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Onset of fracture in random heterogeneous particle chains

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Abstract. In mechanical systems, it is of interest to know the onset of fracture in dependence of the boundary conditions. Here we study a one-dimensional model which allows for an underlying heterogeneous structure in the discrete setting. Such models have recently been studied in the passage to the continuum by means of variational convergence (Γ-convergence). The Γ-limit results determine thresholds of the boundary condition, which mark a transition from purely elastic behavior to the occurrence of a crack. In this article, we provide a notion of fracture in the discrete setting and show that its continuum limit yields the same threshold as that obtained from the Γ-limit. Since the calculation of the fracture threshold is much easier with the new method, we see a good chance that this new approach will turn out useful in applications.

1. Introduction

The mechanical behavior of one-dimensional systems has been of interest for decades. Such systems serve as toy models for higher-dimensional theoretical investigations and are of interest with respect to one-dimensional structures; see, e.g., [8,9,11,12,21]. In order to understand the effective behavior of materials, the systems are studied as the number of particles tends to infinity.

In this article, we focus on the occurrence of cracks and continue a mathematical analysis of the effective behavior of one-dimensional discrete systems in the passage to the continuum. In particular, we strive for insight into the threshold for the overall prescribed length ℓ of a chain. If ℓ is smaller than the threshold, the system will show elastic behavior, whereas cracks are energetically favored if ℓ is larger than the threshold. The interaction potentials between the particles or atoms of the discrete chain are allowed to be in a large class of convex-concave potentials, which include for instance the classical Lennard-Jones potentials. The system is then modeled with

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the help of an energy functional that is the sum of all the interaction potentials; see (2.1). Here we restrict to the interactions of nearest neighbors; for related studies with interactions beyond nearest neighbors we refer to [4, 5, 20].

In view of misplaced atoms or of chains consisting of several different kind of particles, we allow for a random distribution of the interaction potentials; see Assumption 2.1 and (2.2) for details. The limit passage is then also referred to as stochastic homogenization; cf., e.g., [1, 7, 10, 17]. As a special case, also materials with a periodic heterostructure are included; cf. also [15].

An appropriate mathematical technique for the passage of energy functionals from discrete to continuous systems is based on the notion of Γ-convergence, which is a notion of a variational convergence and (under coercivity assumptions) ensures that minimizers of the discrete system converge to minimizers of the system in the continuum limit; see, e.g., [2, 3, 19] and references cited therein. As the number of particles tends to infinity, the energy functional converges to a functional that allows for describing cracks. In particular, it is shown that cracks in the continuum limit emerge if a critical stretch is exceeded. On the other hand, on the discrete level a similar notion of a "critical stretch" or a notion for the onset of a crack has not been introduced so far.

In this article, which is partially based on the PhD thesis [13, Chapter 7] of L. Lauerbach, we focus on the emergence of cracks in atomistic chains. On the level of the continuum limiting model of the chain, "crack" has a clear meaning – it is the point where the continuum deformation features a jump and there is no interaction between the different segments separated by the jump. In contrary, on the level of a discrete chain with $n + 1$ particles, the notion of "crack" cannot be unambiguously defined, since always neighboring particles interact. In the present paper, we introduce a notion of "onset of a crack" at the discrete level for a chain with $n + 1$ particles. For simplicity, we discuss the key idea in the case of a chain with $n + 1$ particles that is composed of (random) potentials that are convex around its ground state and otherwise concave, i.e., for deformations larger than an inflection point z_{frac}. We call a deformation u *elastic* if the individual interaction potentials along the chain are only evaluated in their convex region. In contrary, a deformation that is *not* elastic invokes at least one bond that "lives" in the concave region of the corresponding potential. Next, we consider the energy minimizers u_n of the chain with $n + 1$ particles and prescribed total length $\ell > 0$. If the minimizers u_n are elastic for all $n \in \mathbb{N}$, then we do not expect the occurrence of crack in the continuum limit; while in the other case, we expect that minimizing sequences show a concentration of strain on a finite number of weak bonds and thus a "crack" emerges in the continuum limit. Based on these heuristics, we introduce a "critical stretch" ℓ_n^* for random chains with $n + 1$ particles. Firstly, we prove that it converges, for $n \to \infty$, to the jump-threshold predicted by the zeroth-order Γ-limit of the discrete energy, which has been obtained earlier in [14].

Secondly, we establish a first-order expansion of the critical stretch and show that the coefficients of the expansion term agree with the values predicted by the first-order Γ-limit of the discrete energy derived in [14]. Since the proofs in [14] are technically quite involved, it is interesting to learn that there is a much simpler method for the derivation of the jump threshold in the continuum limit. We expect that the new notions of a fracture point and of a jump threshold in the discrete setting turn out to be useful also in a wider class of applications. They might be compared to the Γ-convergence analysis of weak-membrane and Blake–Zisserman models in [6, 18], which invoke a combination of piecewise affine and piecewise constant interpolations that require the identification of strain concentration on the discrete level as well.

The outline of this article is as follows: in Section 2, we introduce the model in the discrete setting, including the assumptions on the large class of interaction potentials in the random setting. Further, we provide the definition of a critical stretch (Definition 2.1), which corresponds to the jump threshold. We assert the asymptotic behavior of the critical stretch as the number of particles tends to infinity (Theorem 2.1) and compare the limit to the corresponding Γ-convergence results. Moreover, we consider a rescaled setting, define the rescaled jump threshold, and assert its asymptotic behavior as n tends to infinity (Theorem 2.2). Finally, we compare also this result with the corresponding Γ-convergences result. All proofs are provided in Section 3.

2. Setup and main results

We consider a chain of $n+1$ atoms that in a reference configuration are placed at the sites in $\frac{1}{n}\mathbb{Z} \cap [0,1]$; see Figure 1. The deformation of the atoms is referred to as

$$u_n : \frac{1}{n}\mathbb{Z} \cap [0,1] \to \mathbb{R}.$$

For the passage from discrete systems to their continuous counterparts, it is useful to identify the discrete functions with their piecewise affine interpolations, more precisely, with the functions in

$$\mathcal{A}_n := \Big\{ u \in C\big([0,1]\big) : u \text{ is affine on } (i, i+1)\frac{1}{n},\ i \in \{0, 1, \ldots, n-1\}, \\ \text{and monotonically increasing} \Big\}.$$

We shall also consider clamped boundary conditions for the chain and thus introduce for $\ell > 0$ the set

$$\mathcal{A}_{n,\ell} := \big\{ u \in \mathcal{A}_n : u(0) = 0,\ u(1) = \ell \big\}.$$

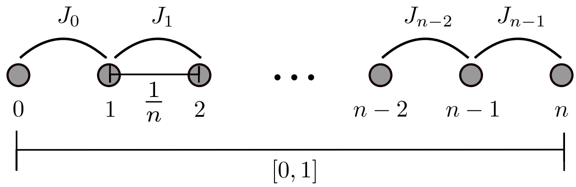

Figure 1. Chain of $n+1$ atoms with reference position $\frac{i}{n}$. The potential J_i describes the nearest neighbor interaction of atom i and $i+1$, $i=0,\dots,n-1$. The characteristic length scale is $\frac{1}{n}$ and the interval is $[0,1]$.

