp})}$, which gives rise to the moduli space of $(1,d)$-polarised abelian surfaces. In fact

$$\widetilde{\mathrm{SO}}^+(L_{2d}^{(0)}) \cong \mathrm{P}\Gamma_d^{(\mathrm{Sp})}$$

by [GH, Proposition 1.2]. We note that in this case

$$[\widetilde{\mathrm{O}}^+(L_{2d}^{(0)}) : \widetilde{\mathrm{SO}}^+(L_{2d}^{(0)})] = 2$$

and that $-\operatorname{id}$ is in neither of these groups. Hence

$$\begin{aligned}\operatorname{vol}_{HM}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(0)})) &= 2\operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(0)})) \\ &= 2^{-4}d^2\prod_{p|d}(1+p^{-2})|B_2B_4|\end{aligned}$$

and by Hirzebruch-Mumford proportionality

$$\dim S_k(\Gamma_d^{(\mathrm{Sp})}) = \frac{d^2}{3\cdot 2^4}\prod_{p|d}(1+p^{-2})|B_2B_4|k^3 + O(k^2).$$

This agrees with [Sa, Proposition 2.2], where this formula was derived for d a prime.

3.6 Lattices associated to Heegner divisors

We shall conclude this section by computing the volume of two lattices of rank $8m+4$. Both of these lattices $K_{2d}^{(m)}$ and $N_{2d}^{(m)}$ arise from the (-2)-reflective part of the ramification divisor of the quotient map

$$\mathcal{D}_{L_{2d}^{(m)}} \to \widetilde{\mathrm{O}}^+_{L_{2d}^{(m)}}\backslash\mathcal{D}_{L_{2d}^{(m)}} = \mathcal{F}_{2d}^{(m)}.$$

For $m=2$ this is the moduli space of K3 surfaces of degree $2d$. For $m=0$ and a prime d we get the moduli of Kummer surfaces associated to $(1,d)$-polarised abelian surfaces (see [GH]). Since the branch locus of the quotient map gives rise to obstructions for extending pluricanonical forms defined by modular forms, knowledge of their volumes is important for the computation of the Kodaira dimension of $\mathcal{F}_{2d}^{(m)}$ (see [GHS2]).

3.6.1 The lattices $K_{2d}^{(m)}$

We consider the lattice

$$K_{2d}^{(m)} = U \oplus mE_8(-1)\oplus\langle 2\rangle\oplus\langle -2d\rangle$$

where d is a positive integer. We first have to determine the local densities for this lattice. Since $\det(K_{2d}^{(m)}) = 4d$, this lattice is equivalent to the following lattices over the p-adic integers for odd primes p:

$$K_{2d}^{(m)}\otimes\mathbb{Z}_p \cong (4m+1)\,U \oplus \begin{cases} U & \text{if } \left(\frac{4d}{p}\right)=1\\ x^2-4dy^2 & \text{if } \left(\frac{4d}{p}\right)=-1.\end{cases}$$

For the local densities we obtain from equations (10) and (11)

$$\begin{aligned}
\alpha_p(K_{2d}^{(m)}) &= P_p(4m+1)(1-\left(\frac{4d}{p}\right)p^{-(4m+2)}) && \text{if } p\not| d\\
\alpha_p(K_{2d}^{(m)}) &= 2p^sP_p(4m+1) && \text{if } p^s\|d\\
\alpha_2(K_{2d}^{(m)}) &= 2^{8m+v(d)}P_2(4m+1)
\end{aligned}$$

where $v(d)=6$ if $d\equiv 1 \mod 4$, $v(d)=7$ if $d\equiv -1 \mod 4$, $v(d)=8$ if $d\equiv 2 \mod 4$, and $v(d)=8+s$ if $d\equiv 0 \mod 4$ and $2^s\|d$.

