THE WORK OF JUNE HUH

GIL KALAI

ABSTRACT

June Huh found striking connections between algebraic geometry and combinatorics, solved central problems in combinatorics that had remained open for decades, and developed a theory of great importance for both fields. June Huh has been awarded the 2022 Fields Medal "for bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron–Rota–Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture." In this paper I will review some of June Huh's contributions.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 05E14; Secondary 52C35, 05B35, 05C31, 14T15, 52B40

KEYWORDS

Matroids, log-concavity, hard Lefschetz theorems, Hodge–Riemann relations

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 1, pp. 50–65
DOI 10.4171/ICM2022/211

June Huh has made groundbreaking contributions in combinatorics and algebraic geometry and his work established profound connections between these two areas. This paper describes some of Huh's main achievements and gives some background, primarily on the combinatorial aspects of his work.

The Heron–Rota–Welsh unimodality conjecture ([33, 54, 63]) asserts that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence. This implies that the coefficients are unimodal. A special case of the conjecture is an earlier conjecture by Read, asserting that the coefficients of the chromatic polynomial of a graph are unimodal. In 2009 June Huh used algebraic geometry to prove Read's unimodality conjecture [34] for graphs, and the more general Heron–Rota–Welsh conjecture for matroids represented over a field of characteristic 0. The case of matroids representable over a field of a nonzero characteristic and the case of general matroids remained open. In 2010 June Huh and Eric Katz [37] found a different algebraic-geometric approach and proved the case of matroids representable over a field of an arbitrary characteristic. Finally, in 2015 the Heron–Rota–Welsh conjecture was proved in full generality by Karim Adiprasito, June Huh, and Eric Katz [2]. For this purpose, it was necessary to extend theorems from algebraic geometry (primarily the Hodge–Riemann relations and the hard Lefschetz theorem) to cases well beyond the scope of algebraic geometry. Huh and his coauthors developed an entirely novel theory of great interest and importance.

June Huh and Botong Wang [39] used connections with algebraic geometry to prove the Dowling–Wilson conjecture. Consider a configuration $\mathcal{P}$ of n points spanning a d-dimensional space. Let w_i be the number of linear spaces of dimension i spanned by the points.

Motzkin conjectured in his 1936 PhD thesis, and proved over the reals in 1951 [49], that $w_1 \le w_{d-1}$. The case of $d = 3$ (in a planar affine formulation) was proved in 1948 by de Bruijn and Erdős, and their abstract combinatorial proof applies to every characteristic.

The Dowling–Wilson "top heavy" conjecture [22] asserts that

$$w_i \le w_{d-i}, \quad i \le [d/2]. \tag{1}$$

An extension of the Dowling–Wilson conjecture for arbitrary matroids (of rank d) was proved by Tom Braden, June Huh, Jacob Matherne, Nicholas Proudfoot, and Botong Wang [13].

The Mason conjecture (on independence numbers) asserts [44] that the sequence of numbers of independent sets of size k of general matroids is log-concave and it comes in several strengths. Following Huh's first result the conjecture was proved by Mathias Lenz [42] for representable real matroids and it was proved for general matroids in [2]. The strong Mason conjecture for arbitrary matroids was proved by June Huh, Benjamin Schröter, and Botong Wang [38] who relied on [2].

These works have led to further advances by several groups of researchers, and I would especially like to mention the solution of the Mihail–Vazirani conjecture on the expansion constant and rapid mixing for random walks on matroids, by Anari, Oveis Gharan, and

Vinzant [4], as well as the works by Brändén and Huh [14] on correlation inequalities for the Potts model.

The structure of this paper is as follows. In Section 1 we discuss chromatic polynomials and Read's conjecture. In Section 2 we discuss matroids and the Heron–Rota–Welsh conjecture. In Section 3 we discuss the Dowling–Wilson conjecture. Section 4 is devoted to algebraic geometry, Hodge theory, and the Grothendieck standard conjectures. In Section 5 we discuss the Mason conjectures and some recent applications and connections. A recent review paper aimed for a general audience on Huh's work and mathematical background was written by Andrei Okounkov [50].

1. GRAPHS, CHROMATIC POLYNOMIALS, AND READ'S CONJECTURE

1.1. The four-color conjecture and chromatic polynomials

A proper coloring of a graph G is a coloring of the vertices of G such that every two adjacent vertices are colored with different colors. Graph coloring is of central importance in graph theory and in graph algorithms.

Theorem 1 (The four-color theorem (Appel and Haken 1976)). *Every planar graph can be properly colored with 4 colors.*

The four-color conjecture was proposed (in a dual form, for planar maps) by Francis Guthrie in 1852 and proved by Kenneth Appel and Wolfgang Haken [6] in 1976.