We consider a discrete energy functional of the form

$$\begin{aligned} \mathcal{A}_{n,\ell} \ni u \mapsto E_n(u) &:= \sum_{i=0}^{n-1} \frac{1}{n} J_i\left(\frac{u\left(\frac{i+1}{n}\right)-u\left(\frac{i}{n}\right)}{\frac{1}{n}}\right) \\ &= \sum_{i=0}^{n-1} \frac{1}{n} J_i\left(n\left(u\left(\frac{i+1}{n}\right)-u\left(\frac{i}{n}\right)\right)\right), \end{aligned} \tag{2.1}$$

where $J_i : (0,\infty) \to \mathbb{R}$ is a potential describing the interaction between the ith atom and its neighbor to the right. We are interested in random heterogeneous chains of atoms, and thus assume that the potentials $\{J_i\}_{i\in\mathbb{Z}}$ are random with a distribution that is stationary and ergodic. We appeal to the following standard setup: let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $(\tau_i)_{i\in\mathbb{Z}}$ a family of measurable maps $\tau_i : \Omega \to \Omega$ such that

- (Group property) $\tau_0\omega = \omega$ for all $\omega \in \Omega$ and $\tau_{i_1+i_2} = \tau_{i_1}\tau_{i_2}$ for all $i_1, i_2 \in \mathbb{Z}$,
- (Stationarity) $\mathbb{P}(\tau_i B) = \mathbb{P}(B)$ for every $B \in \mathcal{F}$, $i \in \mathbb{Z}$,
- (Ergodicity) For all $B \in \mathcal{F}$, it holds that $(\tau_i(B) = B\ \forall i \in \mathbb{Z}) \Rightarrow \mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

We then consider the energy functional

$$E_n : \Omega \times \mathcal{A}_n \to \mathbb{R} \cup \{+\infty\}$$

with

$$E_n(\omega, u) := \sum_{i=0}^{n-1} \frac{1}{n} J\left(\tau_i\omega, n\left(u\left(\frac{i+1}{n}\right)-u\left(\frac{i}{n}\right)\right)\right), \tag{2.2}$$

where the random potential satisfies the following assumptions:

Assumption 2.1. Let $J : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be jointly measurable with $J(\cdot, z) = \infty$ if $z \leq 0$. For $\mathbb{P}$-a.e. $\omega \in \Omega$, the following conditions hold true:

(A1) (Regularity) $J(\omega, \cdot) \in C^3(0,\infty)$.

(A2) (Behavior at 0 and ∞) There exist functions $\psi^+, \psi^- \in C(0,\infty)$, independent of ω, such that

$$\lim_{z\to 0^+} \psi^-(z) = \infty \quad \text{and} \quad \lim_{z\to\infty} \psi^+(z) = 0,$$

and

$$J(\omega,z) \geq \psi^-(z) \text{ for all } 0 < z \leq 1 \quad \text{and} \quad \left|J(\omega,z)\right| \leq \psi^+(z) \text{ for all } z \geq 1.$$

(A3) (Convex-monotone structure) Suppose strict convexity close to 0 in form of

$$z_{\text{frac}}(\omega) := \sup\left\{z > 0 \,:\, J''(\omega,s) := \partial_s^2 J(\omega,s) > 0 \text{ for all } s \in (0,z)\right\} > 0,$$

and assume that $J(\omega,\cdot)$ is monotonically increasing on $[z_{\text{frac}}(\omega),\infty)$.

(A4) (Non-degenerate ground state) Suppose that $J(\omega,\cdot)$ admits a unique minimizer $\delta(\omega) \in (0, z_{\text{frac}}(\omega)]$, called the ground state of $J(\omega,\cdot)$. There exists a constant $c > 0$, independent of ω, such that $\frac{1}{c} > \delta(\omega) > c$ and

$$\forall z \in \delta(\omega) + (-c,c) : c \leq J''(\omega,z) \leq \frac{1}{c} \quad \text{and} \quad \left|J'''(\omega,z)\right| \leq \frac{1}{c}.$$

Next, we introduce the following central quantities for a random heterogeneous chain with $n+1$ particles:

Definition 2.1 (Critical stretch of a chain with $n+1$ particles). Consider the situation of Assumption 2.1. Let $n \in \mathbb{N}$ and $\omega \in \Omega$. The critical stretch $\ell_n^*(\omega)$ is defined as the largest number such that

$$\inf_{\mathcal{A}_n^{\text{el}}(\omega)\cap\mathcal{A}_{n,\ell}} E_n(\omega,\cdot) = \inf_{\mathcal{A}_{n,\ell}} E_n(\omega,\cdot) \quad \text{for all } 0 \leq \ell < \ell_n^*(\omega),$$

where we denote by

$$\mathcal{A}_n^{\text{el}}(\omega) := \left\{u \in \mathcal{A}_n \,:\, \frac{u\left(\frac{i+1}{n}\right) - u\left(\frac{i}{n}\right)}{\frac{1}{n}} \leq z_{\text{frac}}(\tau_i\omega) \text{ for all } i = 0,\ldots,n-1\right\}$$

the set of purely elastic deformations.

The idea behind the above definition is the following: a deformation $u \in \mathcal{A}_n^{\text{el}}(\omega)$ only sees the strictly convex region of the interaction potentials. Thus, we could replace the potentials $J(\tau_i\omega, z)$ in the definition of the energy function E_n by (globally) convex potentials with superlinear growth without changing the energy for deformations in $\mathcal{A}_n^{\text{el}}(\omega)$. As it is well known, such energies do not allow for fracture in the continuum limit. The definition of the critical stretch implies that a prescribed macroscopic stretch (or compression) $\ell < \ell_n^*(\omega)$ can be realized by a deformation in

$\mathcal{A}_n^{\mathrm{el}}(\omega)$ and thus prohibits the formation of a jump, while, for $\ell > \ell_n^*(\omega)$, deformations with minimal energy are required to explore the non-convex region of at least one of the interaction potentials. We may refer to the bonds $[i, i+1]$ that are evaluated outside the convex region as "weak" bonds. If a jump occurs in the limit, then the minimizing sequence shows a concentration of strain in the weak bonds. We thus expect that $\ell_n^*(\omega)$ almost surely converges in the limit $n \to \infty$ to the continuum fracture threshold that can be defined on the level of the continuum Γ-limit; see below. In our first result, we prove that ℓ_n^* indeed converges and we identify its limit, which is the statistical mean of the ground states:

Theorem 2.1. *Let Assumption 2.1 be fulfilled. Then,*

$$\lim_{n\to\infty} \ell_n^*(\omega) = \mathbb{E}[\delta] \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega.$$

(The proof of Theorem 2.1 can be found in Section 3.1.)