From this we obtain that

$$\prod_p \alpha_p(K_{2d}^{(m)})^{-1} = A_2(d)d^{-1}\zeta(2)\zeta(4)\dots\zeta(8m+2)L(4m+2,\left(\frac{4d}{*}\right)), \qquad (14)$$

where

$$A_2(d) = \begin{cases} 2^{-\rho(d)-8m-6} & \text{if } d \equiv 1, 2 \mod 4 \\ 2^{-\rho(d)-8m-7} & \text{if } d \equiv 0, 3 \mod 4. \end{cases}$$

Application of our main formula (7) then gives

$$\mathrm{vol}_{HM}(\mathrm{O}^+(K_{2d}^{(m)})) = 4 \cdot (4d)^{\frac{8m+5}{2}} \cdot \prod_{k=1}^{8m+4} \pi^{-\frac{k}{2}}\Gamma(\frac{k}{2}) \cdot \prod_p \alpha_p(K_{2d})^{-1}. \qquad (15)$$

Combining formulae (14) and (15) and the ζ-identity (12) leads to

$$\mathrm{vol}_{HM}(\mathrm{O}^+(K_{2d}^{(m)})) = \\ C_2(d)d^{\frac{8m+3}{2}}\pi^{-(4m+2)}\Gamma(4m+2)L(4m+2,\left(\frac{4d}{*}\right))\frac{B_2B_4\dots B_{8m+2}}{(8m+2)!!}$$

where

$$C_2(d) = \begin{cases} 2^{-\rho(d)+1} & \text{if } d \equiv 1, 2 \mod 4 \\ 2^{-\rho(d)} & \text{if } d \equiv 0, 3 \mod 4. \end{cases}$$

For applications it is also important to compute the volume with respect to the group $\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})$. For this, we have to know the order of the group of isometries of the discriminant group.

Lemma 3.4 *Let $S = \langle 2\rangle \oplus \langle -2d\rangle$. The order of the discriminant group is*

$$|\,\mathrm{O}(q_S)| = \begin{cases} 2^{1+\rho(d)} & \textit{if } d \equiv -1 \mod 4 \textit{ or } d \textit{ is divisible by } 8 \\ 2^{\rho(d)} & \textit{for all other } d. \end{cases}$$

Proof. We denote the standard generators of $\mathbb{Z}/2d\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ by g and h respectively. We shall first consider automorphisms φ with $\varphi(g) = xg$. Then orthogonality implies $x^2 \equiv 1 \mod 4d\mathbb{Z}$ which means, in particular, that x is odd and $(x, 2d) = 1$. We then have $\varphi(dg) = dg$. We cannot have $\varphi(h) = dg+h$, because orthogonality implies that for the bilinear form B_q, defined by the quadratic form $q = q_S$, we have $B_q(xg, dg + h) = B_q(g, h) = 0$ and hence $-x/2 \equiv 0 \mod \mathbb{Z}$, which shows that x is even, a contradiction. Hence $\varphi(h) = h$ and $\varphi = \varphi' \times \mathrm{id}$ where $\varphi' \in \mathrm{O}(q_R)$ (with $R = \langle -2d\rangle$). In this way we obtain $2^{\rho(d)}$ elements in $\mathrm{O}(q_S)$.

We shall now investigate automorphisms with $\varphi(g) = xg + h$. Then $q(g) = q(\varphi(g))$ implies the condition

$$x^2 \equiv 1 + d \mod 4d\mathbb{Z}.$$

It is not hard to check that this only has solutions if either $d \equiv -1 \mod 4$ or d is divisible by 8. We shall distinguish between the cases d even and d odd. In the first case x must be odd and $(x, 2d) = 1$. Moreover $\varphi(dg) = dg$ and the only possibility for an orthogonal automorphism is $\varphi(h) = dg + h$ and indeed this gives rise to another $2^{\rho(d)}$ orthogonal automorphisms. Now assume d is odd. Then x is even and $(x, d) = 1$. In this case $\varphi(dg) = h$ and the only possibility to obtain an orthogonal automorphism is $\varphi(h) = dg$. Once more, this gives another $2^{\rho(d)}$ orthogonal automorphisms and this proves the lemma. □

By formula (9) it then follows that

$$[\mathrm{PO}(K_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^{+}(K_{2d}^{(m)})] = \begin{cases} 2 & \text{if } d = 1 \\ 2^{\rho(d)} & \text{if } d \equiv 1, 2 \mod 4,\ d > 1 \\ 2^{\rho(d)+1} & \text{if } d \equiv 3 \mod 4. \end{cases}$$

