

SPIN SYSTEMS WITH HYPERBOLIC SYMMETRY: A SURVEY

ROLAND BAUERSCHMIDT AND TYLER HELMUTH

ABSTRACT

Spin systems with hyperbolic symmetry originated as simplified models for the Anderson metal–insulator transition, and were subsequently found to exactly describe probabilistic models of linearly reinforced walks and random forests. In this survey we introduce these models, discuss their origins and main features, some existing tools available for their study, recent probabilistic results, and relations to other well-studied probabilistic models. Along the way we discuss some of the many open questions that remain.

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1. INTRODUCTION

Classical spin systems with spherical symmetry, such as the Ising and classical Heisenberg models, are basic models for magnetism and have been studied extensively over the last century. It is well-understood that the associated symmetry groups play an important role, particularly for the critical and low-temperature behaviour of these models. For example, the discrete \mathbb{Z}_2 symmetry of the Ising model is spontaneously broken at low temperatures, and in this phase truncated correlations decay exponentially. For models with continuous $O(n)$ symmetries, $n \geq 2$, low temperature truncated correlations instead decay polynomially, a reflection of the fact that the symmetry is spontaneously broken to an $O(n - 1)$ symmetry.

Spin systems with hyperbolic symmetry groups are also studied in condensed matter physics, primarily because of their relevance for the Anderson delocalisation–localisation (metal–insulator) transition of random Schrödinger operators and related random matrix models [38, 69, 74]. A rigorous analysis of the Anderson transition remains an outstanding challenge; see Section 3. The essential physical phenomena of the Anderson transition are expected to be captured by the more tractable $\mathbb{H}^{2|2}$ model, a simplified spin system with hyperbolic symmetry [76]. Surprisingly, the $\mathbb{H}^{2|2}$ model and its natural generalisations are intimately connected to probabilistic lattice models. The \mathbb{H}^2 and $\mathbb{H}^{2|2}$ models, motivated by the Anderson transition [34, 72], are exactly related to (linearly) edge-reinforced random walks and vertex-reinforced jump processes, introduced independently in the probability literature in the 1980s [29] and early 2000s [26]. A similar connection exists between the related $\mathbb{H}^{0|2}$ model and random forests [11, 22]; random forests arose earlier (for example) in connection with the Fortuin–Kasteleyn random cluster model [43]. The connections between hyperbolic spin systems and probabilistic phenomena are the main topic of this survey.

More specifically, this survey focuses on probabilistic results in line with the original physical motivation for studying hyperbolic spin systems. In particular, we focus on results for \mathbb{Z}^d (and its finite approximations) for $d \geq 2$. Our perspective is that a central role is played by the continuous symmetry groups of the spin systems. There are other perspectives available, notably that of Bayesian statistics. While the latter perspective has played a role in important results, e.g. [4, 66], and has found use in statistical contexts [5, 6, 31], we will not mention it further. Similarly, there are many related works we cannot discuss; fortunately, many of these are discussed in recent surveys on closely related topics [49, 61, 70, 71].

To set the stage, the remainder of this introduction recalls the *magic formula* for edge-reinforced random walk that led to the discovery of the connections discussed in this survey. Readers familiar with the magic formula may wish to jump to Section 2, where we introduce hyperbolic spin systems, or to Section 3, which discusses the physical background. The probabilistic representations and results for reinforced random walks and random forests are discussed in Sections 4 and 5, respectively, along with questions for the future.

Magic formula for edge-reinforced random walk. Fix $\alpha > 0$, a graph $G = (\Lambda, E)$, and an initial vertex $0 \in \Lambda$. *Edge-reinforced random walk (ERRW)* with $X_0 = 0$ and initial weights α

is the stochastic process $(X_n)_{n \geq 0}$ with transitions

$$\mathbb{P}_0^{\text{ERRW}(\alpha)}[X_{n+1} = j | (X_m)_{m \leq n}, X_n = i] = \frac{(\alpha + L_n^{ij})1_{ij \in E}}{\sum_{k:ik \in E} (\alpha + L_n^{ik})}, \quad (1.1)$$

where L_n^{ij} is the number of times the edge ij has been crossed up to time n (in either direction). The transition rates change rapidly if α is small, and hence this is called the *strong reinforcement* regime. *Weak reinforcement* refers to α being large. The definition can be generalised to edge-dependent weights $\alpha = (\alpha_{ij})$ in a straightforward manner.

Some intuition about ERRW can be gained by considering the case when G is a path on three vertices. Call one edge blue, one edge red, and start an ERRW at the middle vertex. If $\alpha = 2$ then the law of the vector $\frac{1}{2}L_{2n}$ of (half of) the number of crossings of the edges at time $2n$ is the law of a *Pólya urn*. Pólya's urn is the process that starts with an urn containing one red and one blue ball, and then sequentially draws a ball and replaces it with two balls, both of the same colour as the drawn ball. The fundamental fact about Pólya's urn is that $\frac{1}{2n}L_{2n}$ converges to $(U, 1 - U)$ where U is a uniform random variable on $[0, 1]$, i.e. the fraction of crossings of the blue edge is uniform. This can be proven by induction. Note that for an ordinary simple random walk this limit would be deterministic. *A priori* it is hard to predict how ERRW behaves on more complicated graphs. For example, is ERRW transient if simple random walk is transient? Does the answer depend on α ?

It turns out that the connection to Pólya's urn has a far reaching generalisation. The theory of partial exchangeability guarantees that ERRW is a random walk in random environment [30]. A consequence is that $\frac{1}{n}L_n$ has a distributional limit: it is the law of the random environment. Coppersmith and Diaconis discovered that one can give an explicit formula for the limiting law on *any* finite graph. It is surprising that an explicit formula can be obtained; this explains why it has been termed the *magic formula*, see [47, 55].

To precisely formulate this result, recall that an *environment* is a set of conductances $C: E \rightarrow [0, \infty)$ with $\sum_{ij} C_{ij} = 1$. Write C_{ij} for the conductance of the edge $\{i, j\}$ and $C_{ii} = -C_i = -\sum_j C_{ij}$. Associated to C is a reversible Markov chain (simple random walk) with transition probabilities C_{ij}/C_i whose law we denote by $\mathbb{P}_0^{\text{SRW}(C)}$ when started from 0.

Theorem 1.1 (Magic formula for ERRW). *Let $G = (\Lambda, E)$ be finite. Edge-reinforced random walk with $X_0 = 0$ and initial weights $\alpha = (\alpha_{ij})$ is a random walk in random environment:*

$$\mathbb{P}_0^{\text{ERRW}(\alpha)}[\cdot] = \int \mathbb{P}_0^{\text{SRW}(C)}[\cdot] d\mu_\alpha(C). \quad (1.2)$$

The environment μ_α has density proportional to

$$C_0^{\frac{1}{2}} \frac{\prod_{ij \in E} C_{ij}^{\alpha_{ij} - 1}}{\prod_{i \in \Lambda} C_i^{\frac{1}{2}(\alpha_i + 1)}} \sqrt{\det^0 C} \quad (1.3)$$

with respect to Lebesgue measure on the unit simplex in $[0, \infty)^E$, where $\alpha_i = \sum_j \alpha_{ij}$ and $\det^0 C$ is the determinant of any principal cofactor of C .

Note that the matrix-tree theorem implies the determinant in (1.3) can be written as a weighted sum of spanning trees, reflecting that this term is non-local.

Sabot and Tarrès showed how to relate the density (1.3) to the $\mathbb{H}^{2|2}$ model that we will introduce in the next section. This enabled them to leverage powerful results of Disertori, Spencer, and Zirnbauer to establish the existence of a recurrence/transience phase transition for ERRW on \mathbb{Z}^d for $d \geq 3$, see Section 4. In Section 5 we show that connection probabilities in the arboreal gas, a stochastic-geometric model of random forests, can be written in a form very similar to the magic formula. The derivation of this connection probability formula was inspired by [12, 13], which (at least partially) revealed the inner workings of the magic formula: horospherical coordinates (hyperbolic symmetry) and supersymmetric localisation.