For a graph G, let $\chi_G(k)$ be the number of proper colorings of G with k colors. $\chi_G(k)$ is called the chromatic polynomial of the graph G. Chromatic polynomials were introduced by George Birkhoff [11] for planar maps as a possible tool for the study of the four-color conjecture. Later Hassler Whitney extended the definition to general graphs. William Tutte found a far-reaching generalization, now called the Tutte polynomial and also introduced the related Tutte–Grothendieck invariants for graphs, which can be seen as an early bridge between graph theory and algebraic geometry. A starting point of Tutte's work is the deletion–contraction operations. For a graph G and an edge e of G, let $G\backslash e$ denote the graph obtained by deleting the edge e, and G/e denote the graph obtained by contracting the edge e, that is, by merging its two vertices to a single vertex adjacent to neighbors of both. A fundamental relation for chromatic polynomials is

$$\chi_G(k) + \chi_{G/e}(k) = \chi_{G\backslash e}(k). \tag{2}$$

This relation gives an easy inductive proof of the fact that the chromatic polynomial is indeed a polynomial. A graph H is called a minor of a graph G if it can be obtained from G by a sequence of deletions and contractions. Richard Stanley proved [58] that $\chi_G(-1)$ equals the number of acyclic orientations of G.

1.2. Read's conjecture

In 1968 Ronald Read [53] proposed the following conjecture. Suppose that

$$\chi_G(x) = a_n x^n - a_{n-1}x^{n-1} + \cdots + (-1)^i a_{n-i} x^{n-i} + \cdots ; \tag{3}$$

then, the sequence $a_0, a_1, \ldots, a_n$ is unimodal.

A much more general conjecture was posed a short time later by Andrew Heron, Gian-Carlo Rota, and Dominic Welsh. They also conjectured a stronger statement, namely, that the sequence of coefficients is actually log-concave:

$$a_k^2 \geq a_{k-1}a_{k+1}, \quad k = 1, 2, \ldots, n-1. \tag{4}$$

Theorem 2 (June Huh [34]). *The coefficients of the chromatic polynomial $\chi_G(x)$ of every graph G are log-concave.*

The unimodality and log-concavity of sequences arising in combinatorics and algebra have been studied by many researchers and in this context I would like to refer the reader to the survey articles [16, 17, 61]. A stronger property than log-concavity of the coefficients of real polynomials is that of having only real roots. This is not the case for chromatic polynomials of graphs in general (but the location of the roots is still a fascinating topic). Unimodality of the numbers of elements according to their heights in general graded posets is also related to the important "Sperner property" of posets. We note that there are cases where unimodality was expected but failed, e.g., unimodality of face numbers of polytopes [12], and of Young lattices [62].

I first heard about Huh's startling proof of the Read conjecture from a 2011 paper by Jiří Matoušek [45] who regarded this result, among a few other results, as the beginning of a new era in discrete geometry and wrote:

> *"To me, 2010 looks as annus mirabilis, a miraculous year, in several areas of my mathematical interests. Below I list seven highlights and breakthroughs, mostly in discrete geometry, hoping to share some of my wonder and pleasure with the readers."*

Huh's proof relied on connections of the problem to singularities of local analytic functions and ultimately to mixed multiplicities of certain ideals. In his proof Huh related the coefficients of the chromatic polynomial to the Milnor numbers of a complex hyperplane arrangement associated with the graph G and, as we discuss in the next section, his proof extends to arbitrary complex hyperplane arrangements. Huh's connection between chromatic polynomials of graphs and algebraic geometry was, on the one hand, a complete surprise but, on the other hand, it tied in with several developments in and around algebraic combinatorics dating to the mid-1970s. Huh's subsequent discoveries where he further applied algebraic geometry and especially Hodge theory to combinatorics, beautifully combined new and old ideas.

2. MATROIDS AND THE HERON–ROTA–WELSH CONJECTURE

2.1. Matroids

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of points in some vector space. We can associate with X:

- The set of linearly independent subsets of X.
- The set of bases of X (a base is a maximal independent set).
- The set of circuits of X (a circuit is a minimal dependent set).
- The set of flats of X (a flat is a subset that is closed under linear combination).
- The rank function that associates to a subset Y of X the dimension of the vector space spanned by Y.

Matroids were introduced by Hassler Whitney [65] as a generalization of configurations of points in linear spaces or as an abstraction of the notion of linear dependence. Matroid theory is an example of both a highly successful abstraction and a source of very useful and explicit examples. Matroid theory has various connections to the theory of algorithms and mathematical optimization, and also to mathematical logic.

Each of the five notions we mentioned above, independent sets, bases, circuits, flats, and rank functions, give rise to an axiomatic definition of matroids (and all these axiomatic definitions are equivalent). The definition of matroids based on independent sets is given by the following properties:

(1) Subsets of independent sets are themselves independent.

(2) For every subset Y of X, all maximal independent subsets of Y have the same cardinality.

The first property means that the set of independent sets is an abstract simplicial complex while the second property asserts that for every subset Y of the ground set X, the induced complex on Y is pure.

