ENTROPY IN MEAN CURVATURE FLOW

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ABSTRACT

The entropy of a hypersurface is defined by the supremum over all Gaussian integrals with varying centers and scales, thus invariant under rigid motions and dilations. It measures geometric complexity and is motivated by the study of mean curvature flow. We will survey recent progress on conjectures of Colding–Ilmanen–Minicozzi–White concerning the sharp lower bound on entropy for hypersurfaces, as well as their extensions.

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1. INTRODUCTION

In the trailblazing work [34], Colding and Minicozzi define a notion of entropy for hypersurfaces which is given by the supremum over all Gaussian integrals with varying centers and scales (cf. [67]). It is a geometric quantity that measures complexity and is invariant under rigid motions and dilations. In this survey, we discuss recent results on geometric properties of hypersurfaces with low entropy.

Entropy is motivated by the study of mean curvature flow which is a natural analogue of the heat equation in extrinsic curvature flows. Any hypersurface evolves under mean curvature flow in the direction of steepest descent for area, and the flow in general may become singular even before its vanishing. By Huisken's monotonicity formula [53], the entropy is decreasing under mean curvature flow. Thus, the entropy at all future singularities for the flow is bounded from above by that of the initial hypersurface.

By the work of Huisken [53] and Ilmanen [57], all possible blowups at a given singularity for a mean curvature flow are modeled by self-shrinkers which are hypersurfaces that flow in a self-similarly shrinking manner. Despite the abundance of self-shrinkers (see [63,70,71,80]), Colding and Minicozzi [34] study the properties of entropy and prove a striking result that spheres, generalized cylinders, and hyperplanes are the only stable self-shrinkers under mean curvature flow.

Inspired in part by the dynamic approach to mean curvature flow of [34], Colding, Ilmanen, Minicozzi, and White [33] employ a perturbative argument and singularity analysis for mean curvature flow to show that the round sphere minimizes entropy among all closed (i.e., compact without boundary) self-shrinkers. They further conjecture that in dimension less than 7, the round sphere indeed minimizes entropy among all nonflat self-shrinkers and so does it among all closed hypersurfaces.

After reviewing basic properties of entropy in Section 2, we discuss, in Sections 3 and 4, recent progress towards the above conjectures of Colding–Ilmanen–Minicozzi–White, with an emphasis on joint work with Bernstein [10,11]. We conclude our discussion in Section 5 to explain various stability results for round spheres under small perturbation of entropy.

2. ENTROPY FOR HYPERSURFACES

In this section, we discuss related background on the Colding–Minicozzi entropy for hypersurfaces, with an emphasis on its connection with mean curvature flow.

2.1. Basic properties for entropy

Follwing Colding and Minicozzi [34] (cf. [67]), define the entropy for a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ by

$$\lambda[\Sigma] = \sup_{\mathbf{x}_0 \in \mathbb{R}^{n+1}, t_0 > 0} (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4t_0}} \, d\mathcal{H}^n \tag{2.1}$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$.

It is readily checked that $\lambda[\Sigma \times \mathbb{R}^k] = \lambda[\Sigma]$ and, for $\rho > 0$ and $\mathbf{y} \in \mathbb{R}^{n+1}$,

$$\lambda[\rho\Sigma + \mathbf{y}] = \lambda[\Sigma],$$

where $\rho\Sigma + \mathbf{y}$ is a hypersurface given by

$$\rho\Sigma + \mathbf{y} = \{\mathbf{z} \in \mathbb{R}^{n+1} \mid \mathbf{z} = \rho\mathbf{x} + \mathbf{y} \text{ for some } \mathbf{x} \in \Sigma\}.$$

A direct calculation gives $\lambda[\mathbb{R}^n] = 1$. Moreover, Stone [76] computes:

$$2 > \lambda[\mathbb{S}^1] > \frac{3}{2} > \lambda[\mathbb{S}^2] > \cdots > \lambda[\mathbb{S}^n] > \lambda[\mathbb{S}^{n+1}] > \cdots \to \sqrt{2}. \tag{2.2}$$

The definition of entropy can be extended in a straightforward manner to measures and varifolds on a Euclidean space. There are also interesting studies of analogues of the Colding–Minicozzi entropy in noncompact Riemannian manifolds under certain curvature and volume conditions by Sun [78] and in hyperbolic space by Bernstein [9] (see also [90]).

2.2. Mean curvature flow

A one-parameter family of hypersurfaces $\Sigma_t \subset \mathbb{R}^{n+1}$ is a *mean curvature flow* if, for $\mathbf{x} \in \Sigma_t$,

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^{\perp} = \mathbf{H}_{\Sigma_t}, \tag{2.3}$$

where the superscript $\perp$ means the projection to the unit normal $\mathbf{n}_{\Sigma_t}$ on Σ_t, and $\mathbf{H}_{\Sigma_t}$ is the mean curvature given by

$$\mathbf{H}_{\Sigma_t} = -H_{\Sigma_t}\mathbf{n}_{\Sigma_t} = -\operatorname{div}_{\Sigma_t}(\mathbf{n}_{\Sigma_t})\mathbf{n}_{\Sigma_t}.$$

Not only is mean curvature flow a beautiful subject in its own right, it also models various physical phenomena and has potential applications in numerous scientific fields, such as biology, computer imaging, and material sciences (see, e.g., [31, 62, 66, 68]).

By the avoidance principle (see [21, 6.3] and [45, CHAP. 4]), the mean curvature flow starting from any given closed hypersurface becomes singular in finite time. A central topic in the study of mean curvature flow is to understand the asymptotic behavior of the flow near singularities. Namely, suppose $\{\Sigma_t\}_{t\in[0,T)}$ is a mean curvature flow with $T > 0$ the first singular time. Let $\mathbf{x}_i \in \Sigma_{t_i}$ with $\mathbf{x}_i \to \mathbf{x}_0$ and $t_i \to T$ be such that the second fundamental form $|A_{\Sigma_{t_i}}(\mathbf{x}_i)| \to \infty$. If we define

$$\Gamma_s = \frac{1}{\sqrt{T-t}}\Sigma_t \quad \text{and} \quad s = -\log(T-t),$$

then the family $\{\Gamma_s\}_{s\geq -\log T}$ satisfies, for $\mathbf{y} \in \Gamma_s$,

$$\left(\frac{\partial \mathbf{y}}{\partial s}\right)^{\perp} = \mathbf{H}_{\Gamma_s} + \frac{\mathbf{y}^{\perp}}{2}, \tag{2.4}$$

which is called the *rescaled mean curvature flow* associated to the flow $\{\Sigma_t\}_{t\in[0,T)}$. Thus, characterizing the limits of Γ_s as $s \to \infty$ plays a fundamental role in the study of singularity formation for mean curvature flow.