2. HYPERBOLIC SPIN SYSTEMS

This section introduces the hyperbolic spin systems that we will discuss, briefly explains their characteristic symmetries, and discusses how these symmetries manifest themselves if spontaneous symmetry breaking occurs. For precise definitions of the Grassmann and Berezin integrals that are used see, e.g. [13, APPENDIX A].

The $\mathbb{H}^{2|0}$ model. The $\mathbb{H}^2 = \mathbb{H}^{2|0}$ model is defined as follows. We consider the hyperbolic plane \mathbb{H}^2 realised as $\mathbb{H}^2 = \{u = (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}$ and equipped with the Minkowski inner product $u \cdot u' = xx' + yy' - zz'$. For a finite graph $G = (\Lambda, E)$, we consider one spin $u_i \in \mathbb{H}^2$ per vertex $i \in \Lambda$ and define the action

$$H_{\beta,h}(u) = \frac{\beta}{2} \sum_{ij \in E} (u_i - u_j) \cdot (u_i - u_j) + h \sum_{i \in \Lambda} z_i. \quad (2.1)$$

The action also has a straightforward generalisation to edge- and vertex-dependent weights $\beta = (\beta_{ij})$ and $h = (h_i)$, and we will sometimes consider this case. For $\beta > 0$ and $h = 0$, the minimisers of $H_{\beta,0}$ are constant configurations $u_i = u_j$ for all $i, j \in \Lambda$. For $h > 0$, the unique minimiser is $u_i = (0, 0, 1)$ for all i . The \mathbb{H}^2 model is the probability measure on spin configurations whose expectation is given, for bounded $F: (\mathbb{H}^2)^\Lambda \rightarrow \mathbb{R}$, by

$$\langle F \rangle_{\beta,h} = \frac{1}{Z_{\beta,h}} \int_{(\mathbb{H}^2)^\Lambda} \prod_{i \in \Lambda} du_i F(u) e^{-H_{\beta,h}(u)} \quad (2.2)$$

where du_i stands for the Haar measure on \mathbb{H}^2 and $Z_{\beta,h}$ is a normalisation. Parametrising $u_i \in \mathbb{H}^2$ by $(x_i, y_i) \in \mathbb{R}^2$ with $z_i = \sqrt{1 + x_i^2 + y_i^2}$, we can explicitly rewrite (2.2) as

$$\langle F \rangle_{\beta,h} = \frac{1}{Z_{\beta,h}} \int_{(\mathbb{R}^2)^\Lambda} \prod_{i \in \Lambda} \frac{dx_i dy_i}{z_i} F(u) e^{-H_{\beta,h}(u)}. \quad (2.3)$$

The expectation is only normalisable if $h > 0$ (or, more generally, $h_i > 0$ for some vertex i) due to the non-compactness of \mathbb{H}^2 . It is useful to construct a version with $h = 0$ in which the field is fixed (pinned) at some distinguished vertex 0. We denote the *pinned expectation* with pinning $u_0 = (0, 0, 1)$ by $\langle \cdot \rangle_\beta^0$.

The $\mathbb{H}^{0|2}$ model. Now we consider the Grassmann algebra Ω_Λ generated by two generators ξ_i and η_i per vertex $i \in \Lambda$ and set

$$z_i = \sqrt{1 - 2\xi_i\eta_i} = 1 - \xi_i\eta_i, \tag{2.4}$$

and unite these into the formal supervector $u_i = (\xi_i, \eta_i, z_i)$. Thus u_i has two odd (anti-commuting) components ξ_i and η_i and one even (commuting) component z_i . We define $u_i \cdot u_j = -\xi_i\eta_j - \xi_j\eta_i - z_i z_j$, which is again an element of Ω_Λ . These definitions are such that $u_i \cdot u_i = -1$, as in the case of \mathbb{H}^2 spins. Define

$$H_{\beta,h} = \frac{\beta}{2} \sum_{ij \in E} (u_i - u_j) \cdot (u_i - u_j) + h \sum_{i \in \Lambda} z_i. \tag{2.5}$$

For F a polynomial in the ξ_i and η_i set

$$\langle F \rangle_{\beta,h} = \frac{1}{Z_{\beta,h}} \int \left(\prod_{i \in \Lambda} \partial_{\eta_i} \partial_{\xi_i} \frac{1}{z_i} \right) F e^{-H_{\beta,h}}, \tag{2.6}$$

where $\int \prod_{i \in \Lambda} \partial_{\eta_i} \partial_{\xi_i}$ stands for the Grassmann integral, i.e. the top coefficient of the element of the Grassmann algebra to its right. For example,

$$\int \partial_{\xi} \partial_{\eta} e^{-\xi\eta} = \int \partial_{\xi} \partial_{\eta} (1 - \xi\eta) = \int \partial_{\xi} \partial_{\eta} \eta \xi = 1. \tag{2.7}$$

In (2.6) and (2.7) we have used the convention that smooth functions of commuting elements of the algebra are defined by Taylor expansion. By nilpotency the expansion is finite, i.e. a polynomial. The $\mathbb{H}^{0|2}$ model is the expectation (2.6); while this is not a probabilistic expectation, we will soon see that it often carries probabilistic interpretations. Generalisations to edge- and vertex-dependent weights and pinning are straightforward.

The $\mathbb{H}^{2|2}$ model. The $\mathbb{H}^{2|2}$ model is defined as the $\mathbb{H}^{0|2}$ model was, but now beginning with three commuting components x_i, y_i, z_i . Formally, this means the real coefficients of the Grassmann algebra Ω_Λ of the previous section are replaced by smooth functions of x_i and y_i . To each vertex i we associate a formal supervector $u_i = (x_i, y_i, \xi_i, \eta_i, z_i)$, where x_i and y_i are commuting, ξ_i and η_i are generators of a Grassmann algebra, and

$$z_i = \sqrt{1 + x_i^2 + y_i^2 - 2\xi_i\eta_i} = \sqrt{1 + x_i^2 + y_i^2} - \frac{\xi_i\eta_i}{\sqrt{1 + x_i^2 + y_i^2}}. \tag{2.8}$$

As for $\mathbb{H}^{0|2}$, smooth functions of commuting elements of this algebra are defined by Taylor expansion, with the expansion now performed about $(x_i, y_i) \in \mathbb{R}^{2\Lambda}$; the second equality of (2.8) is an example.

The definition (2.8) ensures that z_i has positive degree zero part, and that $u_i \cdot u_i = -1$ for the super inner product $u_i \cdot u_j = x_i x_j + y_i y_j - \xi_i \eta_j - \xi_j \eta_i - z_i z_j$. As previously, we define

$$H_{\beta,h}(u) = \frac{\beta}{2} \sum_{ij \in E} (u_i - u_j) \cdot (u_i - u_j) + h \sum_{i \in \Lambda} z_i, \tag{2.9}$$

and the associated expectation

$$\langle F \rangle_{\beta,h} = \frac{1}{(2\pi)^{|\Lambda|}} \int \left(\prod_{i \in \Lambda} dx_i dy_i \partial_{\eta_i} \partial_{\xi_i} \frac{1}{z_i} \right) F e^{-H_{\beta,h}}. \quad (2.10)$$

This integral combines ordinary integration and Grassmann integration and is an instance of the Berezin integral, sometimes called a superintegral [15]. One computes the Grassmann integral to obtain the top coefficient of the element of the Grassmann algebra; this is a smooth function on $\mathbb{R}^{2\Lambda}$. One then computes the ordinary Lebesgue integral of this function. Again the generalisation to edge- and vertex-dependent weights and pinning is straightforward.