Next, we consider the special case when $\delta(\omega)$ is deterministic, say $\delta(\omega) = 1$ for $\mathbb{P}$-a.e. In that case, we establish a first-order expansion of $\ell_n^*(\omega)$ around its limit $\mathbb{E}[\delta] = 1$ of the form

$$\ell_n^*(\omega) \approx 1 + \sqrt{\frac{1}{n}}\sqrt{\frac{\beta}{\underline{\alpha}}},$$

where β is related to the maximal energy barrier among the random potentials J, and $1/\underline{\alpha}$ is the statistical mean of the curvatures of the random potentials at the ground state.

Theorem 2.2. *Let Assumption 2.1 be satisfied and assume that $\delta(\omega) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Consider the rescaled jump threshold $\gamma_n^*(\omega) := \frac{\ell_n^*(\omega)-1}{\sqrt{\frac{1}{n}}}$. Then*

$$\lim_{n\to\infty} \gamma_n^*(\omega) = \lim_{n\to\infty} \frac{\ell_n^*(\omega) - 1}{\sqrt{\frac{1}{n}}} = \sqrt{\frac{\beta}{\underline{\alpha}}} \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega \in \Omega,$$

where

$$\underline{\alpha} := \left(\mathbb{E}\left[\left(\frac{1}{2}J''(\omega,1)\right)^{-1}\right]\right)^{-1} \quad \textit{and} \quad \beta := \operatorname*{ess\,inf}_{\omega\in\Omega}\big(-J(\omega,1)\big). \tag{2.3}$$

(The proof of Theorem 2.2 can be found in Section 3.2.)

We finally relate the above results to the zeroth- and first-order Γ-limits of E_n subject to clamped boundary conditions, i.e.,

$$E_n^\ell(\omega,\cdot) : L^1(0,1) \to \mathbb{R} \cup \{+\infty\}, \quad E_n^\ell(\omega,u) := \begin{cases} E_n(\omega,u) & \text{if } u \in \mathcal{A}_{n,\ell}, \\ +\infty & \text{else.} \end{cases}$$

The zeroth-order Γ-limit of the discrete energy yields a homogenized energy functional. In the present setting of nearest-neighbor interactions, [14] allows to characterize the homogenized energy functional by

$$E^{\ell}_{\mathrm{hom}}(u) = \int_0^1 J_{\mathrm{hom}}\big(u'(x)\big)\,\mathrm{d}x,$$

where the homogenized energy density map $z \mapsto J_{\mathrm{hom}}(z)$ is convex, lower semicontinuous, monotonically decreasing and satisfies

$$\lim_{z\to 0+} J_{\mathrm{hom}}(z) = +\infty. \tag{2.4}$$

Moreover, the minimum values of $E^{\ell}_n(\omega,\cdot)$ and E^{ℓ}_{hom} satisfy

$$\lim_{n\to\infty} \inf_u E^{\ell}_n(\omega,u) = \min_u E^{\ell}_{\mathrm{hom}}(u) = J_{\mathrm{hom}}(\ell),$$

and therefore can be calculated as

$$\min_u E^{\ell}_{\mathrm{hom}}(u) = J_{\mathrm{hom}}(\ell) = \begin{cases} J_{\mathrm{hom}}(\ell) & \text{for } \ell < \mathbb{E}[\delta], \\ J_{\mathrm{hom}}\big(\mathbb{E}[\delta]\big) & \text{for } \ell \geq \mathbb{E}[\delta]. \end{cases}$$

Hence, the threshold between the elastic and the jump regimes is $\mathbb{E}[\delta]$, which is identical to the limit of $\ell^*_n(\omega)$; see Theorem 2.1. Secondly, we recall a Γ-limit result from [16] for the rescaled energy functional

$$H^{\gamma_n}_n(\omega,v) = \begin{cases} H_n(\omega,v) & \text{if } v \in \mathcal{A}_{n,\gamma_n}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $(\gamma_n)_n$ is a sequence of non-negative numbers with $\gamma_n \to \gamma \geq 0$ and

$$H_n(\omega,v) := \sum_{i=0}^{n-1}\left(J\left(\tau_i\omega, \frac{v\big(\frac{i+1}{n}\big) - v\big(\frac{i}{n}\big)}{\sqrt{\frac{1}{n}}} + \delta(\tau_i\omega)\right) - J\big(\tau_i\omega, \delta(\tau_i\omega)\big)\right).$$

The Γ-limit is shown to be given as

$$H^{\gamma}(v) = \underline{\alpha}\int_0^1 \big|v'(x)\big|^2\,\mathrm{d}x + \beta\#S_v,$$

with homogenized elastic coefficient $\underline{\alpha}$, jump parameter β, $\#S_v$ being the number of jumps of v, and v satisfying boundary conditions which depend on γ. Moreover, it holds true that

$$\lim_{n\to\infty} \inf_v H^{\gamma_n}_n(\omega,v) = \min_v H^{\gamma}(v) = \min\{\underline{\alpha}\gamma^2, \beta\},$$

which yields that the minima of the energy are given by

$$\min_v H^\gamma(v) = \min\{\underline{\alpha}\gamma^2, \beta\} = \begin{cases} \underline{\alpha}\gamma^2 & \text{if } \gamma < \sqrt{\frac{\beta}{\underline{\alpha}}}, \\ \beta & \text{if } \gamma \geq \sqrt{\frac{\beta}{\underline{\alpha}}}. \end{cases}$$

Hence the threshold between elasticity and fracture in the rescaled case is $\sqrt{\frac{\beta}{\underline{\alpha}}}$, which equals the limit of the jump threshold γ_n^* in Theorem 2.2.

In summary, although the techniques by which the results are calculated are completely different, they yield the same result regarding the jump threshold in the continuum setting. The derivation of the limiting jump threshold with help of the newly defined jump threshold in the discrete setting is, however, much easier and thus is of interest for applications. It remains an open problem to analyze corresponding questions in higher dimensional settings. In the following section, we provide the proofs of the above theorems.

3. Proofs

For the upcoming analysis, it is convenient to introduce the notation

$$M_n(\omega, \ell) := \min\left\{\frac{1}{n}\sum_{i=0}^{n-1} J(\tau_i\omega, z^i) : \frac{1}{n}\sum_{i=0}^{n-1} z^i = \ell\right\}$$

to denote the minimum energy of a discrete chain of length ℓ. We begin with an elementary (yet, convenient) reformulation of the critical stretch ℓ_n^* (cf. Definition 2.1).