Observe that the rescaled mean curvature flow $\{\Gamma_s\}_{s\geq -\log T}$ satisfies

$$\frac{d}{ds}\left((4\pi)^{-\frac{n}{2}}\int_{\Gamma_s} e^{-\frac{|\mathbf{y}|^2}{4}}\,d\mathcal{H}^n\right) = -(4\pi)^{-\frac{n}{2}}\int_{\Gamma_s}\left|\mathbf{H}_{\Gamma_s}+\frac{\mathbf{y}^\perp}{2}\right|^2 e^{-\frac{|\mathbf{y}|^2}{4}}\,d\mathcal{H}^n, \tag{2.5}$$

so it is the (negative) gradient flow of the Gaussian surface area

$$F[\Gamma] = (4\pi)^{-\frac{n}{2}}\int_{\Gamma} e^{-\frac{|\mathbf{y}|^2}{4}}\,d\mathcal{H}^n. \tag{2.6}$$

Notice that rewinding the change of variables in (2.5) gives exactly the monotonicity formula discovered by Huisken [53]. And it follows that the entropy $\lambda[\Sigma_t]$ is decreasing in t. Moreover, sending $s\to\infty$, up to passing to a subsequence, Γ_s converges weakly to a critical point, Γ_0, of the functional F that satisfies the Euler–Lagrange equation

$$\mathbf{H}_{\Gamma_0}+\frac{\mathbf{y}^\perp}{2}=\mathbf{0}, \tag{2.7}$$

and thus $\lambda[\Gamma_0]\leq\lambda[\Sigma_0]$ (see [53, 57]). A hypersurface satisfying (2.7) is also called a *self-shrinker*. Observe that $\{\sqrt{-t}\,\Gamma_0\}_{t<0}$ is a Brakke flow [21] that satisfies (2.3) weakly, and we call it a *tangent flow* at $(\mathbf{x}_0,T)$. One may also consider a blowup sequence $\rho_i(\Sigma_{t_i}-\mathbf{x}_i)$ with $\rho_i\to\infty$, and, by Brakke's compactness [21] (see also [55, SECT. 7]), the limit is also a Brakke flow, called a *limit flow* at $(\mathbf{x}_0,T)$.

There is a wild zoo of examples of self-shrinkers (see [4, 63, 64, 70, 71, 80]). However, a long-standing conjecture of Huisken [58, #8] (and of Angenent–Chopp–Ilmanen [5] in $\mathbb{R}^3$) asserts that starting with a generic closed hypersurface, the mean curvature flow develops only spherical and cylindrical singularities. Recently, Colding and Minicozzi have pioneered a number of innovative techniques about entropy and made important progress towards Huisken's conjecture (see [34–41]). Among them, the most relevant to this article is the following result.

Theorem 2.1 ([34, THEOREM 0.12]). *The only smooth embedded entropy-stable self-shrinkers with polynomial volume growth are round spheres, generalized cylinders and hyperplanes.*

Here "entropy-stable" means that there are no perturbations of the self-shrinkers to decrease the entropy. An easy consequence of Theorem 2.1 is that any singularities for mean curvature flow that are not spheres or generalized cylinders may be perturbed away in an appropriate sense. Moreover, it is possible to perturb the initial data to avoid certain unstable singularities for mean curvature flow (see Chodosh–Choi–Mantoulidis–Schulze [27, 28], Sun [77] and Sun–Xue [81, 82]).

Without the smoothness assumption, Colding and Minicozzi show that in dimension less than 7, Theorem 2.1 still holds for oriented F-stationary integral varifolds that have singular sets with locally finite codimension-2 Hausdorff measure [34, THEOREM 0.14]. Furthermore, in [91], Zhu utilizes an α-structural hypothesis in minimal surface theory and extends this result to higher dimensions. Notice that the hypothesis on the size of singular set is expected to hold for any self-shrinkers arising in mean curvature flow [57, PAGE 8].

2.3. Conjectures on the sharp lower entropy bound for hypersurfaces

The dynamic perspective of [34] suggests the following two closely related conjectures of Colding, Ilmanen, Minicozzi, and White (cf. [33, CONJECTURES 0.9 AND 0.10]).

Conjecture 2.2. *For $n \leq 6$, there is an $\varepsilon_0 = \varepsilon_0(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a nonflat self-shrinker not equal to the round sphere, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n] + \varepsilon_0$.*

Conjecture 2.3. *For $n \leq 6$, if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$.*

Both conjectures are known to be true with $n = 1$. Indeed, Conjecture 2.2 follows directly from Abresch–Langer's classification of self-shrinking planar curves [1]. And by work of Gage–Hamilton [50] and Grayson [51], every closed embedded curve in plane evolves under mean curvature flow to a round point which, together with the monotonicity of entropy, proves Conjecture 2.3.

As remarked before, the mean curvature flow starting from any given closed hypersurface becomes singular in finite time and the self-shrinker modeling the singularity of the flow has lower entropy. Thus, Conjecture 2.3 would follow from Conjecture 2.2. Despite Theorem 2.1, one of the difficulties to prove Conjecture 2.2 is that if one perturbs a noncompact self-shrinker, a priori it may flow smoothly without developing singularities.

At last, it may be interesting to think of Conjecture 2.2 as an analogue, in the Gaussian setting, of the question on the sharp lower bound on density for minimal cones (see Ilmanen–White [60] and Marques–Neves [69]).

3. SHARP LOWER BOUND ON ENTROPY FOR SELF-SHRINKERS

In this section, we discuss recent progress towards Conjecture 2.2. First, Brakke's local regularity [21, 6.1] implies that $\mathbb{R}^n$ has the least entropy of all self-shrinkers and, moreover, there is a gap to the next lowest (see also White [88]). As such, Conjecture 2.2 concerns the sharp lower entropy bound with a gap for all *nonflat* self-shrinkers.

Observe that if an immersed hypersurface has entropy strictly less than 2, then it must be embedded. Thus, we always assume embeddedness for the remainder of this section. Moreover, by the Frankel property (see [85, THEOREM 7.4] and [61, THEOREM C]), any embedded self-shrinker is connected.

3.1. Closed self-shrinkers with low entropy

In [33], Colding, Ilmanen, Minicozzi, and White initiate the study of Conjecture 2.2 and prove the following result.

Theorem 3.1 ([33]). *Given n, there exists $\varepsilon = \varepsilon(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed self-shrinker not equal to the round sphere, then*

$$\lambda[\Sigma] \geq \lambda\left[\mathbb{S}^n\right] + \varepsilon. \tag{3.1}$$

Moreover, if

$$\lambda[\Sigma] \le \min\left\{\lambda[\mathbb{S}^{n-1}], \frac{3}{2}\right\}, \tag{3.2}$$

then Σ *is diffeomorphic to* $\mathbb{S}^n$. *(If* $n > 2$*, then* $\lambda[\mathbb{S}^{n-1}] < \frac{3}{2}$ *and the minimum is unnecessary.)*

Outline of the proof. By Abresch–Langer [1], the theorem is vacuously true with $n = 1$; thus, assume $n \ge 2$ below. We also assume $\lambda[\Sigma] \le \min\{\lambda[\mathbb{S}^n], 3/2\}$, as otherwise the theorem follows from inequality (2.2). As Σ is closed and not round, it follows from Colding–Minicozzi's classification of stable self-shrinkers, Theorem 2.1, that Σ is entropy unstable. Thus, there is a nearby hypersurface $\tilde{\Sigma}$ with the following properties:

(1) $\lambda[\tilde{\Sigma}] < \lambda[\Sigma]$;

(2) $\tilde{\Sigma}$ is inside of Σ, i.e., the compact region of $\mathbb{R}^{n+1}$ bounded by Σ contains $\tilde{\Sigma}$;

(3) $H_{\tilde{\Sigma}} - \frac{1}{2}\mathbf{x} \cdot \mathbf{n}_{\tilde{\Sigma}} > 0$ (with a suitable choice of the unit normal of $\tilde{\Sigma}$).