Therefore

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^{+}(K_{2d}^{(m)})) = 2^{\delta_{1,d} - \delta_{4,d(8)}} \frac{B_2 B_4 \dots B_{8m+2}}{(8m+2)!!} \cdot d^{\frac{8m+3}{2}} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{4d}{*}\right)) \tag{16}$$

where $d(8)$ denotes $d \mod 8$ and $\delta_{*,*}$ is the Kronecker symbol.
We want to reformulate this result in terms of generalised Bernoulli numbers. In order to avoid too many different cases, we restrict here to $d \not\equiv 0 \mod 4$ (but it is clear how to remove this restriction). If $d = d_0 t^2$, with d_0 a positive and square-free integer, then the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is equal to

$$D = \begin{cases} d_0 & \text{if } d \equiv 1 \mod 4 \\ 4d_0 & \text{if } d \equiv 2, 3 \mod 4. \end{cases}$$

Note that

$$d^{\frac{8m+3}{2}} = t^{8m+3} D^{\frac{8m+3}{2}} \cdot \begin{cases} 1 & \text{if } d \equiv 1 \mod 4 \\ 2^{-(8m+3)} & \text{if } d \equiv 2, 3 \mod 4. \end{cases} \tag{17}$$

Let χ_D be the quadratic character of this field. Then

$$L(s, \left(\frac{4d}{*}\right)) = L(s, \chi_D) \prod_{p|2t} (1 - \chi_D(p) p^{-s}). \tag{18}$$

The character χ_D is an even primitive character modulo D, and the Dirichlet L-function $L(s, \chi_D)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) D^s L(s, \chi_D) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) D^{\frac{1}{2}} L(1-s, \chi_D). \tag{19}$$

Moreover

$$L(1-k,\chi_D) = -\frac{B_{k,\chi_D}}{k}$$

where B_{k,χ_D} is the corresponding generalised Bernoulli number. Using the functional equation (19) we obtain

$$\begin{aligned}\pi^{-(4m+2)}\Gamma(4m+2)D^{\frac{8m+3}{2}}L(4m+2,\chi_D) &= -2^{4m+1}L(1-(4m+2),\chi_D)\\ &= 2^{4m+1}\frac{B_{4m+2,\chi_D}}{4m+2}. \qquad (20)\end{aligned}$$

Combining (16), (17), (18), (20) and the result of Lemma 3.4 then gives the result

$$\begin{aligned}&\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})) =\\ &\quad F_2(d)t^{8m+3}\frac{B_2B_4\ldots B_{8m+2}}{(8m+2)!!}\frac{B_{4m+2,\chi_D}}{4m+2}\prod_{p|2t}(1-\chi_D(p)p^{-(4m+2)}) \quad (21)\end{aligned}$$

where

$$F_2(d) = \begin{cases} 2^{4m+2} & \text{if } d\equiv 1 \mod 4\\ 2^{-4m-1} & \text{if } d \equiv 2,\ 3 \mod 4.\end{cases}$$

Using this, together with Hirzebruch-Mumford proportionality, we finally find that $\dim S_k(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)}))$ grows as

$$\frac{G_2(d)}{(8m+2)!}\frac{B_2\cdot B_4\ldots B_{8m+2}}{(8m+2)!!}\cdot\frac{B_{4m+2,\chi_D}}{4m+2}t^{8m+3}\prod_{p|2t}(1-\chi_D(p)p^{-(4m+2)})\,k^{8m+2}$$

where

$$G_2(d) = \begin{cases} 2^{4m+2+\delta_{1,d}} & \text{if } d\equiv 1 \mod 4,\\ 2^{-(4m+1)} & \text{if } d \equiv 2,\ 3 \mod 4.\end{cases}$$

3.6.2 The lattices $N_{2d}^{(m)}$

We assume that $d\equiv 1 \mod 4$ and consider the even lattice

$$N_{2d}^{(m)} = U\oplus mE_8(-1)\oplus\begin{pmatrix}2 & 1\\ 1 & \frac{1-d}{2}\end{pmatrix}.$$