Lemma 3.1. *Consider the situation of Assumption 2.1. Let $n \in \mathbb{N}$ and $\omega \in \Omega$. Then, it holds*

$$M_n(\omega, \ell) = \min_{u \in \mathcal{A}_{n,\ell}} E_n(\omega, u). \tag{3.1}$$

Moreover, $\ell_n^*(\omega)$ is the largest number such that for all $0 < \ell < \ell_n^*(\omega)$ there exists $\bar{z} \in \mathbb{R}^n$ satisfying

$$M_n(\omega,\ell) = \frac{1}{n}\sum_{i=0}^{n-1} J(\tau_i\omega, \bar{z}^i), \quad \frac{1}{n}\sum_{i=0}^{n-1}\bar{z}^i = \ell, \quad \bar{z}^i \leq z_{\text{frac}}(\tau_i\omega) \quad \forall i \in \{0,\dots,n-1\}. \tag{3.2}$$

Proof of Lemma 3.1. The identity (3.1) follows by a simple change of variables, that is by setting

$$z^i = n\left(u\left(\frac{i+1}{n}\right) - u\left(\frac{i}{n}\right)\right),$$

and the direct method of the calculus of variations.

Next, we give an argument regarding the characterization of ℓ_n^*. The definition of $\ell_n^*(\omega)$, see Definition 2.1, and (3.1) imply that

$$\inf_{\mathcal{A}_n^{\mathrm{el}}(\omega)\cap\mathcal{A}_{n,\ell}} E_n(\omega,\cdot) = M_n(\omega,\ell) < \infty \quad \forall \ell \in \big(0, \ell_n^*(\omega)\big).$$

Since $\mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ is compact, there exists $\bar{u} \in \mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ such that

$$E_n(\omega,\bar{u}) = \inf_{\mathcal{A}_n^{\mathrm{el}}(\omega)\cap\mathcal{A}_{n,\ell}} E_n(\omega,\cdot).$$

Clearly, $\bar{z} \in \mathbb{R}^n$ defined as $\bar{z}^i = n(\bar{u}(\frac{i+1}{n}) - \bar{u}(\frac{i}{n}))$ satisfies (3.2).

Now we suppose that for some $\ell \geq \ell_n^*$ there exists $\bar{z} \in \mathbb{R}^n$ satisfying (3.2). With help of the same change of variables as above, we find $\bar{u} \in \mathcal{A}_n^{\mathrm{el}}(\omega) \cap \mathcal{A}_{n,\ell}$ satisfying $E_n(\omega,\bar{u}) = M_n(\omega,\ell)$ which contradicts the definition of ℓ_n^*. ∎

Lemma 3.2. *Let Assumption 2.1 be satisfied. Then, $J(\omega,\cdot)$ is increasing on $[\delta(\omega),\infty)$ and it holds that*

$$z_{\mathrm{frac}}^{\sup} := \sup\big\{z_{\mathrm{frac}}(\omega) : \omega \in \Omega\big\} < \infty. \tag{3.3}$$

Proof of Lemma 3.2. For convenience we drop the dependence on ω in our notation and simply write $J(z)$, δ, and z_{frac} instead of $J(\omega,z)$, $\delta(\omega)$, and $z_{\mathrm{frac}}(\omega)$, respectively. We first prove that J is increasing on $[\delta,\infty)$. On $[z_{\mathrm{frac}},\infty)$ this directly follows from (A3). On $[\delta, z_{\mathrm{frac}})$ this follows from the convexity of J on $(0, z_{\mathrm{frac}})$ and the fact that δ minimizes J. Next, we prove (3.3). We first note that (A2) and (A3) imply that

$$\forall z \in (\delta,\infty) : J(\delta) \leq J(z) \leq 0. \tag{3.4}$$

Moreover, (A4) implies that $z_{\mathrm{frac}} \geq \delta + c$. Thus, for all $\eta \in (0,c)$ we obtain

$$\begin{aligned} 0 \geq J(z_{\mathrm{frac}}) &= J(\delta+\eta) + \int_{\delta+\eta}^{z_{\mathrm{frac}}} J'(t)\,\mathrm{d}t \\ &\geq J(\delta+\eta) + J'(\delta+\eta)\big(z_{\mathrm{frac}} - (\delta+\eta)\big), \end{aligned} \tag{3.5}$$

where the second inequality holds, since J' is increasing on $(\delta+\eta, z_{\mathrm{frac}})$ thanks to (A3). (A4) yields

$$J'(\delta+\eta) = J'(\delta+\eta) - J'(\delta) = \int_{\delta}^{\delta+\eta} J''(s)\,ds \geq c\eta.$$

Thus, by rearranging terms in (3.5) and appealing to (3.4) and the previous estimate we get

$$z_{\mathrm{frac}} \leq \delta + \eta - \frac{J(\delta+\eta)}{J'(\delta+\eta)} \leq \delta + \eta - \frac{J(\delta)}{c\eta}. \tag{3.6}$$

It remains to bound $\delta = \delta(\omega)$ and $-J(\delta) = -J(\omega, \delta(\omega))$ by a constant that is independent of ω. From (A4) and (A2), we get

$$\delta \in \left(c, \frac{1}{c}\right) \quad \text{and} \quad -J(\delta) \le \max_{z \in [c, \frac{1}{c}+\eta]} \max\left\{-\psi^-(z), \left|\psi^+(z)\right|\right\} =: d < \infty, \quad (3.7)$$

and thus, (3.6) yields $z_{\text{frac}} \le \frac{1}{c} + \eta + \frac{d}{c\eta}$. ∎

3.1. Proof of Theorem 2.1

Proof of Theorem 2.1. Note that $\omega \mapsto \delta(\omega)$ is (as a minimizer of a measurable function) measurable. Moreover, by (3.7) δ is a non-negative and bounded and thus an L^1-random variable. Thus the ergodic theorem yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta(\tau_i \omega) = \mathbb{E}[\delta], \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} J\left(\tau_i\omega, \delta(\tau_i\omega)\right) = \mathbb{E}\left[J(\delta)\right] \quad (3.8)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. For the rest of the proof, we consider $\omega \in \Omega$ such that (3.8) is valid and drop the dependence on ω. In particular, we set

$$\delta_i := \delta(\tau_i\omega), \quad z^i_{\text{frac}} := z_{\text{frac}}(\tau_i\omega), \quad \text{and} \quad J_i(z) := J(\tau_i\omega, z).$$

Step 1. We show that $\bar{A} := \limsup_{n\to\infty} \ell^*_n \le \mathbb{E}[\delta]$.

Without loss of generality, we suppose that $\bar{A} = \lim_{n\to\infty} \ell^*_n$ and prove $\bar{A} \le \mathbb{E}[\delta]$ by contradiction. Assume that there exists $\varepsilon \in (0, c)$ such that $\bar{A} > \mathbb{E}[\delta] + 3\varepsilon$. By (3.8), we find that $\bar{N} \in \mathbb{N}$ such that

$$\ell^*_n > \frac{1}{n}\sum_{i=0}^{n-1} \delta_i + 2\varepsilon =: k_n \quad \text{for } n > \bar{N}. \quad (3.9)$$

In view of Lemma 3.1, there exists a sequence $(\bar{z}_n)_n$ satisfying for $n \ge \bar{N}$

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}^i_n = k_n, \quad \frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}^i_n) = M_n(k_n), \quad \bar{z}^i_n \le z^i_{\text{frac}} \quad \forall i \in \{0, \dots, n-1\}. \quad (3.10)$$

We claim that

$$\limsup_{n\to\infty} M_n(k_n) \le \mathbb{E}\left[J(\delta)\right], \quad (3.11)$$

$$\liminf_{n\to\infty} M_n(k_n) \ge \mathbb{E}\left[J(\delta)\right] + c_\varepsilon, \quad (3.12)$$

for some $c_\varepsilon > 0$. Clearly, (3.11) and (3.12) yield a contradiction. Hence the assumption $\bar{A} > \mathbb{E}[\delta] + 3\varepsilon$ is wrong and $\bar{A} \le \mathbb{E}[\delta]$ follows by the arbitrariness of $\varepsilon > 0$.