(See [34, COROLLARY 5.15, THEOREM 4.30, AND THEOREM 0.15].)

Next, one may use a Simon-type equation and the parabolic maximum principle to show that, starting from $\tilde{\Sigma}$, the rescaled mean curvature flow, i.e., a family of hypersurfaces $\tilde{\Sigma}_t \subset \mathbb{R}^{n+1}$ flowing by equation (2.4), preserves property (2) and bounds the second fundamental form $A_{\tilde{\Sigma}_t}$ by

$$|A_{\tilde{\Sigma}_t}|^2 \le Ce^{-2t}\left|H_{\tilde{\Sigma}_t} - \frac{1}{2}\mathbf{x} \cdot \mathbf{n}_{\tilde{\Sigma}_t}\right|^2 \tag{3.3}$$

for some constant C depending on $\tilde{\Sigma}$. As $\tilde{\Sigma}_t$ becomes singular in finite time, a (subsequential) limit of blowups of the rescaled flow $\tilde{\Sigma}_t$ at the singularity is given by a (possibly singular) self-shrinker Γ. More crucially, estimate (3.3) gives

$$|A| \le CH \quad \text{on the regular part of } \Gamma. \tag{3.4}$$

Appealing to the monotonicity of entropy, property (1) and the entropy bound of Σ gives that

$$\lambda[\Gamma] \le \lambda[\tilde{\Sigma}] < \lambda[\Sigma] \le \min\left\{\lambda[\mathbb{S}^{n-1}], \frac{3}{2}\right\}.$$

Thus, by Allard's regularity (see [3] or [73]) and estimate (3.4), Γ is a smooth embedded mean-convex self-shrinker. Thus, the classification of mean-convex self-shrinkers of Huisken [53] and of Colding–Minicozzi [34, THEOREM 0.17] implies Γ is of the form $\mathbb{S}^k \times \mathbb{R}^{n-k}$. Furthermore, the entropy bound of Γ ensures Γ is the round sphere. Thus, It follows that $\tilde{\Sigma}_t$ flows smoothly until it vanishes in a round point. Hence, as by construction $\tilde{\Sigma}$ can be chosen to be sufficiently close, in the C^∞ topology, to Σ, it follows that $\lambda[\Sigma] > \lambda[\tilde{\Sigma}] \ge \lambda[\mathbb{S}^n]$ and Σ is diffeomorphic to $\mathbb{S}^n$.

Finally, to see that there is a gap, one argues by contradiction. Suppose there is a sequence of closed self-shrinkers Σ^i that are not round with entropy converging to $\lambda[\mathbb{S}^n]$. Like before, perturbing these self-shrinkers and then applying rescaled mean curvature flow to the perturbations gives a sequence of flows $\tilde{\Sigma}^i_t$ with entropy less than or equal to $\lambda[\Sigma^i]$ and developing a spherical singularity in finite time. By the monotonicity of entropy, rescaling the $\tilde{\Sigma}^i_t$ about the spherical singularity creates a new sequence of rescaled mean curvature flows converging to the static sphere. This contradicts that, by Huisken [53], for i large Σ^i has negative curvature at some point. ∎

Remark 3.2. In the proof of Theorem 3.1, the number $\frac{3}{2}$ in the minimum of (3.2) is only used to rule out the possibility of triple junctions arising in the rescaled mean curvature flow. However, by the orientability and results on mod 2 flat chains [89], the second part of Theorem 3.1 still holds under the weaker assumption that $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$.

3.2. Noncompact self-shrinkers with low entropy

The arguments for Theorem 3.1 fail on noncompact self-shrinkers because perturbing a noncompact self-shrinker and applying rescaled mean curvature flow to the perturbation a priori may yield a rescaled mean curvature flow that has no singularities in finite time. To overcome this issue, it is needed to combine ideas from the proof of Theorem 3.1 and [11, 14].

A starting point is to understand the asymptotic structure of noncompact self-shrinkers. It is shown in [83] that any noncompact self-shrinker in $\mathbb{R}^3$ of finite genus is smoothly asymptotic (at infinity) to a cone or a cylinder (see also [79, APPENDIX A]). Assuming the noncompact self-shrinker has entropy bounded by that of the circle instead, a stronger result is true (cf. [10, PROPOSITION 4.5]).

Lemma 3.3. *If $\Sigma \subset \mathbb{R}^3$ is a noncompact self-shrinker with $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1]$, then one of the following is true:*

(1) *Σ is isometric to a cylinder.*

(2) *There is a regular cone $\mathcal{C} \subset \mathbb{R}^3$ so that Σ is smoothly asymptotic to $\mathcal{C}$, i.e., as $\rho \to 0^+$, the $\rho\Sigma$ converges to $\mathcal{C}$ in $C^\infty_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{\mathbf{0}\})$. In particular, the curvature of Σ is quadratically decaying at infinity.*

Here a regular cone is a proper subset of $\mathbb{R}^{n+1}$ that is invariant under dilations and the link of the cone is a smooth embedded codimension-one submanifold of $\mathbb{S}^n$.

Proof. By definition, there is a cone $\mathcal{C} \subset \mathbb{R}^3$ so that as $\rho \to 0^+$, the $\rho\Sigma$ converges, in the Hausdorff distance, to $\mathcal{C}$. Fix any $\mathbf{y} \in \mathcal{C} \setminus \{\mathbf{0}\}$. Observe that $\sqrt{-t}\,\Sigma$, $t < 0$, is a mean curvature flow converging to $\mathcal{C}$ at time 0. Thus, as $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1] < 2$, it follows from White's stratification theorem [87] that any tangent flow at $(\mathbf{y}, 0)$ is a multiplicity-one self-shrinker of the form $\Gamma \times \mathbb{R}$. Furthermore, by Abresch–Langer's classification of self-shrinking planar curves [1], $\Gamma = \mathbb{R}$ or $\mathbb{S}^1$. If $\Gamma = \mathbb{S}^1$, then Huisken's monotonicity gives Σ splits off a line and, thus, is isometric to a cylinder. If $\Gamma = \mathbb{R}$, then Brakke's local regularity [21] (see also White [88]) implies the flow is regular near $(\mathbf{y}, 0)$. As $\mathbf{y}$ is arbitrary, the second item follows. ∎

Next, it is shown in [11] that there is a topological restriction on asymptotically conical self-shrinkers with entropy less than or equal to that of the round cylinder. This is the key to the proof of Conjecture 2.2 with $n = 2$.