We first have to understand this lattice over the p-adic integers. If $p>2$ then 2 is a p-adic integer and we have the following equality for the anisotropic binary form in $N_{2d}^{(m)}$:

$$\frac{1-d}{2}x^2+2xy+2y^2 = -\frac{d}{2}x^2+2(y+\frac{x}{2})^2.$$

Depending on whether d is a square in $\mathbb{Z}_p^*$ or not, we then obtain from the classification theory of quadratic forms over $\mathbb{Z}_p$ that

$$N_{2d}^{(m)} \otimes \mathbb{Z}_p \cong (4m+1)\,U \oplus \begin{cases} U & \text{if } \left(\frac{d}{p}\right) = 1 \\ -dx^2 + y^2 & \text{if } \left(\frac{d}{p}\right) = -1. \end{cases}$$

We now turn to $p = 2$. Recall that there are only two even unimodular binary forms over $\mathbb{Z}_2$, namely the hyperbolic plane and the form given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This implies that

$$N_{2d}^{(m)} \otimes \mathbb{Z}_2 \cong (4m+1)\,U \oplus \begin{cases} U & \text{if } d \equiv 1 \mod 8 \\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } d \equiv 5 \mod 8. \end{cases}$$

Once again by (10) and (11) we find for the local densities that

$$\begin{aligned} \alpha_p(N_{2d}^{(m)}) &= P_p(4m+1)(1 - \left(\frac{d}{p}\right) p^{-(4m+2)}) && \text{if } p \not| d \\ \alpha_p(N_{2d}^{(m)}) &= 2p^s P_p(4m+1) && \text{if } p^s \| d \\ \alpha_2(N_{2d}^{(m)}) &= 2^{8m+4} P_2(4m+1)(1 - \left(\frac{d}{2}\right) 2^{-(4m+2)}). \end{aligned}$$

We are interested mainly in the group $\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})$. For this we need the next lemma.

Lemma 3.5 *Let*

$$T = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-d}{2} \end{pmatrix}$$

Then $A_T \cong \mathbb{Z}/2d\mathbb{Z}$ *and*

$$|\,\mathrm{O}(q_T)| = 2^{\rho(d)}.$$

Proof. Since $\det(T) = -d$, the discriminant group has order d. In fact, it is cyclic of order d. To see this, let e and f be the basis with respect to which the form is given by the matrix T. Then $(e - 2f)/d$ is in the dual lattice and its class, say h, generates the group A_T. Every homomorphism of A_T is of the form $\varphi(h) = xh$, and it is an isometry if and only if $x^2 \equiv 1 \mod 2d$. This equation has $2^{\rho(d)}$ solutions modulo $d\mathbb{Z}$. □

It now follows from (9) that

$$[\mathrm{PO}(N_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})] = \begin{cases} 2^{\rho(d)} & \text{if } d \equiv 1 \mod 4 \text{ and } d \neq 1 \\ 2 & \text{if } d = 1. \end{cases}$$

By the same calculation as in the preceding example we find now that

$$\operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})) = 2^{\delta_{1,d}-8m-3}\frac{B_2B_4\dots B_{8m+2}}{(8m+2)!!}\cdot$$
$$d^{\frac{8m+3}{2}}\pi^{-(4m+2)}\Gamma(4m+2)L(4m+2,\left(\frac{d}{*}\right)). \tag{22}$$

As above we can use generalised Bernoulli numbers. Hence by Hirzebruch-Mumford proportionality we obtain for $d>1$ that $\dim S_k(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)}))$ grows as

$$\frac{2^{-4m-1}}{(8m+2)!}\frac{B_2\cdot B_4\dots B_{8m+2}}{(8m+2)!!}\cdot\frac{B_{4m+2,\chi_D}}{4m+2}\,t^{8m+3}\prod_{p|t}(1-\chi_D(p)p^{-(4m+2)})\,k^{8m+2}.$$

Here, as before, $d=d_0t^2$, with d_0 square-free, and $D=d_0$ is the discriminant of the quadratic extension $\mathbb{Q}(\sqrt{d})$. For $d=1$ we have an extra factor 2, $t=1$, $\chi_D\equiv 1$ and $B_{4m+2,\chi_D}=B_{4m+2}$. In this case the lattice $N_{2d}^{(m)}$ is unimodular and the formula again agrees with our previous computations in Section 3.3.

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