Note that (2.10) does not have a normalising factor as in the definitions of the \mathbb{H}^2 and $\mathbb{H}^{0|2}$ models, aside from the factor $(2\pi)^{-|\Lambda|}$ that does not depend on the weights. Nonetheless, the expectation is normalised: $\langle 1 \rangle_{\beta,h} = 1$ if $h > 0$. This is due to an internal supersymmetry in the model, which implies $Z_{\beta,h} = (2\pi)^{|\Lambda|}$. More generally, this supersymmetry implies a powerful localisation principle first used in this context in [34].

Theorem 2.1 (SUSY localisation for $\mathbb{H}^{2|2}$). *For $F: \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda \times \Lambda} \rightarrow \mathbb{R}$ smooth and with sufficient decay, and for all edge- and vertex-dependent weights $\beta = (\beta_{ij})$ and $h = (h_i)$ with some $h_i > 0$,*

$$\langle F((z_i), (u_i \cdot u_j)) \rangle_{\beta,h} = F(1, -1). \quad (2.11)$$

On the right-hand side of (2.11), 1 stands for the vector in \mathbb{R}^Λ with all entries equal to 1, and -1 stands for the $|\Lambda| \times |\Lambda|$ matrix with all entries -1 . For example, $\langle z_i \rangle_{\beta,h} = 1$ and $\langle u_i \cdot u_j \rangle_{\beta,h} = -1$.

Beyond: $\mathbb{H}^{n|2m}$. There is a natural generalisation of the above models to the broader class of $\mathbb{H}^{n|2m}$ models with $n + 1$ commuting coordinates and $2m$ anticommuting coordinates. Generalising Theorem 2.1, there is an exact correspondence between observables of the $\mathbb{H}^{n|2m}$ and $\mathbb{H}^{n+2|2m+2}$ models, see [11, SECTION 2]. For developments when $n = 0$, see [25].

Symmetries. The $\mathbb{H}^{n|2m}$ models have continuous symmetries which are analogues of the rotations of the $O(n)$ models. For example, for the hyperbolic plane \mathbb{H}^2 , these symmetries are Lorentz boosts and rotations. The infinitesimal generator of Lorentz boosts in the xz -plane is the linear differential operator T acting as

$$Tz = x, \quad Tx = z, \quad Ty = 0. \quad (2.12)$$

If \mathbb{H}^2 is parametrised by $(x, y) \in \mathbb{R}^2$, then $T = z\partial_x$. For the hyperbolic sigma models, there is an infinitesimal boost $T_i = z_i\partial_{x_i}$ at each vertex i . Haar measure on \mathbb{H}^2 and the action $H_{\beta,0}$ with $h = 0$ are invariant under these symmetries, i.e. $\sum_i T_i H_{\beta,0} = 0$. Analogous symmetries exist for $\mathbb{H}^{0|2}$ and $\mathbb{H}^{2|2}$. If $h > 0$ then $\sum_i T_i H_{\beta,h} \neq 0$, and the symmetries are said to be explicitly broken by the external field. Important consequences of these symmetries are Ward identities. For example, when $n > 0$ (such as for the \mathbb{H}^2 and $\mathbb{H}^{2|2}$ models), for $h > 0$,

$$\frac{\langle z_i \rangle_{\beta,h}}{h} = \sum_j \langle x_i x_j \rangle_{\beta,h}, \quad (2.13)$$

and when $m > 0$ (such as for the $\mathbb{H}^{2|2}$ and $\mathbb{H}^{0|2}$ models),

$$\frac{\langle z_i \rangle_{\beta,h}}{h} = \sum_j \langle \xi_i \eta_j \rangle_{\beta,h}. \tag{2.14}$$

Here x_i and (ξ_i, η_i) stand for an even (bosonic) coordinate and pair of odd (fermionic) coordinates when $n, m > 0$, respectively. The proofs of these identities boil down to integration by parts; see, e.g. [34, APPENDIX B] or [11, LEMMA 2.3].

Spontaneous symmetry breaking. For the \mathbb{H}^2 and $\mathbb{H}^{2|2}$ models on a fixed finite graph, it is a consequence of the non-compactness of the hyperbolic symmetry that, for example, $\langle x_0^2 \rangle_{\beta,h}$ diverges as $h \downarrow 0$. Similarly, for the $\mathbb{H}^{0|2}$ model on a finite graph, symmetry implies that $\langle z_0 \rangle_{\beta,h}$ tends to 0 as $h \downarrow 0$. One of the main questions of statistical physics is whether a symmetry survives in the infinite volume limit, or if it is *spontaneously broken*. To make this precise, it is convenient to consider a finite volume criterion for this question. Consider a sequence of finite graphs Λ that approximate \mathbb{Z}^d in a suitable way (denoted $\Lambda \rightarrow \mathbb{Z}^d$), and let $\langle \cdot \rangle_{\beta,h}$ be the corresponding finite volume expectations. For the \mathbb{H}^2 and $\mathbb{H}^{2|2}$ models, there is spontaneous symmetry breaking (SSB) for a given β if

$$\lim_{h \downarrow 0} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle x_0^2 \rangle_{\beta,h} < \infty, \tag{2.15}$$

and similarly for the $\mathbb{H}^{0|2}$ model there is SSB if

$$\lim_{h \downarrow 0} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle z_0 \rangle_{\beta,h} > 0. \tag{2.16}$$

These notions can be understood by noticing that when the two limits are exchanged the inequalities do not hold: in finite volume the $h = 0$ symmetries are restored in the $h \downarrow 0$ limit, while they are not in infinite volume if SSB occurs. There are other notions of SSB for hyperbolic spin models, but those in (2.15)–(2.16) capture the relevant phenomena from the perspective of the Anderson transition [34, SECTION 4.2], as well as from the perspective of the associated probabilistic models, as will be discussed in Sections 4 and 5.

We briefly summarise when SSB occurs for the \mathbb{H}^2 , $\mathbb{H}^{0|2}$, and $\mathbb{H}^{2|2}$ models. In $d = 2$, (2.15) and (2.16) do not hold for any $\beta > 0$. These results are versions of the Mermin–Wagner theorem [12, 50, 56, 57, 64]. The situation is different in $d \geq 3$. For the \mathbb{H}^2 model, SSB occurs for any $\beta > 0$ as a result of convexity [49]. The $\mathbb{H}^{2|2}$ and $\mathbb{H}^{0|2}$ models, however, have phase transitions: SSB in the form of (2.15) and (2.16), respectively, occurs in $d \geq 3$ if and only if β is sufficiently large [10, 34]. Once SSB is known to occur (or not), it is interesting and physically relevant to ask more precise questions, e.g. about the asymptotics of the correlation functions $\langle x_i x_j \rangle_{\beta,h}$. Sections 4 and 5 will discuss SSB and sharper questions for the $\mathbb{H}^{2|2}$ and $\mathbb{H}^{0|2}$ models.