Theorem 3.4 ([11]). *For $n \geq 2$, let $\Sigma \subset \mathbb{R}^{n+1}$ be a self-shrinker that is smoothly asymptotic to a regular cone $\mathcal{C}$. If $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$, then the link of the asymptotic cone $\mathcal{C}$ separates $\mathbb{S}^n$ into two connected components both diffeomorphic to Σ. As a consequence, the link is connected.*

Outline of the proof. The arguments below may be thought of a natural analog, in the asymptotically conical setting, of the arguments in the proof of Theorem 3.1. However, there is an essential difference: while it is exploited there that the flow of a closed hypersurface must form a singularity in finite time, it is shown below that the flow of an asymptotically conical hypersurface with small entropy must exist without singularities for long-time and the flow eventually becomes star-shaped.

As the theorem is trivially true for hyperplanes, without loss of generality assume $\Sigma \neq \mathbb{R}^n$. By Theorem 2.1, Σ is entropy unstable, so there are two nearby hypersurfaces $\tilde{\Sigma}^{\pm} \subset \mathbb{R}^{n+1}$ such that

(1) $\lambda[\tilde{\Sigma}^{\pm}] < \lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}] = \lambda[\mathbb{S}^{n-1} \times \mathbb{R}]$;

(2) $\tilde{\Sigma}^{+}$ lies in one side of Σ while $\tilde{\Sigma}^{-}$ lies on the other side of Σ;

(3) $H - \frac{1}{2}\mathbf{x} \cdot \mathbf{n} > K(1 + |\mathbf{x}|^2)^{\mu}$ on $\tilde{\Sigma}^{\pm}$ (with respective to the correct orientation) for constants $K > 0$ and $\mu < -1$ both depending on Σ;

(4) $\tilde{\Sigma}^{\pm}$ are both smoothly asymptotic to the cone $\mathcal{C}$.

In the proof of Theorem 3.1, it is convenient to think of self-shrinkers as static points for the rescaled mean curvature flow and show the sign of $H - \frac{1}{2}\mathbf{x} \cdot \mathbf{n}$ is preserved under the flow. Here it is crucial to instead study *shrinker mean curvature relative to the space-time point $X_0 = (\mathbf{x}_0, t_0)$ and at time t* (see also [75])

$$S^{X_0,t} = 2(t_0 - t)H - (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}. \tag{3.5}$$

By the parabolic maximum principle, the sign of shrinker mean curvature is preserved under the mean curvature flow starting with $\tilde{\Sigma}^{\pm}$ at time -1. Notice that shrinker mean curvature makes senses for mean curvature flows which start close to a self-shrinker but that persist up to (and beyond) the singular time of the self-shrinker, which is the key to this proof.

Arguing similarly as in the proof of Theorem 3.1 and invoking property (1) give that the flows $\tilde{\Sigma}^{\pm}_t$ starting with $\tilde{\Sigma}^{\pm}$ at time -1 both exist smoothly for long-time and become star-shaped at time 0. As $\tilde{\Sigma}^{-}_0$ and $\tilde{\Sigma}^{+}_0$ lie on different sides of $\mathcal{C}$ and are both smoothly asymptotic to $\mathcal{C}$, it follows that the link of $\mathcal{C}$ divides $\mathbb{S}^n$ into two components ω^+ and ω^- over which $\tilde{\Sigma}^{+}_0$ and $\tilde{\Sigma}^{-}_0$ are radial graphs, respectively. Thus, $\tilde{\Sigma}^{\pm}_0$ and $\omega^{\pm}$ are diffeomorphic. Hence, by construction, $\tilde{\Sigma}^{-}_0$ and $\tilde{\Sigma}^{+}_0$ are both diffeomorphic to Σ and so are ω^- and ω^+. ∎

Moreover, by the arguments in the proof of Theorem 3.4 and standard topological facts, Theorem 3.4 can be further refined.

Theorem 3.5 ([14, THEOREM 1.2]). *For $n \geq 2$, let $\Sigma \subset \mathbb{R}^{n+1}$ is a self-shrinker smoothly asymptotic to a regular cone $\mathcal{C}$. If $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$, then Σ is contractible and the link of the asymptotic cone $\mathcal{C}$ is a homology $(n-1)$-sphere.*

Immediately, the classification of surfaces and Alexander's theorem [2] gives the following consequence.

Corollary 3.6. *For $2 \leq n \leq 3$, let $\Sigma \subset \mathbb{R}^{n+1}$ be a self-shrinker smoothly asymptotic to a regular cone. If $\lambda[\Sigma] \leq \lambda[\mathbb{S}^{n-1}]$, then Σ is diffeomorphic to $\mathbb{R}^n$.*

We now explain why Conjecture 2.2 is true with $n = 2$. Let $\Sigma \subset \mathbb{R}^3$ be a self-shrinker with $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1]$. If Σ is closed, then, by Theorem 3.1 and remark (3.2), Σ is diffeomorphic to $\mathbb{S}^2$. If Σ is noncompact, then Lemma 3.3 implies that it is either a cylinder or smoothly asymptotic to a regular cone. In the latter case, Corollary 3.6 implies Σ is diffeomorphic to $\mathbb{R}^2$. Thus, by Brendle's classification for genus-zero self-shrinkers [22], Σ is a round sphere, a cylinder or a plane. Hence, it follows that the round sphere has the lowest entropy among all nonflat self-shrinkers in $\mathbb{R}^3$ and cylinder has the second lowest. In particular, this proves Conjecture 2.2 with $n = 2$ and $\varepsilon_0(2) = \lambda[\mathbb{S}^1] - \lambda[\mathbb{S}^2] > 0$.

Furthermore, there is a gap to the third lowest. To see this, suppose there is a sequence of self-shrinkers $\Sigma^i \subset \mathbb{R}^3$ so that $\lambda[\mathbb{S}^1] < \lambda[\Sigma^i] < \lambda[\mathbb{S}^1] + i^{-1}$. Thus, up to passing to a subsequence, the Σ^i converges smoothly to a self-shrinker Σ' with $\lambda[\Sigma'] = \lambda[\mathbb{S}^1]$. By the preceding discussions, Σ' is a cylinder. In particular, Σ' has positive mean curvature. The nature of convergence ensures that, for large i, Σ^i also has positive mean curvature in a large compact set. Hence, by the cylinder rigidity of Colding–Ilmanen–Minicozzi [32], for i large Σ^i is a cylinder, contradicting the entropy bound of Σ^i. Hence, we arrive at the following gap result.