Substep 1.1. Proof of (3.11). Let $z_n \in \mathbb{R}^n$ be given by $z_n^i := \delta_i$ for $i \geq 1$ and $z_n^0 := \delta_0 + 2n\varepsilon$. Since $\frac{1}{n}\sum_{i=0}^{n-1} z_n^i = k_n$, we have

$$M_n(k_n) \leq \frac{1}{n}\sum_{i=1}^{n-1} J_i(\delta_i) + \frac{1}{n}J_0(\delta_0 + 2n\varepsilon)$$
$$= \frac{1}{n}\sum_{i=0}^{n-1} J_i(\delta_i) + \frac{1}{n}\big(J_0(\delta_0 + 2n\varepsilon) - J_0(\delta_0)\big).$$

Hence, (3.11) follows by (A2) and (3.8).

Substep 1.2. Proof of (3.12). Let $\bar{z}_n$ be as in (3.10) and set

$$I_n := \big\{i \in \{0, \dots, n-1\} : \bar{z}_n^i > \delta_i + \varepsilon\big\}.$$

Obviously, it holds that $0 \leq |I_n|/n \leq 1$ and we claim

$$\frac{|I_n|}{n} \geq \frac{\varepsilon}{z_{\text{frac}}^{\sup}} > 0 \quad \text{for all } n \in \mathbb{N}, \tag{3.13}$$

where $z_{\text{frac}}^{\sup} \in (0, \infty)$ is as in Lemma 3.2. Indeed,

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_i + 2\varepsilon = k_n = \frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = \frac{1}{n}\sum_{i\in I_n} \bar{z}_n^i + \frac{1}{n}\sum_{i\notin I_n} \bar{z}_n^i$$
$$\overset{(3.10)}{\leq} \frac{|I_n|}{n} z_{\text{frac}}^{\sup} + \frac{1}{n}\sum_{i=0}^{n-1}(\delta_i + \varepsilon)$$

implies (3.13). Finally, using the monotonicity of J_i on (δ_i, ∞) (see Lemma 3.2) and (A4), we obtain

$$\frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) = \frac{1}{n}\sum_{i\in I_n} J_i(\bar{z}_n^i) + \frac{1}{n}\sum_{i\notin I_n} J_i(\bar{z}_n^i) \geq \frac{1}{n}\sum_{i\in I_n} J_i(\delta_i + \varepsilon) + \frac{1}{n}\sum_{i\notin I_n} J_i(\delta_i)$$
$$\geq \frac{1}{n}\sum_{i\in I_n}\Big(J_i(\delta_i) + \frac{1}{2}c\varepsilon^2\Big) + \frac{1}{n}\sum_{i\notin I_n} J_i(\delta_i) = \frac{1}{n}\sum_{i=0}^{n-1} J_i(\delta_i) + \frac{|I_n|}{n}\frac{1}{2}c\varepsilon^2,$$

where $c > 0$ is as in (A4). Sending $n \to \infty$, we obtain with help of (3.8) and (3.13) the claim (3.12).

Step 2. We claim $\underline{A} := \liminf_{n\to\infty} \ell_n^* \geq \mathbb{E}[\delta]$.

For all $\varepsilon > 0$, we show that

$$\ell_n^* \geq \frac{1}{n}\sum_{i=0}^{n-1} \delta_i - \varepsilon =: k_n \quad \forall n \in \mathbb{N}, \tag{3.14}$$

which in combination with (3.8) implies that $\underline{A} := \liminf_{n\to\infty} \ell_n^* \geq \mathbb{E}[\delta]$ by the arbitrariness of $\varepsilon > 0$.

Let $\bar{z}_n$ be such that

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = k_n, \quad \frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) = M_n(k_n).$$

We show that $\bar{z}_n^i \leq \delta_i < z_{\text{frac}}^i \ \forall i \in \{0,\dots,n-1\}$, which obviously implies (3.14). Indeed, the optimality condition for $\bar{z}_n$ implies that there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{0,\dots,n-1\}$. Since

$$\frac{1}{n}\sum_{i=0}^{n-1}(\bar{z}_n^i - \delta_i) \leq -\varepsilon,$$

there exists $\hat{i} \in \{0,\dots,n-1\}$ such that $\bar{z}_n^{\hat{i}} \in (0,\delta_i)$ and thus $J_{\hat{i}}'(\bar{z}_n^{\hat{i}}) < 0$. Hence $J_i'(\bar{z}_n^i) < 0$ for all $i \in \{0,\dots,n-1\}$. Since $J_i' \geq 0$ on (δ_i,∞) by Lemma 3.2, we conclude that $\bar{z}_n^i \leq \delta_i \leq z_{\text{frac}}^i$ and thus $\ell_n^* \geq k_n$ by Lemma 3.1. ∎

3.2. Proof of Theorem 2.2

We begin with a preliminary structure result for minimizers of the minimum problem in the definition of $M_n(\omega, 1+n^{-\frac{1}{2}}D)$ for some $D > 0$; see (3.1).

Proposition 3.3. *Let Assumption 2.1 be satisfied and assume that $\delta(\omega) = 1$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Fix $D > 0$. There exist $\bar{N} \in \mathbb{N}$ and a sequence (N_n) satisfying $N_n \to \infty$ such that the following statements hold true for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \geq \bar{N}$.*

Let $\bar{z}_n \in \mathbb{R}^n$ be such that

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = 1 + n^{-\frac{1}{2}}D \quad \text{and} \quad \frac{1}{n}\sum_{i=0}^{n-1} J(\tau_i\omega, \bar{z}_n^i) = M_n(\omega, 1+n^{-\frac{1}{2}}D). \tag{3.15}$$

Then, it holds that

$$\bar{z}_n^i \in [1, 1+c^{-2}n^{-\frac{1}{2}}D] \cup [N_n,\infty) \quad \text{for all } i \in \{0,\dots,n-1\}, \tag{3.16}$$

where $c > 0$ is as in (A4).

Proof of Proposition 3.3. We consider $\omega \in \Omega$ such that $\delta(\tau_i\omega) = 1 \ \forall i \in \mathbb{N}$ and drop the dependence on ω. Moreover, we use the shorthand notation $z_{\text{frac}}^i := z_{\text{frac}}(\tau_i\omega)$ and $J_i(z) := J(\tau_i\omega, z)$.