For an abstract simplicial complex K on a ground set X, we can define its dual (also called its blocker) by

$$K^* = \{S \subset X : X \backslash S \not\subset M\}.$$

If M is a matroid, we can define its dual as the matroid whose independent set complex is the dual of the independent set complex of M.

2.2. From graphs to matroids

Let G be a (connected) graph on n vertices $\{v_1, v_2, \ldots, v_n\}$, and suppose that $e_1, e_2, \ldots, e_n$ is the standard basis in an n-dimensional vector space over a field F. We associate to every edge $e = \{v_i, v_j\}$, $i < j$ the vector $e_i - e_j$. Remarkably, we get the same matroid for every field we start with. This matroid is called the graphic matroid associated

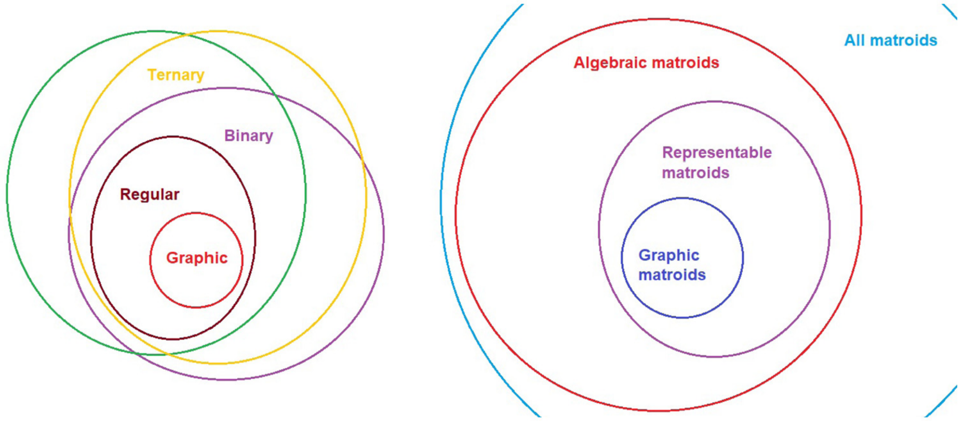

FIGURE 1
Important classes of matroids. (Right) Matroids also provide an abstraction of the notion of algebraic dependence. The large and mysterious class of algebraic matroids consists of matroids that can be represented by algebraic dependence relations over some field. (Left) Tutte characterized graphic matroids in terms of forbidden minors. Regular matroids are those matroids that can be represented over every field, and Paul Seymour [56] developed a structure theory for this class. Jim Geelen, Bert Gerards, and Geoff Whittle (see [28]) have recently proved that matroids represented over every field are characterized by a finite list of forbidden minors.

with G. It is easy to see that in this case, bases correspond to spanning trees, circuits correspond to simple cycles, independent sets correspond to spanning forests, and the rank function for subgraph H that corresponds to a set of edges is n minus the number of the connected components of H.

If M is a graphic matroid, the dual matroid need not be graphic. However, for planar graphs the dual matroid is the matroid associated to the dual graph. The notion of deletion and contraction extend from graph theory to matroid theory. (Indeed, these two operations are dual under matroid duality.)

2.3. Rank functions, characteristic polynomials, and the Heron–Rota–Welsh conjecture

The rank function of a matroid associates a nonnegative integer $r(Y)$ to every subset $Y \subset X$, with the following properties:

(i) $r(\emptyset) = 0$,

(ii) $r(A \cup B) \leq r(A) + r(B) - r(A \cap B)$,

(iii) $r(A) \leq r(A \cup \{b\}) \leq r(A) + 1$.

The characteristic function of a matroid M with ground set X is defined as follows:

$$\chi_M(\lambda) := \sum_{S \subseteq E} (-1)^{|S|} \lambda^{r(M)-r(S)}. \tag{5}$$

If M is a graphic matroid for the graph G, then $\chi_M(\lambda)$ is the chromatic polynomial of G.

Theorem 3 (Adiprasito, Huh, and Katz [2]). *The coefficients of the characteristic polynomial of a matroid M are log-concave.*

June Huh [34] proved the results for matroids (regarded as hyperplane arrangements) representable over a field of characteristic 0 and, as we mentioned above, the proof uses the Milnor numbers of the arrangement. The proof by Huh and Katz [37] for the case of an arbitrary characteristic relied on the intersection theory of "wonderful compactification" defined by Corrado De Concini and Claudio Procesi [21] for complements of hyperplane arrangements combined with an inequality of Askold Khovanskii and Bernard Teissier.

Adiprasito, Huh, and Katz [2] proved the full result. This requires far-reaching extensions of results from algebraic geometry to cohomology rings of algebraic varieties that do not exist. Here is the description of one of the early steps in the argument: the original definition of De Concini and Procesi of the "wonderful compactification" applied to realizable matroids, but Feichtner and Yuzvinsky defined in 2004 [25] a commutative ring associated to an arbitrary matroid that specializes to the cohomology ring of a wonderful compactification in the realizable case.