Corollary 3.7 ([11, COROLLARY 1.2]). *There is a $\delta > 0$ so that if $\Sigma \subset \mathbb{R}^3$ is a self-shrinker not equal to a round sphere, a cylinder or a plane, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^1] + \delta$.*

4. SHARP LOWER BOUND ON ENTROPY FOR CLOSED HYPERSURFACES

In this section, we discuss a complete resolution of Conjecture 2.3, which asserts that round spheres have the least entropy of closed hypersurfaces of dimension less than 7. By definition (see (2.1)) $\lambda[\Sigma] \geq 1$ for any hypersurface $\Sigma \subset \mathbb{R}^{n+1}$. In fact, Chen [25] shows that $\lambda[\Sigma] = 1$ if and only if Σ is a hyperplane. Though a hyperplane can be approximated by closed hypersurfaces, Conjecture 2.3 claims that the entropy of any closed hypersurface is strictly larger than 1.

In [10], Bernstein and the author use a weak mean curvature flow and analyze terminal singularities to confirm Conjecture 2.3 (compare Ketover–Zhou [65]) After that, Zhu [91] further elaborates on the Colding–Minicozzi classification of stable self-shrinkers (see Theorem 2.1) to extend Conjecture 2.3 to all dimensions.

Theorem 4.1 ([10, 91]; cf. [65]). *If $\Sigma \subset \mathbb{R}^{n+1}$ is a closed hypersurface, then $\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$ with equality if and only if Σ is a round sphere.*

Despite the discussion of Section 2.3, the proof of Theorem 4.1 that we explain below is independent of Conjecture 2.2. We give necessary background on some key ingredients that may be of independent interest and then sketch the proof of Theorem 4.1.

4.1. Weak mean curvature flow

Among various notions of weak mean curvature flow, the most relevant to this article are the Brakke flow and level set flow. Following Ilmanen [55] (cf. [21]), a Brakke flow is a one-parameter family of Radon measures on $\mathbb{R}^{n+1}$ which satisfy equation (2.3) in a certain weak form. The Brakke flow ensures that the mass of the measures decreases along the flow. A Brakke flow is *integral* if, at almost all times, the flow is an integer rectifiable Radon measure. Thinking of hypersurfaces $\Sigma \subset \mathbb{R}^{n+1}$ as measures $\mathcal{H}^n \lfloor \Sigma$, any smooth mean curvature flow is an integral Brakke flow.

Motivated by work of Osher–Sethian [72] in numerical analysis, the theory of level set flows has been established independently by Chen–Giga–Goto [26] and Evans–Spruck [46–49] (cf. [54, 55]). A level set flow is a family of hypersurfaces obtained in the following way. First, embed a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ as the 0-level set of a Lipschitz function on $\mathbb{R}^{n+1}$. Then evolving the function in the way that, intuitively, every level set of the function flows by mean curvature yields a family of Lipschitz functions on $\mathbb{R}^{n+1}$. The level set flow of Σ is given by the 0-level set of the family of functions. It is shown, for instance, in [46], that the level set flow is well defined in the sense that it is independent of the choice of initial functions and coincides with the smooth mean curvature flow as long as the latter exists.

For the purposes of this article, Brakke flows have two important properties. The first is that Huisken's monotonicity formula [53] also holds for Brakke flows (see [57] and [86]). The second is the powerful regularity of Brakke [21] for such flows. A major technical difficulty in using Brakke flows is that there is a great deal of nonuniqueness as, by construction, Brakke flows are allowed to vanish instantaneously. On the other hand, the level set flow satisfies a strong maximum principle and thus is unique. In [55], Ilmanen uses an elliptic regularization procedure to construct a multiplicity-one Brakke flow that is supported on any given nonfattening level set flow (cf. Evans–Spruck [49]). Here a level set flow is *nonfattening* if the flow does not develop nonempty interiors. Observe the nonfattening condition is generic. Thus, it suffices to consider the closure (under Brakke's compactness) of the set of integral Brakke flows constructed by the elliptic regularization procedure.

4.2. Noncollapsed self-shrinkers and Brakke flows

An important notion of being noncollapsed for self-shrinkers and, more generally, for flows is introduced in [10] and is used to ensure nonvanishing. A self-shrinking measure on

$\mathbb{R}^{n+1}$ is an integer n-rectifiable Radon measure μ on $\mathbb{R}^{n+1}$ such that the associated varifold is F-stationary.

Definition 4.2. A self-shrinking measure μ on $\mathbb{R}^{n+1}$ is *noncollapsed* if there are $\mathbf{y} \in \mathbb{R}^{n+1}$ and $R > 4\sqrt{n}$ so that

(1) $\operatorname{spt}(\mu)$ is regular (i.e., smooth properly embedded) in the (open) ball $B_R(\mathbf{y})$;

(2) $\operatorname{spt}(\mu)$ separates $B_R(\mathbf{y}) \subset \mathbb{R}^{n+1}$ into two connected components Ω_+ and Ω_- containing closed balls $\bar{B}_{2\sqrt{n}}(\mathbf{x}_+)$ and $\bar{B}_{2\sqrt{n}}(\mathbf{x}_-)$, respectively.

The measure μ is *strongly noncollapsed* if $\mu \times \mu_{\mathbb{R}^k}$ is noncollapsed for all $k \geq 0$, where $\mu_{\mathbb{R}^k}$ is the k-dimensional Hausdorff measure on $\mathbb{R}^k$.

For instance, if μ is a noncompact self-shrinking measure on $\mathbb{R}^3$ with $\lambda[\mu] < 3/2$, then μ is strongly noncollapsed (cf. Lemma 3.3). On the other hand, the avoidance principle implies compact self-shrinking measures on $\mathbb{R}^{n+1}$ are all collapsed.

In an analogous way, define (strongly) noncollapsed Brakke flows as follows.

Definition 4.3. An integral Brakke flow $\mathcal{K} = \{\mu_t\}_{t \geq t_0}$ in $\mathbb{R}^{n+1}$ is *noncollapsed at time τ* if there are $(\mathbf{y}, s) \in \mathbb{R}^{n+1} \times (t_0, \tau)$, $R > 4\sqrt{n(\tau - t_0)}$, and $0 < \varepsilon < \min\{\tau - s, s - t_0\}$ so that

(1) $\mathcal{K}$ is regular in $B_R(\mathbf{y}) \times (s - \varepsilon, s + \varepsilon)$;

(2) $\operatorname{spt}(\mu_s)$ separates $B_R(\mathbf{y}) \subset \mathbb{R}^{n+1}$ into two connected components Ω_+ and Ω_- containing closed balls $\bar{B}_{2\sqrt{n(\tau-s)}}(\mathbf{x}_+)$ and $\bar{B}_{2\sqrt{n(\tau-s)}}(\mathbf{x}_-)$, respectively.

The Brakke flow $\mathcal{K}$ is *strongly noncollapsed at time τ* if $\{\mu_t \times \mu_{\mathbb{R}^k}\}_{t \geq t_0}$ is noncollapsed at time τ for all $k \geq 0$.