Horospherical coordinates. An important tool for the study of the above models are horospherical coordinates for the superspaces $\mathbb{H}^{n|2m}$ with $n \geq 2$ [33, 34]. For the hyperbolic plane \mathbb{H}^2 these are coordinates $(t, s) \in \mathbb{R}^2$ such that

$$x = \sinh(t) - \frac{1}{2}e^t |s|^2, \quad y = e^t s, \quad z = \cosh(t) + \frac{1}{2}e^t |s|^2. \tag{2.17}$$

For the space \mathbb{H}^n , these coordinates generalise by taking $s = (s^i) \in \mathbb{R}^{n-1}$. For the superspaces $\mathbb{H}^{n|2m}$, in addition there are m pairs Grassmann coordinates $\psi = (\psi^i), \bar{\psi} = (\bar{\psi}^i)$ such that

$$\begin{aligned} x &= \sinh(t) - \frac{1}{2}e^t |s|^2 - e^t \psi \bar{\psi}, & y &= e^t s, & \xi &= e^t \psi, & \eta &= e^t \bar{\psi}, \\ z &= \cosh(t) + \frac{1}{2}e^t |s|^2 + e^t \psi \bar{\psi}, \end{aligned} \tag{2.18}$$

where we are using the abbreviation $\psi \bar{\psi} = \sum_{i=1}^m \psi^i \bar{\psi}^i$ if there are m Grassmann components. In these coordinates the action becomes

$$\begin{aligned} H_{\beta,h} &= \beta \sum_{ij} \left((\cosh(t_i - t_j) - 1) + \frac{1}{2}e^{t_i+t_j} |s_i - s_j|^2 + e^{t_i+t_j} (\psi_i - \psi_j)(\bar{\psi}_i - \bar{\psi}_j) \right) \\ &+ h \sum_i \left((\cosh(t_i) - 1) + \frac{1}{2}e^{t_i} |s_i|^2 + e^{t_i} \psi_i \bar{\psi}_i \right), \end{aligned} \tag{2.19}$$

and the hyperbolic reference measure is $dt ds \partial_{\bar{\psi}} \partial_{\psi} e^{(n-2m-1)\sum_i t_i}$, where $\partial_{\bar{\psi}} \partial_{\psi}$ denotes Grassmann integration if $m > 0$. A crucial feature of (2.19) is that the s and $\psi, \bar{\psi}$ variables appear quadratically in $H_{\beta,h}$ and hence can be integrated out via exact Gaussian computations. The t -marginal is thus proportional to the *positive* measure $e^{-\tilde{H}_{\beta,h}} dt$ where

$$\begin{aligned} \tilde{H}_{\beta,h}(t) &= \beta \sum_{ij} (\cosh(t_i - t_j) - 1) + h \sum_i (\cosh(t_i) - 1) \\ &+ \frac{n - 2m - 1}{2} \left(\log \det(-\Delta_{\beta(t)} + h(t)) - 2 \sum_i t_i \right). \end{aligned} \tag{2.20}$$

In (2.20), $-\Delta_{\beta(t)} + h(t)$ is the t -dependent matrix acting as

$$(-\Delta_{\beta(t)} f + h(t) f)_i = - \sum_{j \sim i} \beta e^{t_i+t_j} (f_j - f_i) + h e^{t_i} f_i. \tag{2.21}$$

The t -dependent weights $\beta_{ij}(t) = \beta e^{t_i+t_j}$ and $h_i(t) = h e^{t_i}$ generalise immediately to edge- and vertex-dependent reference weights. The determinant in (2.20) arises from the Gaussian integration over the s and $\psi, \bar{\psi}$ variables. Since the t -field is distributed according to a positive measure, one can use standard tools from analysis. This is useful since, e.g. for all $\mathbb{H}^{n|2m}$ models,

$$\langle z_i \rangle_{\beta,h} = \langle z_i + x_i \rangle_{\beta,h} = \langle e^{t_i} \rangle_{\beta,h}, \tag{2.22}$$

where the first identity used that $\langle x_i \rangle_{\beta,h} = 0$, by symmetry. For the pinned expectations, analogous representations hold with $h = 0$ and $t_0 = 0$.

3. PHYSICAL BACKGROUND: ANDERSON TRANSITION

This section briefly discusses the origins of hyperbolic spin systems as simplified models for the Anderson delocalisation–localisation transition. For a more detailed survey about this, we refer in particular to [71]. Further excellent surveys include [70] and [38, 58] for a physics perspective. For general background on the Anderson transition, see [1].

Consider a random matrix $H = (H(i, j))_{i, j \in \Lambda}$ such as the Anderson Hamiltonian $H = H_\beta = -\beta\Delta + V$ where $V = (V_i)_{i \in \Lambda}$ is an i.i.d. Gaussian potential, Λ is a discrete torus approximating \mathbb{Z}^d , and Δ is the lattice Laplacian on Λ . The fundamental question is to determine whether or not the spectrum of H contains an absolutely continuous part in the infinite volume limit, and very closely related to this, if the eigenfunctions (often called states in this context) of H are extended or localised. Extended states correspond to a metallic phase while localised states correspond to an insulating phase. To discuss this further, define the two-point correlation function

$$\tau_{\beta, E, h}(j, k) = \mathbb{E} |(H_\beta - E - ih)^{-1}(j, k)|^2, \quad j, k \in \Lambda, \quad (3.1)$$

where $i = \sqrt{-1}$. The existence of extended states for energies near E is essentially implied by $\lim_{h \downarrow 0} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \tau_{\beta, E, h}(j, j) < \infty$. For the Anderson model, it is a long-standing conjecture that this occurs in $d \geq 3$ for E inside the spectrum of $-\beta\Delta$ when β is sufficiently large. In the same setting, the more precise quantum diffusion conjecture asserts

$$\lim_{h \downarrow 0} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \tau_{\beta, E, h}(j, k) \approx D(E, \beta)(-\Delta)^{-1}(j, k) \sim C(E, \beta)|j - k|^{-(d-2)}, \quad j, k \in \mathbb{Z}^d \quad (3.2)$$

for some constants C, D , and where the asymptotics hold for $|j - k| \rightarrow \infty$. This gives a hint that the conjecture might be difficult: the two-point function decays slowly, like that of the massless Gaussian free field. Such behaviour also occurs for fluctuations of spontaneously broken continuous symmetries (Goldstone modes). In [38, 74] it was argued that the origin of extended states is the existence of SSB for a (complicated) spin model with hyperbolic symmetry, and that quantum diffusion is exactly the associated Goldstone mode. The spin model is based on the supersymmetric approach to the replica trick for computing the two-point function.

We briefly indicate some parallels between the present discussion and Section 2. The elementary identity

$$\frac{1}{h} \mathbb{E} \operatorname{Im}(H_\beta - E - ih)^{-1}(j, j) = \sum_k \mathbb{E} |(H_\beta - E - ih)^{-1}(j, k)|^2, \quad (3.3)$$

which is also valid without expectations, is analogous to the Ward identities (2.13)–(2.14). Thus the role of $\langle z_j \rangle$ is played by $\mathbb{E} \operatorname{Im}(H_\beta - E - ih)^{-1}(j, j)$. In the limit $h \downarrow 0$ this is π times the density of states $\rho(E)$, i.e. the asymptotic eigenvalue distribution. The role of the two-point functions $\langle x_j x_k \rangle$ or $\langle \xi_j \eta_k \rangle$ is played by $\tau_{\beta, E, h}(j, k) = \mathbb{E} |(H_\beta - E - ih)^{-1}(j, k)|^2$. The absolute values in the latter correlation function are essential and the origin of the hyperbolic symmetry [71, SECTION 2.3]. The non-compactness of the hyperbolic symmetry manifests itself in the high temperature phase: the unboundedness of $\tau_{\beta, E, h}(j, k)$ as $h \downarrow 0$ signals an absence of delocalisation. The stronger notion of localisation corresponds to

$$\tau_{\beta, E, h}(j, k) \approx \frac{e^{-c|j-k|}}{h}. \quad (3.4)$$

The divergence as $h \downarrow 0$ is analogous to the behaviour of the $\mathbb{H}^{2|2}$ model, see Section 2, and is different from that of spin systems with compact symmetry. For further discussion, see [71].

Dictionary. The analogies between the expected behaviours of the Anderson model and the probabilistic models described in the next two sections are summarised below. In $d = 2$, $\beta_c = \infty$, while $\beta_c < \infty$ for $d \geq 3$.