Let me quote from [2]: "After the completion of [37], it was gradually realized that the validity of the Hodge–Riemann relations for the Chow ring of M is a vital ingredient for the proof of the log-concavity conjectures. While the Chow ring of M could be defined for arbitrary [matroid] M, it was unclear how to formulate and prove the Hodge–Riemann relations. From the point of view of [25], the ring $A^*(M)_{\mathbb{R}}$ is the Chow ring of a smooth, but noncompact toric variety $X(\Sigma_M)$, and there is no obvious way to reduce to the classical case of projective varieties."

We will discuss some of the algebraic geometry aspects in Section 4. We note that the algebraic results of [2] actually apply to more general geometric objects well beyond matroids. For more on matroid theory see [7, 32, 43, 51, 64].

3. THE DOWLING–WILSON CONJECTURE

3.1. Background: Theorems by de Bruijn–Erdős, Motzkin, Greene, and Ryser's linear algebraic proof

Theorem 4. *A set of n points in the plane not all on the same line determines at least n lines.*

Here we say that a configuration of points determines a line ℓ if the line contains two (distinct) points from the configuration.

Proof. The Gallai–Sylvester theorem asserts that there exists a line that contains precisely two points of the configuration. The theorem now follows by induction when you delete one of these two points from the configuration. ■

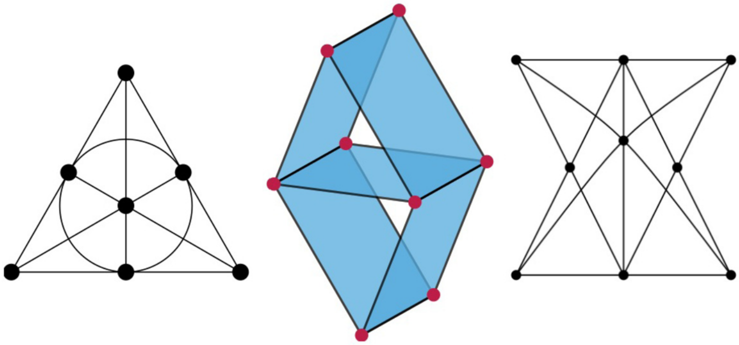

FIGURE 2
Important examples of matroids. From left to right: The Fano matroid, the Vámos matroid, and the non-Pappus matroid. The points of the Fano plane violate the Gallai–Sylvester theorem, hence it is not representable over the reals. As a matter of fact, the Fano matroid is representable over a field F if and only the characteristic of F is 2. The Vámos matroid is not algebraic. Pappus ancient theorem implies that the non-Pappus matroid is not representable over any field. Bernt Lindström proved that it is algebraic. Picture credit: Wikipedia and the "matroid union" blog.

The assertion of the Gallai–Sylvester theorem does not apply over characteristic two as seen by the Fano plane, nor does it apply for the complex plane. By contrast, the proof by Nicolaas de Bruijn and Paul Erdős uses an abstract combinatorial reasoning that is based only on the very first axiom of Euclid: "Every two points span a unique line." An algebraic proof of the theorem was given by Herbert Ryser [55].

Proof. Ryser's proof Consider the 0–1 *incidence matrix* with rows corresponding to points in the configuration and columns to lines determined by these points. Suppose that the columns of the incidence matrix are $c_1, c_2, \dots, c_m$. Note that the inner product of every two distinct rows is one. Write $b_i = \langle c_i, c_i \rangle$ for the number of points on the ith line ($b_i > 1$). Suppose that

$$\sum \alpha_i c_i = 0.$$

We write

$$0 = \left\langle \sum \alpha_i c_i, \sum \alpha_i c_i \right\rangle = \sum \alpha_i^2 (b_i - 1) + \left(\sum \alpha_i\right)^2.$$

It follows that the rows are linearly independent and therefore we must have $m \leq n$. ∎

Ryser's proof was a starting point for many algebraic proofs in combinatorics. We leave it as an exercise to show that it implies that there is bijection $\psi(p)$ from points to lines such that $p \in \psi(p)$.

Theodore Motzkin considered the theorem in higher dimensions. He conjectured (already in his 1936 thesis) that n points in a d-dimensional space that affinely span the space span at least n hyperplanes. Motzkin himself proved the result, as well as an extension of the Gallai–Sylvester theorem for configurations in higher-dimensional real vector spaces [49]. Curtis Greene [30] proved a stronger theorem: there is a one-to-one map ψ from every point p to a hyperplane containing p.

Let us now move to matroids of rank d. (Note that affine dependence of points in a d-dimensional vector space describe a matroid of rank $d+1$.) In 1974 Thomas Dowling and Richard Wilson conjectured that

$$w_i \leq w_{d-i}, \quad \text{whenever } i < d - i. \tag{6}$$

This conjecture is referred to as the *top-heavy conjecture*.