Note that if μ is a self-shrinking measure that is (strongly) noncollapsed, then the associated Brakke flow is (strongly) noncollapsed at time 0. A key observation is that being (strongly) noncollapsed at a time is an open condition for integral Brakke flows. Thus, given an integral Brakke flow $\mathcal{K}$ with finite entropy, if a tangent flow of $\mathcal{K}$ at $(\mathbf{y}, \tau)$ is (strongly) noncollapsed at time 0, then $\mathcal{K}$ is (strongly) noncollapsed at time τ.

There is a general structural result for self-shrinking measures with entropy less than that of a round sphere. It follows from an inductive argument and White's stratification theorem [87].

Proposition 4.4 ([10, PROPOSITION 4.12]). *For $n \geq 2$, if μ is a self-shrinking measure on $\mathbb{R}^{n+1}$ with $\lambda[\mu] < \lambda[\mathbb{S}^n]$, then one of the following holds:*

(1) *μ has compact support.*

(2) *μ is strongly noncollapsed.*

(3) *There is a self-shrinking measure ν on $\mathbb{R}^{n+1}$ so that $\lambda[\nu] \leq \lambda[\mu]$ and $\nu = \hat{\nu} \times \mu_{\mathbb{R}^{n-k}}$ for $\hat{\nu}$ a compact self-shrinking measure and $1 \leq k \leq n-1$.*

4.3. Outline of the proof of Theorem 4.1

By work of Gage–Hamilton [50] and Grayson [51], the claim is true with $n = 1$. To that end, assume $n \geq 2$. Argue by contradiction, then suppose $\lambda[\Sigma] < \lambda[\mathbb{S}^n]$. Let $\mathcal{K} = \{\mu_t\}_{t\geq 0}$ be the integral Brakke flow in $\mathbb{R}^{n+1}$ with $\mu_0 = \mathcal{H}^n \lfloor \Sigma$. By the spheres comparison and avoidance principle, the extinction time of $\mathcal{K}$, $T_0(\mathcal{K})$ satisfies

$$0 < T_0(\mathcal{K}) = \sup\bigl\{t \geq 0 \mid \operatorname{spt}(\mu_t) \neq \emptyset\bigr\} < \infty.$$

Appealing to Definition 4.3 and the avoidance principle implies that $\mathcal{K}$ is collapsed at time $T_0(\mathcal{K})$. As being noncollapsed is an open condition for Brakke flows, any tangent flow of $\mathcal{K}$ at time $T_0(\mathcal{K})$ is collapsed at time 0. Recall from Section 2.2 that tangent flows are given by self-shrinking measures. Thus, it is enough to show, for all $1 \leq k \leq n$, the set, $\mathcal{CSM}_k(\lambda[\mathbb{S}^n])$, of all compact self-shrinking measures on $\mathbb{R}^{k+1}$ that have entropy less than $\lambda[\mathbb{S}^n]$ is an empty set, because it would follow from Proposition 4.4 that all tangent flows are strongly noncollapsed at time 0, giving a contradiction.

On $\mathbb{R}^2$, all self-shrinking measures with entropy less than $3/2$ are smooth embedded and thus have been classified by Abresch–Langer [1]. Thus, by a direct computation, the claim is true with $k = 1$. Suppose, inductively, the claim holds for all $1 \leq k \leq l-1$. Assume $l < n+1$, as otherwise we are done. Argue by contradiction, then suppose the claim were false for $k = l$. Take an entropy minimizing sequence of compact self-shrinking measures ν_i on $\mathbb{R}^{l+1}$ with $\lambda[\nu_i] < \lambda[\mathbb{S}^n]$. Then, up to passing to a subsequence, the ν_i converges to a self-shrinking measure ν_0 with $\lambda[\nu_0] < \lambda[\mathbb{S}^n]$. As any compact self-shrinking measure is collapsed, the ν_i are all collapsed. By the openness of being noncollapsed, ν_0 is also collapsed. Thus, by the inductive hypothesis and Proposition 4.4, ν_0 has compact support. Our construction and the entropy bound ensure ν_0 is entropy stable and has a singular set of codimension at least 2. Hence, appealing to Colding–Minicozzi [34, THEOREM 0.14] when $l \leq 6$ while to Zhu [91, THEOREM 1.2] when $l \geq 7$ gives $\nu_0 = \mathcal{H}^l \lfloor \mathbb{S}^l$, contradicting $\lambda[\nu_0] < \lambda[\mathbb{S}^l]$.

It remains only to characterize the equality case. Suppose $\lambda[\Sigma] = \lambda[\mathbb{S}^n]$. If Σ is not (modulo translations and dilations) a self-shrinker, then applying mean curvature flow to Σ for short time yields a closed hypersurface $\tilde{\Sigma}$ with $\lambda[\tilde{\Sigma}] < \lambda[\mathbb{S}^n]$. This contradicts what we have just shown. Thus, modulo translations and dilations, Σ is a self-shrinker. Moreover, Σ is entropy stable, as otherwise one finds a perturbation of Σ so that the perturbation is a closed hypersurface with strictly less entropy, giving a contradiction. Thus, the classification of entropy stable self-shrinkers, Theorem 2.1, implies Σ is a round sphere.

5. STABILITY FOR THE ENTROPY INEQUALITY

We continue to discuss a natural follow-up question to Theorem 4.1 that whether a closed hypersurface with entropy sufficiently close to the lowest is itself close to a round sphere. There are various perspectives of this question. For instance, Wang [84] proves a forward analogue of Brakke's clearing-out lemma [21] and establish an explicit relationship between a certain normalized Hausdorff distance of a surface to a round sphere and the difference between their entropy (cf. [12]). It is also interesting to approach this question from a topological viewpoint. Indeed, an immediate application of mean curvature flow and Corollary 3.7 is that any closed surface in $\mathbb{R}^3$ with entropy less than or equal to that of a round cylinder has genus zero. In particular, such a surface is isotopic to $\mathbb{S}^2$.

A conditional isotopic stability result is also true in general dimensions. To state the hypotheses, we follow relevant notions of [14]. For $\Lambda > 0$, let $\mathcal{RMC}_n^*(\Lambda)$ be the set of nonflat regular minimal cones $\mathcal{C} \subset \mathbf{R}^{n+1}$ with $\lambda[\mathcal{C}] < \Lambda$, and let $\mathcal{S}_n^*(\Lambda)$ be the set of nonflat self-shrinkers $\Sigma \subset \mathbb{R}^{n+1}$ with $\lambda[\Sigma] < \Lambda$. The first hypothesis is

$$\mathcal{RMC}_k^*(\Lambda) = \emptyset \quad \text{for all } 3 \le k \le n. \tag{$\star_{n,\Lambda}$}$$

As all regular minimal cones in $\mathbb{R}^2$ consist of unions of rays, $\mathcal{RMC}_1^*(\Lambda) = \emptyset$. Similarly, as geodesics in $\mathbb{S}^2$ are great circles, $\mathcal{RMC}_2^*(\Lambda) = \emptyset$. The second hypothesis is

$$\mathcal{S}_{n-1}^*(\Lambda) = \emptyset. \tag{$\star\star_{n,\Lambda}$}$$

As round cylinders are nonflat self-shrinkers, ($\star\star_{n,\Lambda}$) holds only if $\Lambda \le \lambda[\mathbb{S}^{n-1}]$. Bernstein and the author [17] and Chodosh–Choi–Mantoulidis–Schulze [27] employ different strategies to prove the following conditional result in general dimensions.