Step 1. We show that

$$0 \leq J'(\bar{z}_n^i) \leq \frac{1}{c}Dn^{-\frac{1}{2}} \quad \text{for all } i \in \{0,\dots,n-1\}, \tag{3.17}$$

where $c > 0$ is as in (A4).

By the optimality condition for $\bar{z}_n$, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{0, \dots, n-1\}$. Since

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = 1 + n^{-\frac{1}{2}} D,$$

there exists $i_1 \in \{0, \dots, n-1\}$ such that $\bar{z}_n^{i_1} \geq 1 + n^{-\frac{1}{2}} D > 1$. Lemma 3.2 and the assumption $\delta(\tau_i \omega) = 1$ imply that J_i is increasing on $(1, \infty)$ and thus we have $\Lambda \geq 0$. Moreover, there exists $i_2 \in \{0, \dots, n-1\}$ such that $\bar{z}_n^{i_2} \leq 1 + n^{-\frac{1}{2}} D$. For n sufficiently large such that $n^{-\frac{1}{2}} D < c$, where $c > 0$ as in (A4), we have (using that $J_i'(1) = 0$)

$$0 \leq \Lambda = J_{i_2}'(\bar{z}_n^{i_2}) = \int_1^{\bar{z}_n^{i_2}} J_{i_2}''(t)\,dt \overset{(A4)}{\leq} \frac{1}{c} n^{-\frac{1}{2}} D.$$

Since $\Lambda = J_i'(\bar{z}_n^i)$ for all $i \in \{0, \dots, n-1\}$, the claim (3.17) follows.

Step 2. Argument for (3.16).

We firstly observe that (3.17) implies that $1 \leq \bar{z}_n^i$ for all $i \in \{0, \dots, n-1\}$ (recall $J_i'(z) < 0$ on $(0, 1)$). The remaining estimates of (3.16) are proven in three steps.

Substep 2.1. We claim that for n sufficiently large, $\bar{z}_n^i \leq z_{\text{frac}}^i$ implies that

$$\bar{z}_n^i \leq 1 + c^{-2} n^{-\frac{1}{2}} D,$$

where $c > 0$ is as in (A4). Indeed, using $J_i''(s) > 0$ on $(0, z_{\text{frac}}^i)$ and (A4), we deduce from $\bar{z}_n^i \leq z_{\text{frac}}^i$ and n sufficiently large that

$$c^{-1} D n^{-\frac{1}{2}} \overset{(3.17)}{\geq} J_i'(\bar{z}_n^i) = \int_1^{\bar{z}_n^i} J_i''(t)\,dt \overset{(A4)}{\geq} c \min\{\bar{z}_n^i - 1, c\}.$$

From the above inequality, we deduce that $\bar{z}_n^i - 1 \geq c$ implies that $n \leq D^2/c^6$. Hence, $\bar{z}_n^i - 1 < c$ and thus $1 \leq \bar{z}_n^i \leq 1 + c^{-2} D n^{-\frac{1}{2}}$ for $n > D^2/c^6$.

Substep 2.2. There exists $M < \infty$, depending only on $\psi^-(1)$ from (A2) and $c > 0$ from (A4), such that

$$\sup_{n \in \mathbb{N}} |I_n^w| \leq M, \quad \text{where } I_n^w := \left\{ i \in \{0, \dots, n-1\} : \bar{z}_n^i \geq z_{\text{frac}}^i \right\}. \tag{3.18}$$

Suppose that $|I_n^w| \geq 2$ and consider some $i_n \in I_n^w$. Define

$$\hat{z}_n^i := \begin{cases} \bar{z}_n^i & \text{if } i \notin I_n^w, \\ 1 & \text{if } i \in I_n^w \setminus \{i_n\}, \\ 1 + \sum_{i \in I_n^w} (\bar{z}_n^i - 1) & \text{if } i = i_n. \end{cases} \tag{3.19}$$

By construction, we have $\sum_{i=0}^{n-1} \bar{z}_n^i = \sum_{i=0}^{n-1} \hat{z}_n^i$ and thus by (3.15)

$$0 \geq \sum_{i=0}^{n-1} \big(J_i(\bar{z}_n^i) - J_i(\hat{z}_n^i)\big) = \sum_{i \in I_n^w \setminus \{i_n\}} \big(J_i(\bar{z}_n^i) - J_i(1)\big) + J_{i_n}(\bar{z}_n^{i_n}) - J_{i_n}(\hat{z}_n^{i_n}). \tag{3.20}$$

By the monotonicity of J_i on $(1,\infty)$, (A3), and (A4) in the form

$$J_i(z_{\text{frac}}^i) - J_i(1) \geq J_i(1+c) - J_i(1) = \int_1^{1+c} \int_1^s J_i''(t)\,dt\,ds \geq \frac{1}{2}c^3$$

(where $c > 0$ is as in (A4)), we find

$$J_i(\bar{z}_n^i) - J_i(1) \geq J_i(z_{\text{frac}}^i) - J_i(1) \geq \frac{1}{2}c^3 := \eta \quad \forall i \in I_n^w. \tag{3.21}$$

Moreover, using $\hat{z}_n^{i_n} \geq 1$ and thus $J_{i_n}(\hat{z}_n^{i_n}) \leq 0$ (which follows from the monotonicity of J_i on $(1,\infty)$ and (A2)), we obtain

$$J_{i_n}(\bar{z}_n^{i_n}) - J_{i_n}(\hat{z}_n^{i_n}) \geq J_{i_n}(1) \overset{(A2)}{\geq} \psi^-(1). \tag{3.22}$$

Combining (3.20)–(3.22), we deduce the uniform bound $|I_n^w| \leq 1 - \eta^{-1}\psi^-(1)$.

Substep 2.3. We show that there exists (N_n) satisfying $N_n \to \infty$ as $n \to \infty$ such that $\bar{z}_n^i \geq N_n$ for all $i \in I_n^w$, where I_n^w is defined in (3.18).

We argue by contradiction and assume that there exists $A \in [1,\infty)$ and an index $\hat{i} \in I_n^w$ such that $\bar{z}_n^{\hat{i}} \leq A$. For n sufficiently large, we show that this contradicts (3.15). Define

$$\tilde{z}_n^i := \begin{cases} 1 & \text{if } i = \hat{i}, \\ \bar{z}_n^i + \big(n - |I_n^w|\big)^{-1}(\bar{z}_n^{\hat{i}} - 1) & \text{if } i \notin I_n^w, \\ \bar{z}_n^i & \text{if } i \in I_n^w \setminus \{\hat{i}\}. \end{cases} \tag{3.23}$$

By construction, we have $\sum_{i=0}^{n-1} \tilde{z}_n^i = \sum_{i=0}^{n-1} \bar{z}_n^i$. Since $\bar{z}_n$ is a minimizer (see (3.15)),

$$0 \geq \sum_{i=0}^{n-1} \big(J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)\big) = J_{\hat{i}}(\bar{z}_n^{\hat{i}}) - J_{\hat{i}}(1) + \sum_{i \notin I_n^w} \big(J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)\big).$$