3.2. The proof of the Dowling–Wilson conjecture

Theorem 5 (Braden, Huh, Matherne, Proudfoot, and Wang 2020 [13]). *Let M be a matroid, and let $\mathcal{L}^k(M)$ denote the set of k-flats of M; then, for any k, j, $k \leq j \leq \operatorname{rank}(M) - k$:*

(1) *The cardinality of $\mathcal{L}^k(M)$ is at most the cardinality of $\mathcal{L}^j(M)$.*

(2) *There is an injective map ψ from $\mathcal{L}^k(M)$ to $\mathcal{L}^j(M)$, satisfying $F \subset \psi(F)$.*

An additional result from the same paper asserts that if Γ is any group acting on M, then

(3) There is an injective map ψ from $\mathbb{Q}\mathcal{L}^k(M)$ to $\mathcal{L}^j(M)$, of permutation representation of Γ.

The case of representable matroids was proved earlier by Huh and Wang 2017 [39]. The paper [13] also gives consequences for Kazhdan–Lusztig polynomials of matroids (introduced by Elias and Proudfoot).

We note that it is still an outstanding open question (even for representable matroids) that the sequence $w_1, w_2, \ldots, w_n$ is log-concave. It is not even known for rank-3 matroids that

$$w_2^2 \geq w_1 w_3, \tag{7}$$

and this is referred to as the "point–lines–planes" conjecture. A stronger form of this conjecture (due to Mason) asserts that

$$w_2^2 \geq \frac{3}{2} \frac{w_1 - 1}{w_1 - 2} w_1 w_3.$$

In 1982 Paul Seymour [57] proved this conjecture for matroids having no five points on a line.

4. THE CONNECTION WITH HODGE THEORY AND ALGEBRAIC GEOMETRY

4.1. Three fundamental ideas and other ingredients from the proof of the Heron–Rota–Welsh conjecture

"I like the solution even more than the problem."

June Huh at a lecture at ICERM, 2015.

In his 2015 lecture at ICERM (see also [36]), June Huh explained three fundamental ideas that were used in the proof of the general Heron–Rota–Welsh conjecture:

(1) The idea of Bernd Sturmfels that a matroid can be viewed as a tropical linear space.

Indeed, tropical geometry provided both a necessary framework and insights into the solution. Briefly, tropical mathematics replaces traditional addition with the operation of "taking the minimum," and multiplication with ordinary addition. This idea arose in several areas of mathematics and in physics and it played an important role in enumerative algebraic geometry. (For more details, see [36] and Section 5.4 and appendix C of [50].)

(2) The idea of Richard Stanley [60] that a polarized Hodge structure on the cohomology of projective toric varieties produces important combinatorial inequalities.

Here the main example was the g-theorem for convex polytopes ([10, 46, 59]) where Stanley used the hard Lefschetz theorem for the cohomology ring. Another notable example was Stanley's proof of the Erdős–Moser conjecture.

(3) The idea of Peter McMullen [47, 48] that the g-conjecture can be proved entirely within the realm of convex polytope theory using the "flip connectivity" of simplicial polytopes of a given dimension.

Any two simplicial polytopes are connected by a sequence of "flips" (also known as "Pachner moves") and McMullen proved that the validity of the hard Lefschetz theorem and the Hodge–Riemann relations are preserved under flips.

In the same lecture, June Huh mentioned quite a few more ideas by many people working in algebraic combinatorics and in algebraic geometry that play a role in the proof. We already mentioned Tessier, Khovanskii, De Concini and Procesi, and Fleischer and Yuzvinsky, and Huh mentioned also Federico Ardila and Caroline Klivans [8], Angela Gibney and Diane Maclagan [29], Kalle Karu [40], and William Fulton and Robert MacPherson [26, 27]. Of course, the proof involved a large number of additional original (at times crazy) ideas by Adiprasito, Huh, and Katz themselves.

4.2. Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations

Hodge theory gives rise to three conjectures (PD), (HL), and (HR), referred to as the standard conjectures, for certain algebras associated with geometric and combinatorial objects:

(PD) stands for the Poincaré duality, and it asserts that certain vector spaces A_i and A_{d-i} are dual (and thus have the same dimension).

(HD) stands for hard Lefschetz theorem and it asserts that certain linear maps ϕ_k from A_k to $A_k + 1$ have the property that their composition from A_i all the way to A_{d-i} is an injection.

(HR) stands for the Hodge–Riemann relations. (PD) and (HD) imply that a certain bilinear form is nondegenerate and (HR) is a stronger statement that this form is definite.

For the case of smooth projective algebraic variety M, we can consider its cohomology ring $A_i = H^{2i}(M)$. (For the case of singular algebraic varieties, that come into play in the strongest versions of the Dowling–Wilson conjecture, we need to use intersection cohomology.)