Theorem 5.1 ([17, THEOREM 1.3]; cf. [27, THEOREM 10.1]). *Fix $n \ge 3$ and $\Lambda \le \lambda[\mathbb{S}^{n-1}]$. If ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) both hold and Σ is a closed connected hypersurface in $\mathbb{R}^{n+1}$ with $\lambda[\Sigma] \le \Lambda$, then Σ is smoothly isotopic to $\mathbb{S}^n$.*

Remark 5.2. By Marques–Neves' proof of the Willmore conjecture (see [69, THEOREM B]) $\mathcal{RMC}_3^*(\lambda[\mathbb{S}^2]) = \emptyset$. And Corollary 3.7 ensures $\mathcal{S}_2^*(\lambda[\mathbb{S}^2]) = \emptyset$. Thus, ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$) are both fulfilled with $n = 3$ and $\Lambda = \lambda[\mathbb{S}^2]$.

5.1. Overview of the proof of Theorem 5.1

The basic idea is, again, to apply mean curvature flow to Σ and then analyze the behavior of the flow near singularities. Let $\mathcal{K}$ be the Brakke flow starting at Σ. If, modulo translations and dilations, Σ is a self-shrinker, then the claim follows from Theorem 3.1. Otherwise, the entropy is strictly decreasing under the flow. Then it is shown in [14, SECT. 3] that, by hypotheses ($\star_{n,\Lambda}$) and ($\star\star_{n,\Lambda}$), appealing to Allard's regularity [3] and White's stratification theorem [87] implies that any singularities for $\mathcal{K}$ are modeled by smooth embedded multiplicity-one self-shrinkers, which are either compact and, by Theorem 3.1, smoothly isotopic to $\mathbb{S}^n$, or noncompact asymptotically conical.

Due to the lack of a classification for self-shrinkers of simple topology in general dimensions, it seems very difficult to rule out the possibility of these asymptotically conical singularities for $\mathcal{K}$. However, as suggested by Theorem 2.1, such singularities are unstable under mean curvature flow and are expected to be perturbed away in an appropriate sense. This strategy has been carried out in [27]. Namely, it is shown that the ancient mean curvature flow that lies on one-side of a given asymptotically conical self-shrinker exists uniquely for long-time. As an application, perturbing Σ and applying mean curvature flow to the perturbation gives a mean curvature flow that is smooth until it disappears in a round point. The claim follows immediately from this.

The strategy of [17] is distinct from that of [27] and relies on the study of self-expanding solutions to mean curvature flow [13,15–20]. There are two key ingredients. One is an application of a forward analogue of Huisken's monotonicity formula for flows emerging from a conical singularity [15] (see also Section 5.2) to show that taking a second blowup gives self-expanding flows. Another is a topological uniqueness for self-expanders asymptotic to a given cone with entropy less than that of a round cylinder [16] (see also Section 5.3). Thus, combining these with a suitable bubble-tree blowup argument implies $\mathcal{K}$ is smooth at almost all times and stay in the same isotopic class whenever it is smooth. Hence, as near its extinction point $\mathcal{K}$ is isotopic to the shrinking spheres, it follows that Σ is isotopic to $\mathbb{S}^n$.

5.2. Forward monotonicity formula for flows coming out of cones

Huisken's monotonicity formula implies any tangent flows are backwardly self-shrinking. On the other hand, it is unknown that whether tangent flows forward in time are self-expanding or not. Nonetheless, suppose $\mathcal{T} = \{\mu_t\}_{t\in\mathbb{R}}$ is a tangent flow with $\mu_0 = \mathcal{H}^n \lfloor \mathcal{C}$ for $\mathcal{C}$ a regular cone in $\mathbb{R}^{n+1}$. Let $\mathcal{T}' = \{\mu'_t\}_{t\in\mathbb{R}}$ be a tangent flow of $\mathcal{T}$ at $(\mathbf{0}, 0)$. Thus, $\mathcal{T}'$ is self-shrinking for negative times and equal to μ_t for all $t \le 0$. To that end, we explain the reason for that $\mathcal{T}'$ is self-expanding for positive times.

Consider the (forward) rescaled flow $\{\nu_s\}_{s\in\mathbb{R}}$ associated to $\mathcal{T}\lfloor\mathbb{R}^{n+1}\times(0,\infty)$ about $(\mathbf{0}, 0)$. Appealing to [56] and [27, SECT. 8] (cf. [44]) gives, for all $s > 0$, that $\operatorname{spt}(\nu_s)$ is trapped between two self-expanders Γ_- and Γ_+, both smoothly asymptotic to $\mathcal{C}$. Here self-expanders are critical points for the expander energy functional

$$E[\Gamma] = \int_\Gamma e^{\frac{|\mathbf{x}|^2}{4}}\, d\mathcal{H}^n. \tag{5.1}$$

Following a suggestion of Ilmanen [56], define the *relative expander entropy* of ν_s relative to Γ_- by

$$E_{\mathrm{rel}}[\nu_s, \Gamma_-] = \lim_{R\to\infty}\left(\int_{B_R} e^{\frac{|\mathbf{x}|^2}{4}}\, d\nu_s - \int_{\Gamma_-\cap B_R} e^{\frac{|\mathbf{x}|^2}{4}}\, d\mathcal{H}^n\right). \tag{5.2}$$

In [15], Bernstein and the author employ calibration type arguments to show the limit in (5.2) exists and is finite. Prior to that, this relative functional has been studied by Ilmanen–Neves–Schulze [59] in the curve case. Deruelle and Schulze [43] investigate this relative functional in general dimensions and exploit the convergence rate between two self-expanders [8] to show it is well defined and finite for pairs of self-expanders asymptotic to the same cone.

Furthermore, it is shown in [15] that there is a monotonicity formula for the relative expander entropy for flows emerging from a conical singularity. Applying this formula to $\{\nu_s\}_{s\in\mathbb{R}}$ yields that, for $s_1 < s_2$,

$$E_{\mathrm{rel}}[\nu_{s_2}, \Gamma_-] - E_{\mathrm{rel}}[\nu_{s_1}, \Gamma_-] = -\int_{s_1}^{s_2}\int\left|\mathbf{H} - \frac{\mathbf{x}^\perp}{2}\right|^2 e^{\frac{|\mathbf{x}|^2}{4}}\, d\nu_s\, ds. \tag{5.3}$$

As a consequence, given a sequence $s_i \to -\infty$, there is a subsequence s_j so that the ν_{s_j} converges to a critical point of the functional E. This implies that $\mathcal{T}'$ is self-expanding for positive times.