By (3.21), we have $J_{\hat{i}}(\bar{z}_n^{\hat{i}}) - J_{\hat{i}}(1) \geq \eta(c) > 0$. To obtain a contradiction, it suffices to show that the second term on the right-hand side vanishes as n tends to infinity. This can be seen as follows: on the one hand, we have $\bar{z}_n^i \in [1, 1 + c^{-2}n^{-\frac{1}{2}}D]$ for all $i \notin I_n^w$ by Substep 2.1, and on the other hand, we have

$$(n - |I_n^w|)^{-1}(\bar{z}_n^{\hat{i}} - 1) \leq (n - M)^{-1}(A - 1),$$

thanks to $|I_n^w| \leq M$. Hence, $\bar{z}_n^i, \tilde{z}_n^i \in [1, 1 + \frac{c}{2}]$ for n sufficiently large (depending only on c, D, M, and A). Now, a quadratic Taylor expansion of J_i at $\bar{z}_n^i$ yields (using $|J''(z)| \leq c^{-1}$ for $z \in [1, 1+c)$; see (A4))

$$\begin{aligned}\sum_{\substack{i=1\\ i\notin I_n^w}}^{n} \big|J_i(\bar{z}_n^i) - J_i(\tilde{z}_n^i)\big| &\leq \sum_{i=0}^{n-1} \big(\big|J_i'(\bar{z}_n^i)\big|(n-M)^{-1}(A-1) + c^{-1}(n-M)^{-2}(A-1)^2\big) \\ &\overset{(3.17)}{\leq} n(n-M)^{-1}c^{-1}(A-1)\big(n^{-\frac{1}{2}}D + (A-1)(n-M)^{-1}\big) \\ &\leq Cn^{-\frac{1}{2}},\end{aligned}$$

where $C < \infty$ depends only on A, c, D, and M. ■

Proof of Theorem 2.2. By the ergodic theorem, it holds that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} J''(\tau_i\omega, 1)^{-1} = \mathbb{E}\big[J''(1)^{-1}\big], \qquad \lim_{n\to\infty} \beta_n(\omega) = \beta \tag{3.24}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, where β is defined in (2.3) and

$$\beta_n(\omega) := \min\big\{-J(\tau_i\omega, 1) : i \in \{0, \ldots, n-1\}\big\}. \tag{3.25}$$

In Step 3 below, we provide an argument for the limit $\beta_n \to \beta$.

In Steps 1 and 2, we consider $\omega \in \Omega$ such that (3.24) and the conclusion of Proposition 3.3 are valid. Moreover, we drop the dependence on ω and use the shorthand notation $z_{\mathrm{frac}}^i := z_{\mathrm{frac}}(\tau_i\omega)$ and $J_i(z) := J(\tau_i\omega, z)$.

Step 1. We prove $\bar{A} := \limsup_{n\to\infty} \gamma_n^* \leq \sqrt{\frac{\beta}{\underline{\alpha}}}$ by contradiction: assume that there exists $\varepsilon > 0$ and $\bar{N} \in \mathbb{N}$ such that

$$\ell_n^* > 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\varepsilon) =: k_n \quad \text{for } n > \bar{N}. \tag{3.26}$$

In view of Lemma 3.1, there exists $(\bar{z}_n)_n$ satisfying

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = k_n, \quad \frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) = M_n(k_n), \quad \bar{z}_n^i \leq z_{\mathrm{frac}}^i \quad \forall i \in \{0, \ldots, n-1\}. \tag{3.27}$$

We show that

$$\limsup_{n\to\infty} n\left(M_n(k_n) - \frac{1}{n}\sum_{i=0}^{n-1} J_i(1)\right) \leq \beta, \tag{3.28}$$

$$\liminf_{n\to\infty} n\left(\frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) - \frac{1}{n}\sum_{i=0}^{n-1} J_i(1)\right) \geq \beta(1+\varepsilon)^2. \tag{3.29}$$

Clearly, (3.28) and (3.29) contradict (3.27) for n sufficiently large.

Substep 1.1. Argument for (3.29).

We claim that there exists $K < \infty$ such that for all n sufficiently large

$$n\left(\frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) - \frac{1}{n}\sum_{i=0}^{n-1} J_i(1)\right) \geq \left(\frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{1}{2}J_i''(1)\right)^{-1}\right)^{-1}\frac{\beta}{\underline{\alpha}}(1+\varepsilon)^2 - \frac{K}{\sqrt{n}}, \tag{3.30}$$

where $\overline{\alpha}$ and β are defined in (2.3). Note that (3.24) and (3.30) imply (3.29).

We prove (3.30). By (3.26), (3.27), and Proposition 3.3 (applied with $D = \sqrt{\frac{\beta}{\underline{\alpha}}}(1+\varepsilon)^2)$, we get

$$1 \leq z_n^i \leq 1 + n^{-\frac{1}{2}}C \tag{3.31}$$

for some $C < \infty$ independent of n. Hence, a Taylor expansion yields

$$\sum_{i=0}^{n-1} J_i(\bar{z}_n^i) = \sum_{i=0}^{n-1} J_i(1) + \frac{1}{2}\sum_{i=0}^{n-1} J_i''(1)(\bar{z}_n^i - 1)^2 + \frac{1}{6}\sum_{i=0}^{n-1} J_i'''(\xi_n^i)(\bar{z}_n^i - 1)^3, \tag{3.32}$$

where $\xi_n^i \in [1, \bar{z}_n^i]$. To estimate the second term on the right-hand side, note that Cauchy–Schwarz' inequality yields

$$\left(\sum_{i=0}^{n-1}(\bar{z}_n^i - 1)\right)^2 \leq \left(\frac{1}{2}\sum_{i=0}^{n-1} J_i''(1)(\bar{z}_n^i - 1)^2\right)\left(\sum_{i=0}^{n-1}\left(\frac{1}{2}J_i''(1)\right)^{-1}\right).$$

Combined with the identity $\sum_{i=0}^{n-1}(\bar{z}_n^i - 1) = n(k_n - 1) = \sqrt{n}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\varepsilon)$, we get

$$\left(\frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{1}{2}J_i''(1)\right)^{-1}\right)^{-1}\frac{\beta}{\underline{\alpha}}(1+\varepsilon)^2 \leq \frac{1}{2}\sum_{i=0}^{n-1} J_i''(1)(\bar{z}_n^i - 1)^2. \tag{3.33}$$

Moreover, (3.31) and (A4) imply for n sufficiently large that

$$\frac{1}{6}\sum_{i=0}^{n-1} J_i'''(\xi_n^i)(\bar{z}_n^i - 1)^3 \geq -\frac{C^3}{6c\sqrt{n}}. \tag{3.34}$$

Clearly, (3.32)–(3.34) imply (3.30) (with $K = \frac{C^3}{6c}$).

Substep 1.2. Argument for (3.28).