In [35] June Huh considered five examples (we are somewhat imprecise here): the cohomology of a compact Kähler manifold, the ring of algebraic cycles modulo homological equivalence on a smooth projective variety, McMullen's algebra generated by the Minkowski summands of a simple convex polytope, the combinatorial intersection cohomology of a convex polytope, the reduced Soergel bimodule of a Coxeter group element, and the Chow ring of a matroid. The only case among these examples where the standard conjectures are not known is in their original appearance in Grothendieck's work [31] toward the Weil conjectures. The example of Soergel bimodules is related to the celebrated 2014 solution of the Kazhdan–Lusztig conjecture for general Coxeter groups by Ben Elias and Geordie Williamson [23]. While it may be premature to expect it, it is not premature to hope that some connections will be found between the combinatorial appearances of the standard conjectures and their appearances in representation theory and number theory.

Remarks. (1) The proof of the Heron–Rota–Welsh conjecture by June Huh and his collaborators largely exploits "positivity," namely the Hodge–Riemann relations. For another central problem in algebraic combinatorics, the "g-conjecture for spheres," positivity is no longer available, and remarkable techniques to replace it and thus prove the conjecture were recently developed first by Adiprasito [1] (the "Hall–Laman property"), subsequently by Stavros Argyrios Papadakis and Vasiliki Petrotou [52] (the "anisotropy property"), and ultimately by Adiprasito, Papadakis, and Petrotou [3]. (See also Kalle and Elizabeth Xiao [41] for a simplified proof.)

(2) The work of Karu ([40]) on a hard Lefshetz theorem for general polytopes (see also [9,18]), of Elias and Williamson [23] on the Kazhdan–Lusztig conjecture, and of Braden, Huh, Matherne, Proudfoot, and Wang [13] on the Dowling–Wilson conjecture rely on (HL) and (HR), not for (combinatorial extensions of) the ordinary homology but for (combinatorial extensions of) Goresky and MacPherson's intersection homology.

5. THE STRONG MASON CONJECTURE (ON INDEPENDENCE NUMBERS), AND RELATED DEVELOPMENTS AND APPLICATIONS

5.1. Mason conjecture, regular strength, strong, and ultra-strong

Let M be an n-element matroid and let $i_k(M)$ denote the number of independent sets of M of size k. The Mason conjecture [44] comes in several strengths.

The Mason conjecture:

$$i_k^2(M) \geq i_{k-1}(M) i_{k+1}(M).$$

The strong Mason conjecture:

$$i_k^2(M) \geq \left(1 + \frac{1}{k}\right) i_{k-1}(M) i_{k+1}(M).$$

The ultra-strong Mason conjecture:

$$i_k^2(M) \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) i_{k-1}(M) i_{k+1}(M).$$

Mathias Lenz showed [42] how to derive the Mason conjecture for representable matroids, based on the work of Huh and Katz. Adiprasito, Huh, and Katz showed how to derive the Mason conjecture from their Hodge theory techniques and Huh, Schröter, and Wang extended these techniques to prove the strong Mason conjecture. The ultra-strong conjecture was proved in parallel by direct combinatorial reasoning by Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant [5] and based on Hodge theory by Brändén and Huh [14, 15].

June Huh's results of the past decade have led to much further research on the unimodality and log-concavity of various sequences arising in combinatorics. In some cases new combinatorial proofs were found. Let me refer the reader to recent papers by Swee Hong Chan and Igor Pak [19, 20].

5.2. The Mihail–Vazirani conjecture

For a matroid M on a ground set X, consider a graph whose vertices are all bases of the matroids and two bases are adjacent if their symmetric difference has two elements. Milena Mihail and Umesh Vazirani conjectured that for every set Y of vertices in this graph, the number of edges between Y to its complement $\bar{Y}$ is at least $\min(|Y||, \bar{Y}|)$.

If M consists of the elements of the standard basis in $\mathbb{R}^d$ and their negatives, then the graph we obtain is the graph of the discrete n-dimensional discrete cube and the assertion of the Mihail–Vazirani conjecture is a well-known isoperimetric inequality of the discrete cube.

In a pioneering 1992 paper, Tomás Feder and Milena Mihail [24] proved the conjecture for balanced matroids. In 2018 Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant [4] proved the Mihail–Vazirani conjecture. Their proof relied on the Adiprasito–Huh–Katz paper, although gradually they were able to find elementary proofs not depending on Hodge theory of crucial inequalities they needed. Their result leads to a polynomial-time algorithm to approximate the number of bases in a matroid.

CONCLUSION

June Huh found striking connections between algebraic geometry and combinatorics, solved central problems in combinatorics that had remained open for decades, and developed a theory of great importance for both fields. In my review, I naturally concentrated on the combinatorial side of the story. I did not describe in this review the connection with tropical geometry, a major area both in algebraic combinatorics and algebraic geometry. The reader is also referred to Huh's papers to learn about the theory of Lorentzian polynomials developed by June Huh and his coauthors.

It is a great pleasure to congratulate June Huh for his spectacular achievements.

FUNDING

Supported by ERC grant 834735 and by an ISF grant 2669/21.

REFERENCES

[1] K. Adiprasito, Combinatorial Lefschetz theorems beyond positivity. 2019, arXiv:1812.10454v4.