	$\beta < \beta_c$	$\beta > \beta_c$
Anderson Model	localised (insulating) phase	extended (metallic) phase
VRJP	positive recurrent phase	transient phase
Arboreal gas	subcritical percolation phase	percolating phase

Logically, there is the possibility of non-extended states that are not localised, which would correspond to a null-recurrent phase for the VRJP and a phase of the arboreal gas where infinite clusters do not occur, but the cluster size distribution has infinite mean.

4. LINEARLY REINFORCED WALKS AND $\mathbb{H}^{2|2}$

Formulas arising in the study of the $\mathbb{H}^{2|2}$ model (e.g. (2.21)) have interpretations in terms of random walks, and similarities with ERRW did not go unnoticed [34, SECTION 1.5]. This was given an explanation by Sabot and Tarrès [65]; the explanation passes through another reinforced random walk, which we now introduce. Fix edge weights $\beta_{ij} > 0$ for each edge $ij \in E$, and set $\beta_{ij} = 0$ if $ij \notin E$. The *vertex-reinforced jump process (VRJP)* with $X_0 = 0$ is the continuous-time self-interacting random walk with transition probabilities

$$\mathbb{P}_0^{\text{VRJP}(\beta)}[X_{t+dt} = j | (X_s)_{s \leq t}, X_t = i] = \beta_{ij} L_t^j, \quad L_t^j = 1 + \int_0^t 1_{X_s=j} ds. \quad (4.1)$$

The quantity L_t^j is the *local time* at j at time t , up to the shift by 1. In words, then, conditionally on the shifted local times at time t and that $X_t = i$, a VRJP jumps to site j with probability proportional to $\beta_{ij} L_t^j$. Thus previously vertices visited are preferred. The amount of local time accrued at i before jumping away has the distribution of an exponential random variable with rate $\sum_j \beta_{ij} L_t^j$. With this in mind, large edge weights β_{ij} heuristically correspond to weak reinforcement: jumps occur quickly and do not alter the local time profile too much.

Sabot and Tarrès gave an exact formula for the (properly scaled) limiting local times of the VRJP, and explained that this distribution is also the distribution of the t -field of the $\mathbb{H}^{2|2}$ model. They further showed that the magic formula for the ERRW follows from this result, see Section 4.4 below. Similarly to the ERRW, the VRJP can be expressed as a continuous-time random walk in a random environment. The next theorem is a slightly informal statement of this result. The precise formulation requires looking at the VRJP in the correct time parameterisation; see [65]. For a symmetric square matrix A with $\sum_j A_{ij} = 0$ for all rows i , we write $\det^0(A)$ for the value of any principal cofactor of A , e.g. the determinant with the first row and column of A removed.

Theorem 4.1 (Magic formula for VJRP [65]). Let $G = (\Lambda, E)$ be a finite graph with $|\Lambda| = N$. In the exchangeable time parameterisation of the VRJP,

$$\mathbb{P}_0^{\text{VJRP}(\beta)}[\cdot] = (2\pi)^{-\frac{N-1}{2}} \int_{\mathbb{R}^{\Lambda \setminus 0}} \mathbb{P}_0^{\text{SRW}(c(t))}[\cdot] e^{-\frac{\beta}{2} \sum_{i,j} \cosh(t_i - t_j)} (\det^0(-\Delta_{\beta(t)}))^{\frac{1}{2}} \prod_{k \in \Lambda \setminus 0} e^{-t_k} dt_k, \quad (4.2)$$

where $\mathbb{P}_0^{\text{SRW}(c(t))}$ is the distribution of a continuous-time simple random walk with conductances $c(t)_{ij} = \beta e^{t_i + t_j}$ started at 0.

The measure on the right-hand side of (4.2) is exactly the horospherical t -marginal of the $\mathbb{H}^{2|2}$ model (with $h = 0$ and pinned at 0). The existence of a phase transition between a transient and a recurrent phase of the VRJP on \mathbb{Z}^d for $d \geq 3$ now essentially follows from the following earlier results for the $\mathbb{H}^{2|2}$ model (and extensions to the pinned model):

Theorem 4.2 (SSB for $\mathbb{H}^{2|2}$ [34]). Let $d \geq 3$ and $\beta \geq \beta_1$. There exists $C_\beta > 0$ such that

$$\lim_{h \downarrow 0} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle \cosh(t_i)^8 \rangle_{\beta, h} \leq C_\beta. \quad (4.3)$$

Similar statements hold for other observables and for the pinned model.

Theorem 4.3 (Localisation for $\mathbb{H}^{2|2}$ [33]). Let $d \geq 1$ and $\beta \leq \beta_0$. There exist $C_\beta, c_\beta > 0$ such that

$$\langle x_i x_j \rangle_{\beta, h} \leq \frac{C_\beta}{h} e^{-c_\beta |i-j|}. \quad (4.4)$$

Similar statements hold for other observables and for the pinned model.

The existence of a recurrent phase for small β has also been proved more directly from the definition of the VRJP [4]. A proof of transience that only uses the random walk point of view seems challenging, and would be of interest.

4.1. Hyperbolic symmetry and the VRJP

A more direct and general connection between hyperbolic spin systems and the VRJP was found later [12]. Towards this, observe (as was already done in [65]) that the joint process (X_t, L_t) of the VRJP and its local time is a Markov process, where $L_t = (L_t^j)_{j \in V}$. The infinitesimal generator \mathcal{L} of the joint process acts on $g: V \times [0, \infty)^V \rightarrow \mathbb{R}$ by

$$\mathcal{L}g(i, \ell) = \sum_j \beta_{ij} \ell_j (g(j, \ell) - g(i, \ell)) + \frac{\partial g(i, \ell)}{\partial \ell_i}. \quad (4.5)$$

Write $\mathbb{E}_i^{\text{VRJP}(\beta, \ell)}$ for the expectation of the joint process with initial vertex i and local times $\ell = (\ell_i)_{i \in \Lambda}$. The definition (4.1) corresponds to $\ell_i = 1$ for all i .

To connect the VRJP to hyperbolic symmetry, consider the \mathbb{H}^2 model, for example, and recall the infinitesimal generator T_i of Lorentz boosts in the $x_i z_i$ -plane acting at vertex i from (2.12). Then under mild hypotheses on G , integration by parts and (2.12) yield

$$-\sum_j \int_{(\mathbb{H}^2)^\Lambda} (\mathcal{L}G(j, z))_{x_i x_j} e^{-H_{\beta, 0}(u)} \prod_{k \in \Lambda} du_k = \int_{(\mathbb{H}^2)^\Lambda} (T_i x_i) G(j, z) e^{-H_{\beta, 0}(u)} \prod_{k \in \Lambda} du_k. \quad (4.6)$$

Thus boosts are adjoint to the generator of the VJRP. A consequence is the next theorem.

Theorem 4.4. *Consider the \mathbb{H}^2 model. If $F: \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ decays fast enough, then*

$$\langle x_i x_j F(z) \rangle_{\beta,0} = \left\langle z_i \int_0^\infty dt \mathbb{E}_i^{\text{VRJP}(\beta,z)} [F(L_t) 1_{X_t=j}] \right\rangle_\beta. \quad (4.7)$$

Sketch of proof. Normalise (4.6) and choose $G(j, \ell) = G_t(j, \ell) = \mathbb{E}_j^{\text{VRJP}(\beta,\ell)} F(L_t)$. Since (X_t, L_t) is a Markov process with generator \mathcal{L} , we have $\mathcal{L}G_t = \partial_t G_t$. Integrating the resulting identity over t in $(0, \infty)$ gives the result. ■

Theorem 4.4 shows that \mathbb{H}^2 quantities can be computed in terms of the averages of VRJP quantities, the average being over the initial local time of the VRJP. This average is inconvenient for studying the VRJP itself. The computations above, however, immediately generalise to other hyperbolic spin models. For the $\mathbb{H}^{2|2}$ model, one can in addition use Theorem 2.1 to exactly compute the undesirable average. The result is the following theorem.