For every $n \in \mathbb{N}$, we choose $\hat{i}_n \in \{0, \ldots, n-1\}$ such that $-J_{\hat{i}_n}(1) = \beta_n$ (see (3.25)) and define $z_n \in \mathbb{R}^n$ as

$$z_n^i = \begin{cases} 1 & \text{if } i \in \{0, \ldots, n-1\} \setminus \{\hat{i}_n\}, \\ 1 + n(k_n - 1) & \text{if } i = \hat{i}_n. \end{cases}$$

Since $\frac{1}{n}\sum_{i=0}^{n-1} z_n^i = k_n = 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\varepsilon)$, we have

$$n\Big(M_n(k_n) - \frac{1}{n}\sum_{i=0}^{n-1} J_i(1)\Big) \le J_{\hat{i}_n}\big(1 + n(k_n - 1)\big) - J_{\hat{i}_n}(1)$$
$$\le \psi^+\Big(1 + \sqrt{n}\sqrt{\frac{\beta}{\underline{\alpha}}}(1+\varepsilon)\Big) + \beta_n,$$

where the second inequality holds by (A2) and the choice of $\hat{i}_n$. Now, (3.28) follows from (3.24) and assumption (A2).

Step 2. Proof of $\underline{A} := \liminf_{n\to\infty} \gamma_n^* \ge \sqrt{\frac{\beta}{\underline{\alpha}}}$.

We show that, for every $\varepsilon > 0$, there exists $\bar{N} \in \mathbb{N}$ such that

$$\ell_n^* \ge 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1-\varepsilon) =: k_n \quad \text{for } n > \bar{N}. \tag{3.35}$$

Note that (3.35) implies that $\liminf_{n\to\infty} \gamma_n^* \ge \sqrt{\frac{\beta}{\underline{\alpha}}}(1-\varepsilon)$ for all $\varepsilon > 0$, and thus the claim.

Let $(\bar{z}_n)_n$ be a sequence satisfying for all $n \in \mathbb{N}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \bar{z}_n^i = k_n, \quad M_n(k_n) = \frac{1}{n}\sum_{i=0}^{n-1} J_i(\bar{z}_n^i). \tag{3.36}$$

To prove (3.35), we only need to show that

$$z_n^i \le z_{\text{frac}}^i \quad \text{for all } i \in \{0,\dots,n-1\} \text{ for } n \text{ sufficiently large,} \tag{3.37}$$

depending only on $\underline{\alpha}$ β, c, and $\varepsilon > 0$.

Substep 2.1. We show that

$$\limsup_{n\to\infty} n\Big(M_n(k_n) - \frac{1}{n}\sum_{i=0}^{n-1} J_i(1)\Big) \le \beta(1-\varepsilon). \tag{3.38}$$

Set

$$\hat{z}_n^i := 1 + n^{-\frac{1}{2}}\sqrt{\frac{\beta}{\underline{\alpha}}}(1-\varepsilon)\Big(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\alpha_i}\Big)^{-1}\frac{1}{\alpha_i},$$

where $\alpha_i := \frac{1}{2}J_i''(1)$. By construction, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} \hat{z}_n^i = k_n, \quad 0 \le \hat{z}_n^i - 1 \le n^{-\frac{1}{2}}C, \tag{3.39}$$

where $C < \infty$ depends only on $\underline{\alpha}$, β, and $c > 0$ from (A4) (note that (A4) implies that $\alpha_i \le \frac{1}{2c}$ and $\frac{1}{\alpha_i} \le \frac{2}{c}$). Hence, a Taylor expansion of J_i at 1 and (A4) yield for n

sufficiently large

$$\sum_{i=0}^{n-1}\big(J_i(\hat{z}_n^i)-J_i(1)\big)\le\sum_{i=0}^{n-1}\alpha_i(\hat{z}_n^i-1)^2+\frac{1}{6c}\sum_{i=0}^{n-1}(\hat{z}_n^i-1)^3$$
$$\le\frac{\beta}{\underline{\alpha}}(1-\varepsilon)^2\Big(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\alpha_i}\Big)^{-1}+\frac{C^3}{6c}n^{-\frac{1}{2}},$$

where $C<\infty$ is the same as in (3.39). Finally, (3.24) implies that $(\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{\alpha_i})^{-1}\le\underline{\alpha}(1+\varepsilon)$ for n sufficiently large and thus (3.38) follows.

Substep 2.2. We now prove (3.37) by contraposition. Suppose that $\bar{z}_n^{\hat{i}}>z_{\mathrm{frac}}^{\hat{i}}$ for some $\hat{i}\in\{0,\dots,n-1\}$. Then, Proposition 3.3 yields $\bar{z}_n^{\hat{i}}\ge N_n$ for some (N_n) with $N_n\to\infty$, and thus $J_{\hat{i}}(\bar{z}_n^{\hat{i}})\ge-\sup_{s\ge N_n}\psi^+(s)$ by (A2). Hence, with $J_i(\bar{z}_n^i)\ge J_i(1)$ and $-J_{\hat{i}}(1)\ge\beta$, we therefore get

$$\sum_{i=0}^{n-1}\big(J_i(\bar{z}_n^i)-J_i(1)\big)\ge J_{\hat{i}}(\bar{z}_n^{\hat{i}})-J_{\hat{i}}(1)\ge\beta-\sup_{s\ge N_n}\psi^+(s).$$

Since $\sup_{s\ge N_n}\psi^+(s)\to0$ for $n\to\infty$, the above lower bound combined with the upper bound (3.38) and (3.36) yields a contradiction for n sufficiently large, and thus (3.37) follows.

Step 3. Argument for $\beta_n\to\beta$ almost surely in (3.24).

The sequence $(\beta_n(\omega))_n\subset\mathbb{R}$ is decreasing and it holds that $\beta_n(\omega)\ge\beta$ for all $n\in\mathbb{N}$. Hence, there exists $\widehat{\beta}(\omega)\ge\beta$ such that

$$\lim_{n\to\infty}\beta_n(\omega)=\widehat{\beta}(\omega)\ge\beta.$$

It remains to show that $\widehat{\beta}(\omega)=\beta$ for $\mathbb{P}$-a.e. $\omega\in\Omega$. We argue by contradiction and therefore suppose that there exist $\varepsilon>0$ and a set $\Omega'\subset\Omega$ with positive measure such that $\widehat{\beta}(\omega)\ge\beta+\varepsilon$ for all $\omega\in\Omega'$. Then we obtain for all $\omega\in\Omega'$ that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_{\{-J(\tau_i\omega,1)\le\beta+\frac{1}{2}\varepsilon\}}(\tau_i\omega)=0,$$

where χ_A denotes the indicator function. Clearly, this contradicts the ergodic theorem and the definition of β in the form

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_{\{-J(\tau_i\omega,1)\le\beta+\frac{1}{2}\varepsilon\}}=\mathbb{E}[\chi_{\{-J(1)\le\beta+\frac{1}{2}\varepsilon\}}]>0\quad\text{for }\mathbb{P}\text{-a.e. }\omega\in\Omega.$$

Hence the theorem is proven. ∎

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