[2] K. Adiprasito, J. Huh, and E. Katz, Hodge theory for combinatorial geometries. *Ann. of Math.* **188** (2018), 381–452.

[3] K. Adiprasito, S. A. Papadakis, and V. Petrotou, Anisotropy, biased pairings, and the Lefschetz property for pseudomanifolds and cycles. 2021, arXiv:2101.07245v2.

[4] N. Anari, S. O. Gharan, and C. Vinzant, Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids. *Duke Math. J.* **170** (2021), 3459–3504. (Preliminary version, FOCS 2018.)

[5] N. Anari, S. Oveis Gharan, K. Liu, and C. Vinzant, Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids. 2018, arXiv:1811.01600.

[6] K. Appel and W. Haken, *Every planar map is four-colorable*, with the collaboration of John Koch. Contemp. Math. 98, Providence, RI, American Mathematical Society, 1985.

[7] F. Ardila, G. Denham, and J. Huh, Lagrangian geometry of matroids. *J. Amer. Math. Soc.* (2022).

[8] F. Ardila and C. Klivans, The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B* **96** (2006), 38–49.

[9] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, and L. Kaup, Combinatorial duality and intersection product: a direct approach. *Tohoku Math. J.* **57** (2005), 273–292.

[10] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes. *J. Combin. Theory Ser. A* **31** (1981), 237–255.

[11] G. Birkhoff, A determinant formula for the number of ways of coloring a map. *Ann. of Math.* **14** (1912), 42–46.

[12] A. Björner, The unimodality conjecture for convex polytopes. *Bull. Amer. Math. Soc.* **4** (1981), 187–188.

[13] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang, Singular Hodge theory for combinatorial geometries. 2022, arXiv:2010.06088v3.

[14] P. Brändén and J. Huh, Hodge–Riemann relations for Potts model partition functions. 2018, arXiv:1811.01696.

[15] P. Brändén and J. Huh, Lorentzian polynomials. 2019, arXiv:1902.03719.

[16] F. Brenti, *Unimodal, log-concave and Pólya frequency sequences in combinatorics*. Mem. Amer. Math. Soc. 413, 1989.

[17] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. In *Jerusalem Combinatorics '93*, edited by H. Barcelo and G. Kalai, pp. 71–89, Contemp. Math. 178, American Mathematical Society, Providence, RI, 1994.

[18] P. Bressler and V. A. Lunts, Hard Lefschetz theorem and Hodge–Riemann relations for intersection cohomology of nonrational polytopes. *Indiana Univ. Math. J.* **54** (2005), 263–307.

[19] S. H. Chan and I. Pak, Log-concave poset inequalities. 2021, arXiv:2110.10740v2.

[20] S. H. Chan and I. Pak, Introduction to the combinatorial atlas. 2022, arXiv:2203.01533.

[21] C. De Concini and C. Procesi, Wonderful models of subspace arrangements. *Selecta Math. (N.S.)* **12** (1995), 459–494, 36–70.

[22] T. Dowling and R. Wilson, Whitney number inequalities for geometric lattices. *Proc. Amer. Math. Soc.* **47** (1975), 504–512.

[23] B. Elias and G. Williamson, The Hodge theory of Soergel bimodules. *Ann. of Math.* **180** (2014), 1089–1136.

[24] T. Feder and M. Mihail, Balanced matroids. In *Proc. 24th Annual ACM Symposium on Theory of Computing*, pp. 26–38, ACM Press, 1992.

[25] E. M. Feichtner and S. Yuzvinsky, Chow rings of toric varieties defined by atomic lattices. *Invent. Math.* **155** (2004), 515–536.

[26] W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels, Intersection theory on spherical varieties. *J. Algebraic Geom.* **4** (1995), 181–193.

[27] W. Fulton and B. Sturmfels, Intersection theory on toric varieties. *Topology* **36** (1997), 335–353.

[28] J. Geelen, B. Gerards, and G. Whittle, Solving Rota's conjecture. *Notices Amer. Math. Soc.* **61** (2014), 736–743.

[29] A. Gibney and D. Maclagan, Lower and upper bounds on nef cones. *Int. Math. Res. Not.* **14** (2012), 3224–3255.

[30] C. Greene, A rank inequality for finite geometric lattices. *J. Combin. Theory* **9** (1970), 357–364.

[31] A. Grothendieck, Standard conjectures on algebraic cycles. In *Algebraic Geometry*, pp. 193–199, Oxford University Press, 1969.

[32] Y. O. Hamidoune and I. Salaün, On the independence numbers of a matroid. *J. Combin. Theory Ser. B* **47** (1989), 146–152.

[33] A. Heron, Matroid polynomials. In *Combinatorics (Proceedings of the Conference on Combinatorial Mathematics, Mathematics Institute)*, pp. 164–202, Oxford, 1972.

[34] J. Huh, Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. *J. Amer. Math. Soc.* **25** (2012), 907–927.