Theorem 4.5. *Consider the $\mathbb{H}^{2|2}$ model. Then for any $F: \mathbb{R}^\Lambda \rightarrow \mathbb{R}$ that decays fast enough,*

$$\langle x_i x_j F(z) \rangle_{\beta,0} = \int_0^\infty dt \mathbb{E}_i^{\text{VRJP}(\beta)} [F(L_t) 1_{X_t=j}]. \quad (4.8)$$

In particular, $\langle x_i^2 \rangle_{\beta,h}$ is the expected time the VRJP started from i spends at i when killed at rate $h > 0$. This relation can be used to prove the VRJP is recurrent in two dimensions, irrespective of the reinforcement strength $\beta > 0$, by proving a Mermin–Wagner theorem for the $\mathbb{H}^{2|2}$ model [12]. Informally, Mermin–Wagner theorems assert that continuous symmetries cannot be spontaneously broken in $d = 1, 2$. As discussed earlier, for the $\mathbb{H}^{2|2}$ model SSB corresponds to a finite variance, i.e. transience.

Isomorphism theorems. Theorems 4.4 and 4.5 are examples of *isomorphism theorems*, meaning identities relating the local time field of a stochastic process to a spin system. The first example of such a result related simple random walk to the Gaussian free field and was obtained by Brydges, Fröhlich, and Spencer [19]. They were inspired by Symanzik [73]. The formulation as a distributional identity is due to Dynkin [36]; sometimes the result is called the BFS–Dynkin isomorphism. A host of other isomorphism theorems have been found in Gaussian settings, see [53]. Other isomorphism theorems for the VRJP can be obtained by the approach above, and it is possible to obtain Theorem 4.1 in this way, see [13]. Isomorphisms for the VRJP can also be obtained by expressing the VRJP as a mixture of Markov processes and using isomorphism theorems for the Markov processes; see [23].

4.2. Random Schrödinger representation and STZ field

In [33], it was observed that after conjugation by the diagonal matrix $e^{-t} = (e^{-t_i})_i$, the matrix $-\Delta_{\beta(t)} + h(t)$ in (2.21) becomes a Schrödinger operator with t -dependent potential:

$$e^{-t} \circ (-\Delta_{\beta(t)} + h(t)) \circ e^{-t} = -\Delta_\beta + V(t), \quad V_i(t) = \sum_j \beta_{ij} (e^{t_j - t_i} - 1) + h_i e^{-t_i}. \quad (4.9)$$

This point of view led to the proof of Theorem 4.3. It was later recognised that this random Schrödinger point of view can be used to obtain a powerful representation of the t -field [67]. For the pinned $\mathbb{H}^{2|2}$ model with $h = 0$ and $t_0 = 0$, the t -field measure (2.20) can be written in terms of $-\Delta_\beta + V(t)$ using that

$$e^{-\tilde{H}_\beta(t)} = e^{-\frac{1}{2} \sum_i V_i(t)} (\det(-\Delta_\beta + V(t)))^{1/2}. \quad (4.10)$$

This suggests it might be useful to change variables from t to $V(t)$. This change of variables is not directly well-defined when $t_0 = 0$ since the set of V such that $(-\Delta_\beta + V)$ is positive definite is $|\Lambda|$ -dimensional. This can be sidestepped by treating V as the fundamental variable, i.e. considering

$$e^{-\frac{1}{2} \sum_i V_i} (\det(-\Delta_\beta + V))^{-1/2} \mathbf{1}(-\Delta_\beta + V \text{ is positive definite}) dV. \quad (4.11)$$

The random vector $B_i = \frac{1}{2}(V_i + \sum_j \beta_{ij})$ is often called the ‘ β -field,’ but since we use β for edge weights (inverse temperature), we will denote it by B instead and call it the *STZ field*.

Theorem 4.6. *The Laplace transform of the STZ field is given by*

$$\mathbb{E}e^{-\langle \lambda, B \rangle} = \prod_i \frac{1}{(\lambda_i + 1)^{1/2}} \prod_{ij} e^{-\beta_{ij}(\sqrt{\lambda_i + 1} \sqrt{\lambda_j + 1} - 1)}. \quad (4.12)$$

Moreover, the t -field (pinned at any vertex) can be recovered in distribution from B .

In particular, the theorem implies the STZ field is 1-dependent. In [68], this remarkable property of the STZ field was used to construct an infinite volume version on \mathbb{Z}^d , and applied to characterise transience and recurrence of the VRJP in terms of a 0/1 law.

4.3. Phase diagram of the VRJP

The most basic qualitative question one can ask about the VRJP is whether it is recurrent or transient for a given reinforcement strength $\beta > 0$. This may in principle depend on the precise notion of recurrence used, as the VRJP is non-Markovian. As discussed above, for $d \geq 3$ the existence of a phase in which the VRJP is almost surely recurrent was established in [4, 65], and an almost surely transient phase in [65]. For $d = 2$, recurrence for all $\beta > 0$ in the sense of infinite expected local time at the initial vertex was established in [12]. Proofs of almost sure recurrence followed shortly [58, 64]. Similar results had previously been established for the ERW [12, 56, 68].

The qualitative behaviour of the VRJP is almost completely understood on \mathbb{Z}^d due to the following remarkable correlation inequality of Poudevigne.

Theorem 4.7. *For the $\mathbb{H}^{2|2}$ model and any convex function f , the expectation $\langle f(e^{t_j}) \rangle_\beta^0$ is increasing in all weights $\beta = (\beta_{ij})$.*

The proof of Theorem 4.7 relies on the STZ field [62]. This inequality implies that transience is a monotone property with respect to the constant initial reinforcement parameter β . Combined with the results of the previous paragraph, this implies that the VRJP has a sharp transition from almost sure recurrence to almost sure transience on \mathbb{Z}^d for con-

stant β : recurrence for $\beta < \beta_c(d)$ and transience for $\beta > \beta_c(d)$. The behaviour at β_c is open. Poudevigne's correlation inequality also leads to a proof of recurrence in $d = 2$.

4.4. Further discussion

Back to edge-reinforced random walk. The connection of the ERRW to the $\mathbb{H}^{2|2}$ model is somewhat less direct than for the VRJP: it turns out that the ERRW is an average of VRJPs [65]. Somewhat more precisely, ERRW with initial edge weights α can be obtained from the VRJP with initial edges weights β if the β_{ij} are chosen to be independent Gamma random variables with mean α_{ij} . While this additional randomness presents some difficulties, the existence of a transient phase for the ERRW in $d \geq 3$ was obtained by similar methods to that of the VRJP [32]. In terms of the spin model, the Gamma-distributed random edge weights correspond to replacing the exponential $e^{\sum_{ij} \beta_{ij}(u_i \cdot u_j + 1)}$ by $\prod_{ij} (-u_i \cdot u_j)^{-\alpha_{ij}}$ in the (super)measure. Such a product weight is often called a Nienhuis interaction.

Interestingly, the recurrence of the ERRW in two dimensions was obtained before the recurrence of the VRJP. This was possible due to insights of Merkl and Rolles, who directly proved a Mermin–Wagner-type theorem for the ERRW by making use of the magic formula [57]. Merkl and Rolles were able to conclude recurrence of the ERRW on \mathbb{Z}^2 for strong reinforcement if each edge of the lattice was replaced by a long path. Sabot and Zeng's proved recurrence on \mathbb{Z}^2 for all reinforcement strengths by obtaining a characterisation of recurrence in terms of the STZ field [68], and showing that an estimate from [57] implies recurrence. The ergodic properties of the STZ field play a crucial role in this argument.