5.3. Topological uniqueness for self-expanders with low entropy

It is illustrated by Angenent–Chopp–Ilmanen [5] that there is an open set of regular cones so that for each cone in the set there are at least two self-expanders asymptotic to the cone (cf. [13]). However, it is proved in [16] that given a regular cone with sufficiently small entropy all self-expanders asymptotic to the cone are in the same isotopic class.

Theorem 5.3 ([16]). *For $0 < \Lambda \leq \lambda[\mathbb{S}^{n-1}]$, let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a regular cone with $\lambda[\mathcal{C}] < \Lambda$ and assume one of the following holds:*

(1) $2 \leq n \leq 6$ *and* $\Lambda = \lambda[\mathbb{S}^{n-1}]$.

(2) $n \geq 7$ *and* ($\star_{n,\Lambda}$) *holds.*

If $\Gamma_1, \Gamma_2 \subset \mathbb{R}^{n+1}$ are two self-expanders both smoothly asymptotic to $\mathcal{C}$, then Γ_1 and Γ_2 are a.c.-isotopic with fixed cone.

Here two asymptotically conical hypersurfaces are said to be *a.c.-isotopic with fixed cone* if there is an isotopy of hypersurfaces that respects the asymptotically conical behavior and fixes the asymptotic cone.

Outline of the proof of Theorem 5.3. It follows from the main result of [20] that the space of asymptotically conical expanders is an infinite-dimensional smooth Banach manifold. Thus, invoking Smale's version [74] of the Sard theorem gives a residual set $\mathscr{R}$ of regular cones so that for each cone in the set any self-expanders smoothly asymptotic to the cone are nondegenerate in the sense that there are no nontrivial normal Jacobi fields that fix the asymptotic cone. In particular, degenerate asymptotically conical self-expanders can be perturbed with varying asymptotic cones to nondegenerate ones. As such, we focus below on generic cones $\mathcal{C} \in \mathscr{R}$. Our goal is to construct Morse flow lines joining any two self-expanders both smoothly asymptotic to $\mathcal{C}$.

Denote by $\mathcal{ACH}_n(\mathcal{C})$ the space of hypersurfaces in $\mathbb{R}^{n+1}$ that are smoothly asymptotic to the cone $\mathcal{C}$. It is convenient to define an order on $\mathcal{ACH}_n(\mathcal{C})$ as follows. First fix a choice of unit normals $\mathbf{n}_{\mathcal{L}}$ on the link $\mathcal{L}$ of $\mathcal{C}$. We then let $\omega_+ \subset \mathbb{S}^n$ be the open set so that $\partial\omega_+ = \mathcal{L}$ and $\mathbf{n}_{\mathcal{L}}$ points into ω_+. For $\Sigma \in \mathcal{ACH}_n(\mathcal{C})$, let $\Omega_+(\Sigma) \subset \mathbb{R}^{n+1}$ be the open set so that $\partial\Omega_+(\Sigma) = \Sigma$ and the blowdowns of Ω_+ in $\mathbb{S}^n$ converge as sets to ω_+. For $\Sigma_1, \Sigma_2 \in \mathcal{ACH}_n(\mathcal{C})$, we say $\Sigma_1 \preceq \Sigma_2$ provided $\Omega_+(\Sigma_2) \subset \Omega_+(\Sigma_1)$.

Let $\mathcal{ACE}_n(\mathcal{C}) \subset \mathcal{ACH}_n(\mathcal{C})$ be the subset consisting of self-expanders. If $\Gamma \in \mathcal{ACE}_n(\mathcal{C})$ is unstable, then there are two eternal rescaled mean curvature flows that deform Γ to two stable elements $\Gamma_\pm \in \mathcal{ACE}_n(\mathcal{C})$ with $\Gamma_- \preceq \Gamma \preceq \Gamma_+$. Moreover, if $\Gamma' \in \mathcal{ACE}_n(\mathcal{C})$ is stable and $\Gamma' \preceq \Gamma$ (respectively, $\Gamma \preceq \Gamma'$), then $\Gamma' \preceq \Gamma_- \preceq \Gamma$ (respectively, $\Gamma \preceq \Gamma_+ \preceq \Gamma'$). The hypotheses ensure that the eternal flows are smooth. On the other hand, if $\Gamma_0, \Gamma_1 \in \mathcal{ACE}_n(\mathcal{C})$ are (strictly) stable and $\Gamma_0 \preceq \Gamma_1$, then appealing to a min–max construction for relative expander entropy [18] yields an element $\Gamma_2 \in \mathcal{ACE}_n(\mathcal{C})$ with $\Gamma_2 \neq \Gamma_i$ for $i \in \{0, 1\}$ and $\Gamma_0 \preceq \Gamma_2 \preceq \Gamma_1$. Again, the hypotheses guarantee the smoothness of the min–max self-expander. Hence, arguing by induction on the cardinality of the subset consisting of stable elements of $\mathcal{ACE}_n(\mathcal{C})$ (see [19]), it follows that every element of $\mathcal{ACE}_n(\mathcal{C})$ can be deformed via rescaled mean curvature flows through a finite number of intermediate elements of $\mathcal{ACE}_n(\mathcal{C})$ to the lowest (with respect to the order $\preceq$), implying the claim. ■

6. FURTHER DISCUSSIONS

Instead of assuming low entropy, Hershkovits and White prove a sharp relation between the entropy and topology of closed self-shrinkers for all dimensions [52]. This may be thought of as an extension of Theorem 3.1. Thinking of self-shrinkers as a special class of ancient the mean curvature flow, combining with work of Angenent–Daskalopoulos–Sesum [6,7], Bernstein–Wang [11], and Brendle–Choi [23,24], Choi, Haslhofer. and Hershkovits [29] classify the ancient mean curvature flow in $\mathbb{R}^3$ with entropy less than or equal to that of a cylinder. There is an analogous classification for ancient mean curvature flows in higher dimensions under the assumption that the flows are smoothly asymptotic at time $-\infty$ to a round cylinder [30].

In general, Conjecture 2.2 is wide open, in part because it is unknown whether there is a complete classification for self-shrinkers of dimension at least 3 with simple topology (compare Brendle [22]) It would be also interesting to study analogous questions in higher codimensions. We refer the interested reader to [41] and references therein.

Very recently, Daniels–Holgate [42] combines [29] and [30] with suitable barriers to construct smooth mean curvature flows with surgery that approximate weak mean curvature flows with only spherical and neck-pinch singularities. Together with [28], this implies that any closed hypersurface in $\mathbb{R}^4$ that has entropy less than or equal to $\lambda[\mathbb{S}^1 \times \mathbb{R}^2]$ is smoothly isotopic to $\mathbb{S}^3$, which, together with Theorem 5.1, sheds some light on the smooth Schoenflies conjecture for $\mathbb{R}^4$.

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