[35] J. Huh, Combinatorial applications of the Hodge–Riemann relations. In *Proceedings of the International Congress of Mathematicians, Vol. IV. Invited lectures*, pp. 3093–3111, World Scientific Publishers, Hackensack, NJ, 2018.

[36] J. Huh, Tropical geometry of matroids. In *Current Developments in Mathematics 2016*, pp. 1–46, International Press, 2018.

[37] J. Huh and E. Katz, Log-concavity of characteristic polynomials and the Bergman fan of matroids. *Math. Ann.* **354** (2012), 1103–1116.

[38] J. Huh, B. Schröter, and B. Wang, Correlation bounds for fields and matroids. *J. Eur. Math. Soc. (JEMS)* **24** (2022), 1335–1351.

[39] J. Huh and B. Wang, Enumeration of points, lines, planes, etc. *Acta Math.* **218** (2017), 297–317.

[40] K. Karu, Hard Lefschetz theorem for nonrational polytopes. *Invent. Math.* **157** (2004), 419–447.

[41] K. Karu and E. Xiao, On the anisotropy theorem of Papadakis and Petrotou. 2022, arXiv:2204.07758v2.

[42] M. Lenz, The f-vector of a representable-matroid complex is log-concave. *Adv. in Appl. Math.* **51** (2013), 543–545.

[43] L. Lovász, Matroid matching and some applications. *J. Combin. Theory Ser. B* **28** (1980), 208–236.

[44] J. Mason, Matroids: Unimodal conjectures and Motzkin's theorem. In *Combinatorics (Proceedings of the Conference on Combinatorial Mathematics)*, pp. 207–220, Mathematics Institute, Oxford, 1972.

[45] J. Matoušek, The dawn of an algebraic era in discrete geometry? In *Proceedings of the 27th European Workshop on Computational Geometry (EuroCG'11)*, 2011.

[46] P. McMullen, The numbers of faces of simplicial polytopes. *Israel J. Math.* **9** (1971), 559–570.

[47] P. McMullen, On simple polytopes. *Invent. Math.* **113** (1993), 419–444.

[48] P. McMullen, Weights on polytopes. *Discrete Comput. Geom.* **15** (1996), 363–388.

[49] T. Motzkin, The lines and planes connecting the points of a finite set. *Trans. Amer. Math. Soc.* **70** (1951), 451–464.

[50] A. Okounkov, Combinatorial geometry takes the lead. In *Proceedings of the International Congress of Mathematicians, Vol. 1*, pp. 414–458, EMS Press, 2022.

[51] J. Oxley, *Matroid theory*. Oxf. Grad. Texts Math., Oxford University Press, Oxford, 2011.

[52] S. A. Papadakis and V. Petrotou, The characteristic 2 anisotropicity of simplicial spheres. 2020, arXiv:2012.09815.

[53] R. Read, An introduction to chromatic polynomials. *J. Combin. Theory* **4** (1968), 52–71.

[54] G.-C. Rota, Combinatorial theory, old and new. In *Actes du Congress International des Mathématiciens (Nice, 1970)*, Tome 3, pp. 229–233, Gauthier-Villars, Paris, 1971.

[55] H. J. Ryser, An extension of a theorem of de Bruijn and Erdős on combinatorial designs. *J. Algebra* **10** (1968), 246–261.

[56] P. D. Seymour, Decomposition of regular matroids. *J. Combin. Theory Ser. B* **28** (1980), 305–359.

[57] P. D. Seymour, On the points-lines-planes conjecture. *J. Combin. Theory Ser. B* **33** (1982), 17–26.

[58] R. P. Stanley, Acyclic orientations of graphs. *Discrete Math.* **5** (1973), 171–178.

[59] R. P. Stanley, The number of faces of a simplicial convex polytope. *Adv. Math.* **35** (1980), 236–238.

[60] R. P. Stanley, Combinatorial applications of the hard Lefschetz theorem. In *Proceedings of the International Congress of Mathematicians, Vols. 1, 2 (Warsaw, 1983)*, pp. 447–453. PWN, Warsaw, 1984.

[61] R. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph Theory and its Applications: East and West (Jinan, 1986)*, Ann. New York Acad. Sci., vol. 576, pp. 500–535, New York, 1989.

[62] D. Stanton, Unimodality and Young's lattice. *J. Combin. Theory Ser. A* **54** (1990), 41–53.

[63] D. Welsh, Combinatorial problems in matroid theory. In *Combinatorial Mathematics and its Applications (Oxford, 1969)*, pp. 291–306, Academic Press, London, 1971.

[64] D. Welsh, *Matroid Theory*. London Math. Soc. Monogr. Ser. 8, Academic Press, London, 1976.

[65] H. Whitney, On the abstract properties of linear dependence. *Amer. J. Math.* **57** (1935), 509–533.

GIL KALAI

Hebrew University of Jerusalem, Jerusalem, Israel, and Reichman University, Herzliya, Israel, kalai@math.huji.ac.il