Beyond \mathbb{Z}^d . There are also results for the VRJP beyond \mathbb{Z}^d . The existence of a transition on trees was proven in [27], and on non-amenable graphs in [4]. A fairly complete understanding on trees has been obtained, see [7] and references therein.

Future directions. There remain many open questions. What is the critical behaviour of the VRJP and the $\mathbb{H}^{2|2}$ model on \mathbb{Z}^d , $d \geq 3$? Is there an upper critical dimension? For $d = 3$ aspects of this question were studied numerically in [35], and evidence was found for the existence of a multifractal structure in the $\mathbb{H}^{2|2}$ model. Multifractal structure is also expected near the Anderson transition for random Schrödinger operators. For the regular tree (Bethe lattice), further remarkable critical behaviour was observed, in part numerically, in [75]. This reference concerns a more complicated sigma model, but the main predictions also apply to the $\mathbb{H}^{2|2}$ model [35]. On \mathbb{Z}^2 the VRJP is believed to be *positive* recurrent, i.e. exponentially localised, but this important conjecture about the $\mathbb{H}^{2|2}$ model and the VRJP remains open. The heuristic for positive recurrence is based on the (marginal) renormalisation group flow and goes along with the prediction of asymptotic freedom at short distances [34, SECTION 4.3]. Analogous predictions based on similar heuristics exist for the $2d$ Heisenberg model, the $2d$ Anderson model, $4d$ non-abelian Yang–Mills theory, and the $2d$ arboreal gas (discussed below). Another question is to understand the VRJP in $d \geq 3$ with non-constant initial local times: Theorem 4.4 and results of [72] suggest the VRJP is always transient if started with initial local times given by the z -field of the \mathbb{H}^2 model. Understanding the properties of the z -field that destroy the phase transition would be interesting.

5. THE ARBOREAL GAS AND $\mathbb{H}^{0|2}$

The arboreal gas is the uniform measure on (unrooted spanning) forests of a weighted graph. More precisely, given an undirected graph $G = (\Lambda, E)$, a forest $F = (\Lambda, E(F))$ is an acyclic subgraph of G having the same vertex set as G . Given an edge weight $\beta > 0$ (inverse temperature) and a vertex weight $h \geq 0$ (external field), the probability of an edge set F under the arboreal gas measure is

$$\mathbb{P}_{\beta,h}[F] = \frac{1}{Z_{\beta,h}} \beta^{|E(F)|} \prod_{T \in F} (1 + h|V(T)|) \mathbf{1}(F \text{ is a forest}) \quad (5.1)$$

where $T \in F$ denotes that T is a tree in the forest F , i.e. a connected component of F . We write $\mathbb{P}_\beta = \mathbb{P}_{\beta,0}$. As for the VRJP, the generalisation to edge- and vertex-dependent weights $\beta = (\beta_{ij})$ and $h = (h_i)$ is straightforward, and is sometimes useful.

The arboreal gas arises naturally in the context of the q -state random cluster model (q -RCM), which we recall is the model defined by (5.1) by omitting the indicator function and instead weighting each component by a factor $q > 0$. In particular, $q = 1$ is Bernoulli bond percolation. On a finite graph, the arboreal gas edge weight β' is the limit of the q -RCM as $q, \beta \rightarrow 0$ such that $\beta/q \rightarrow \beta'$, and it is natural to think of the arboreal gas as the 0-RCM. The most fundamental question about the arboreal gas is whether or not it has a percolation phase transition. It is straightforward to establish a subcritical phase when β is small: the arboreal gas can be stochastically dominated by bond percolation [43, THEOREM 3.21], and for β small the domination is by subcritical percolation.

5.1. Phase transitions for the arboreal gas

The existence of a supercritical phase for the arboreal gas is a more subtle question than for the q -RCM with $q > 0$. One way to see this subtlety is based on symmetries. To discuss this, recall that for $q \in \{2, 3, \dots\}$ there is a connection between the q -RCM and the q -state Potts model [40]. In particular, spin–spin correlations in the q -state Potts model are equivalent to connection probabilities in the q -RCM. The results of [22, 45] extend this relationship to $q = 0$: the $\mathbb{H}^{0|2}$ model is a spin representation of the arboreal gas.

Theorem 5.1. *Let $\langle \cdot \rangle_\beta$ and \mathbb{P}_β denote the $\mathbb{H}^{0|2}$ and arboreal gas measures on a finite graph. For vertices $i, j \in \Lambda$,*

$$\mathbb{P}_\beta[i \leftrightarrow j] = -\langle u_i \cdot u_j \rangle_\beta. \quad (5.2)$$

Moreover, the partition functions of the $\mathbb{H}^{0|2}$ model and the arboreal gas coincide.

Strictly speaking, the $\mathbb{H}^{0|2}$ formulation of Theorem 5.1 first occurred in [11] as a reformulation of [22, 45]; the hyperbolic point of view plays an important role in the proof of Theorem 5.3 below. Theorem 5.1 suggests the existence of a supercritical phase for the arboreal gas may depend on the dimension, as strong connection probabilities corresponds to a symmetry breaking phase transition for the $\mathbb{H}^{0|2}$ model. Unlike the q -Potts models with $q \in \{2, 3, \dots\}$, this model possesses a continuous symmetry, so one might expect a Mermin–Wagner theorem to prevent such a transition in $d = 2$. This is indeed true:

Theorem 5.2 ([11, THEOREM 1.3]). *Let $d = 2$. For any $\beta > 0$, there exists $c(\beta) > 0$ such that $\mathbb{P}_\beta^\Lambda[0 \leftrightarrow j] \leq |j|^{-c(\beta)}$ for any $\Lambda \subset \mathbb{Z}^2$.*

It is possible to predict Theorem 5.2 without knowing about the $\mathbb{H}^{0|2}$ spin representation as follows [28]. The critical value of β for the q -RCM on \mathbb{Z}^2 with $q \geq 1$ is known to be $\beta_c(q) = \sqrt{q}$ [14], and this self-dual point is predicted to be the critical point for all $q > 0$. Since the arboreal gas $\mathbb{P}_{\beta'}$ is the limit of the q -RCM with $\beta = \beta'q$, if the location of the critical point is continuous as $q \downarrow 0$, it follows that $\beta_c(0) = \infty$. These heuristics support the conjecture that connection probabilities of the $2d$ arboreal gas decay exponentially for any $\beta > 0$. Independent support for this conjecture can be obtained by renormalisation group heuristics, almost exactly as for the $2d$ VRJP [22].

Turning the preceding paragraph into a rigorous proof would be very interesting. It would also be interesting to have a probabilistic proof (in terms of forests) that the arboreal gas does not have a phase transition on \mathbb{Z}^2 . The proof of Theorem 5.2 given in [11] follows different lines. A key step is the following, which reduces the proof to an adaptation of [64, THEOREM 1].

Theorem 5.3 (Magic formula for arboreal gas). *Let $G = (\Lambda, E)$ be a finite connected graph. For vertices $0, j \in \Lambda$,*

$$\mathbb{P}_\beta[0 \leftrightarrow j] = \frac{1}{Z_\beta} \int_{\mathbb{R}^{\Lambda \setminus 0}} e^{t_j} e^{-\frac{\beta}{2} \sum_{i,k} \cosh(t_i - t_k)} (\det^0(-\Delta_{\beta(t)}))^{3/2} \prod_{k \in \Lambda \setminus 0} e^{-3t_k} dt_k, \quad (5.3)$$

where $\det^0(-\Delta_{\beta(t)})$ denotes any principal cofactor of $-\Delta_{\beta(t)}$.

In outline, the proof of Theorem 5.3 consists of three steps: Theorem 5.1, rewriting the $\mathbb{H}^{0|2}$ expectation in terms of $\mathbb{H}^{2|4}$ by SUSY localisation, and then changing to horospherical coordinates and integrating out all but the t -field. The magic formula for the VRJP from Theorem 4.1 has a strikingly similar form, but with the two occurrences of $3s$ in (5.3) replaced by $1s$. This difference in powers is due to there being two additional Grassmann Gaussian integrals for $\mathbb{H}^{2|4}$ as compared to $\mathbb{H}^{2|2}$.

In three and more dimensions the arboreal gas does, however, undergo a percolation phase transition. To state a precise theorem, let $\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ denote a torus of side-length L^N with L large. The next theorem immediately implies that there is a macroscopic tree occupying most of the torus with large probability.

Theorem 5.4 ([10, THEOREM 1.1]). *Let $d \geq 3$. If β is sufficiently large, then there exists $\theta_d(\beta) = 1 - O(1/\beta)$, $D(\beta) > 0$, and $\kappa > 0$ such that*

$$\mathbb{P}_{\beta}^{\Lambda_N}[0 \leftrightarrow j] = \theta_d(\beta)^2 + D(\beta)(-\Delta)^{-1}(0, j) + O\left(\frac{1}{\beta|j|^{d-2+\kappa}}\right) + O\left(\frac{1}{\beta L^{\kappa N}}\right). \quad (5.4)$$

Similar asymptotics hold for other correlation functions.

The polynomial correction in Theorem 5.4 is the hallmark of critical behaviour in statistical mechanics, and is a manifestation of the Goldstone mode associated with the broken continuous symmetry of the $\mathbb{H}^{0|2}$ model at low temperatures. The proof of Theorem 5.4 relies essentially on the $\mathbb{H}^{0|2}$ representation (Theorem 5.1), and is based on a

combination of Ward identities and a renormalisation group analysis. The renormalisation group analysis is based in part on methods developed previously in different contexts, in particular [8, 9, 18, 20, 21].

5.2. Further discussion

In contrast to the VRJP, even the qualitative phase diagram of the arboreal gas remains incomplete: we do not know the existence of a β_c such that percolation occurs if $\beta > \beta_c$, and does not if $\beta < \beta_c$. It is also more difficult to discuss the arboreal gas directly in the infinite volume limit than the VRJP; the analogue of the STZ field does not have finite dependence and is less obviously useful, and useful correlation inequalities to this end remain conjectural, see below. Nonetheless, many open questions beckon.

Critical behaviour. There is strong evidence that the upper critical dimension of the arboreal gas is $d = 6$, just as for bond percolation, and that the critical behaviour is governed by more conventional critical behaviour as compared to the $\mathbb{H}^{2|2}$ model [28], see also [39, 48].

Comparison with percolation. Recall that the analogue of Theorem 5.4 for Bernoulli percolation has an *exponentially* decaying correction [24]. Informally, this means that supercritical percolation with the giant removed behaves like subcritical percolation. This can be given a more precise meaning in the simpler setting of the Erdős–Rényi random graph, i.e. Bernoulli percolation on the complete graph K_N , where it is known as the discrete duality principle [2, SECTION 10.5].

The polynomial correction in Theorem 5.4 shows that the arboreal gas does not satisfy a duality principle. Rather, its supercritical phase behaves like a critical model off the giant. This can again be given a more precise formulation on the complete graph K_N and on the wired regular tree, where detailed results are known [37, 51, 54, 63]. In particular, the exact cluster distribution can be determined: on K_N in the supercritical phase there is a unique giant tree, and an unbounded number of trees of size $\Theta(N^{2/3})$.

It is natural to predict that the macroscopic behaviour of the arboreal gas on the d -dimensional tori Λ_N with $d \geq 3$ is similar to that on the complete graph. In particular, one expects a unique giant tree. The next-order critical corrections can also be expected to be similar, at least when $d > 6$. In particular, the second biggest tree should then have size comparable to $|\Lambda_N|^{2/3}$. Similar results have been established for critical Bernoulli percolation in high dimensions, see [44, CHAPTER 13]. More ambitiously, we expect the order statistics of the rescaled cluster size distribution to be universal, i.e. the same as on the complete graph as determined in [51, 54]. This conjecture may be easier to explore in other settings first, e.g. on expanders, where a phase transition can be established by elementary methods [42].

Infinite-volume geometry and the UST. There is a large body of literature in probability theory concerning *uniform spanning forests* (USF), meaning weak infinite-volume limits of uniform spanning tree (UST) measures on finite graphs, see [52, CHAPTER 10]. To avoid confusion with the arboreal gas (sometimes also called the USF [46]), we will call these infinite-volume limits the UST on \mathbb{Z}^d . While the component structure of the UST on a finite

graph is not particularly interesting, the infinite volume limit is: Pemantle proved that there is a unique connected component on \mathbb{Z}^d for $d \leq 4$, and infinitely many connected components on \mathbb{Z}^d for $d > 4$ [59]. This happens as ‘long connections’ can be lost in the weak limit.

On a finite graph, the UST measure is the limit $\beta \rightarrow \infty$ of the arboreal gas with edge weights β , and it is natural to wonder if the arboreal gas at low temperatures $\beta \gg 1$ has similar properties to the UST in infinite volume. For global properties, this can evidently only happen when there is a percolation transition. We are therefore lead to ask: for $d \geq 3$ at low temperatures, is it the case that for $d = 3, 4$ the infinite-volume arboreal gas has a unique infinite tree, while for $d > 4$ there are infinitely many infinite trees? Is it the case that the infinite components of the arboreal gas are topologically one-ended, as for the UST?

Negative correlation. A key tool in studying the q -RCM with $q \geq 1$ is that it is *positively associated*: for increasing functions $f, g: \{0, 1\}^E \rightarrow \mathbb{R}$, the covariance of f and g is non-negative. This is a special case of the FKG inequality [41]. Positive association fails for $q < 1$ and for the arboreal gas. It is believed, but not known, that these models are in fact *negatively associated*: for $f, g: \{0, 1\}^E \rightarrow \mathbb{R}$ depending on disjoint sets of edges, the covariance of f and g is non-positive. Negative association is more subtle than positive association, and the development of flexible, yet powerful, theoretical frameworks is an active subject [3, 16, 17, 60]. While some of this theory applies to the arboreal gas, it remains open to prove even the special case of *negative correlation*: for distinct edges $e, f \in E$,

$$\mathbb{P}_\beta[e, f \in F] \leq \mathbb{P}_\beta[e \in F]\mathbb{P}_\beta[f \in F]. \tag{5.5}$$

Negative correlation for all weights is equivalent to all connection probabilities $\mathbb{P}_\beta[0 \leftrightarrow j] = \langle e^{t_j} \rangle_\beta^0$ being increasing in all weights $\beta = (\beta_{ij})$, where the right-hand side is in terms of the t -field of the pinned $\mathbb{H}^{0|2}$ model. The analogue for the $\mathbb{H}^{2|2}$ model is precisely Poudevigne’s inequality, Theorem 4.7. Does this inequality extend to other $\mathbb{H}^{n|2m}$ models?

6. CONCLUDING REMARKS

This survey has focused on the connections between hyperbolic spin systems and probabilistic models that share phenomenology with the Anderson transition, including a number of open questions. It is also worth repeating a question from [49]: are there other models of random walk that are related to spin systems? A partial answer was given in [13], but we expect there is more to be discovered; see, e.g. [66]. Similarly, one may search for probabilistic representations of $\mathbb{H}^{n|2m}$ models for values of n, m not discussed here.

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ROLAND BAUERSCHMIDT

DPMMS, University of Cambridge, Cambridge CB3 0WB, UK, rb812@cam.ac.uk

TYLER HELMUTH

Mathematical Sciences, Durham University, Durham DH1 3LE, UK,
tyler.helmuth@durham.ac.uk

