



Some recent developments on the geometry of random spherical eigenfunctions

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Abstract. A lot of efforts have been devoted in the last decade to the investigation of the high-frequency behaviour of geometric functionals for the excursion sets of random spherical harmonics, i.e., Gaussian eigenfunctions for the spherical Laplacian $\Delta_{\mathbb{S}^2}$. In this survey, we shall review some of these results, with particular reference to the asymptotic behaviour of variances, phase transitions in the nodal case (*Berry's cancellation phenomenon*), the distribution of the fluctuations around the expected values, and the asymptotic correlation among different functionals. We shall also discuss some connections with the Gaussian kinematic formula, with Wiener chaos expansions, and with recent developments in the derivation of quantitative central limit theorems (the so-called Stein–Malliavin approach).

1. Introduction

Spherical eigenfunctions are defined as the solutions of the Helmholtz equation

$$\Delta_{\mathbb{S}^2} f_\ell + \lambda_\ell f_\ell = 0, \quad f_\ell : \mathbb{S}^2 \rightarrow \mathbb{R}, \quad \ell = 1, 2, \dots,$$

where $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian and $\{-\lambda_\ell = -\ell(\ell + 1)\}_{\ell=1,2,\dots}$ is the set of its eigenvalues. A random structure can be constructed easily by assuming that the eigenfunctions $\{f_\ell(\cdot)\}$ follow a Gaussian isotropic random process on \mathbb{S}^2 . More precisely, for each $x \in \mathbb{S}^2$, we take $f_\ell(x)$ to be a Gaussian random variable defined on a suitable probability space $\{\Omega, \mathfrak{F}, \mathbb{P}\}$; without loss of generality, we assume $\{f_\ell(\cdot)\}$ to have mean zero, unit variance, and covariance function given by

$$\begin{aligned} \mathbb{E}[f_\ell(x)f_\ell(y)] &= P_\ell(\langle x, y \rangle), \quad x, y \in \mathbb{S}^2, \\ P_\ell(t) &:= \frac{1}{2^\ell \ell!} \frac{d^\ell}{dt^\ell} (t^2 - 1), \quad t \in [-1, 1], \end{aligned}$$

where $\{P_\ell(\cdot)\}$ denotes the family of Legendre polynomials: this is the only covariance

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structure to ensure that the random eigenfunctions are isotropic, that is, invariant in law with respect to the action of the group of rotations $SO(3)$. Random spherical eigenfunctions, also known as random spherical harmonics, arise in a huge number of applications, especially in connection with mathematical physics: in particular, their role in quantum chaos has drawn strong interest in the last two decades, starting from the seminal papers [7, 8, 43, 61]; also, they represent the Fourier components of isotropic spherical random fields, whose analysis has an extremely important role in cosmology (see, e.g., [35]). Of course, random spherical harmonics are just a special case of a much richer literature on random eigenfunctions on general manifolds; special interest has been drawn for instance by *arithmetic random waves*, i.e., random eigenfunctions on the torus \mathbb{T}^d , which were introduced in [52] and then studied, among others, in [9, 10, 19, 26, 27, 33, 36, 53, 54]; see also [17, 55] and the references therein. Although some of the results that we shall discuss have related counterparts on the torus, on the higher-dimensional spheres, on more general compact manifolds, and in the Euclidean case, we will stick mainly to \mathbb{S}^2 for brevity and simplicity.

A lot of efforts have been spent in the last decade to characterize the geometry of the excursion sets of random spherical harmonics, which are defined as

$$A_u(f_\ell; \mathbb{S}^2) := \{x \in \mathbb{S}^2 : f_\ell(x) \geq u\}, \quad u \in \mathbb{R}. \quad (1.1)$$

A classical tool for the investigation of these sets is given by the so-called Lipschitz–Killing curvatures (or, equivalently, by Minkowski functionals; see [1]), which in dimension 2 correspond to the Euler–Poincaré characteristic, (half of) the boundary length and the excursion area. A general expression for their expected values (covering much more general Gaussian fields than random eigenfunctions) is given by the *Gaussian kinematic formula* (see [1, 58]). Over the last decade, more refined characterizations for random spherical harmonics have been obtained, including neat analytic expressions (in the high-energy limit $\lambda_\ell \rightarrow \infty$) for the fluctuations around their expected values and the correlation among these different functionals; much of the literature has been concerned with the *nodal* case, corresponding to $u = 0$, to which we shall devote special attention. In this survey, we shall review some of these results and present some open issues for future research.

2. The Gaussian kinematic formula for Lipschitz–Killing curvatures on excursions sets

2.1. The Kac–Rice formula and the expectation metatheorem

The first modern attempt to investigate the geometry of random processes and fields can probably be traced back to the groundbreaking work by Kac (1943) and Rice

(1945) [25, 49] on the zeroes of stochastic processes. Their pioneering argument can be introduced as follows: let $f(\cdot, \cdot) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous stochastic process satisfying regularity conditions; our aim is to derive the expected cardinality of its zero set in some finite interval (say $[0, T]$), i.e., the mean of

$$N_0([0, T]) := \text{Card} \{t \in [0, T] : f(t) = 0\}.$$

Now assume that $\{f(\cdot)\}$ is C^1 with probability one, such that $f(0), f(T) \neq 0$ and

$$\{t : f(t) = 0, f'(t) = 0\} = \emptyset;$$

then the following result (*Kac's counting lemma*) can be established easily (see [3, p. 69]):

$$N_0([0, T]) = \lim_{\varepsilon \rightarrow 0} \int_0^T \frac{1}{2\varepsilon} \mathbb{I}_{(-\varepsilon, \varepsilon)}(f(t)) |f'(t)| dt,$$

where as usual \mathbb{I}_A denotes the indicator function of the set A . With further efforts and assuming that all exchanges of integrals and limits can be justified, one obtains also

$$\mathbb{E}[N_0([0, T])] = \int_0^T \mathbb{E}[|f'(t)| \mid f(t) = 0] p_{f(t)}(0) dt, \tag{2.1}$$

where $\mathbb{E}[\cdot|\cdot]$ denotes as usual the conditional expected value and $p_f(\cdot)$ the marginal density of $f(\cdot)$, which is assumed to exist and admit enough regularity conditions (in the overwhelming majority of the literature and in this whole survey, $f(\cdot)$ will indeed be assumed to be Gaussian); (2.1) is the simplest example of the *Kac–Rice formula*.

The basic idea behind the Kac–Rice approach has proved to be extremely fruitful, leading to an enormous amount of applications and generalizations. In particular, in the research monographs [1, 3], (slightly different) versions of a general *expectation metatheorem* (in the terminology of [1]) are proved. More precisely, let us take M to be a compact, d -dimensional oriented C^1 manifold with a C^1 Riemannian metric g . Assume that $f : M \rightarrow \mathbb{R}^d$ and $h : M \rightarrow \mathbb{R}^k$ are vector-valued random fields which satisfy suitable regularity conditions (see [1, 3] for more details and [56] for some very recent developments). Let $B \subset \mathbb{R}^k$ be a subset with boundary dimension smaller than or equal to $k - 1$; then define

$$N_u(f, h, M, B) = \{t \in M : f(t) = u, h(t) \in B\}, \quad u \in \mathbb{R}^d.$$

The following extension of the Kac–Rice formula holds.

Theorem 2.1 ([1, 3]). *It holds that*

$$\mathbb{E}[N_u(f, h, M, B)] = \int_M \mathbb{E}[|\det \{\nabla f(t)\}| \mathbb{I}_B(h(t)) \mid f(t) = u] p_{f(t)}(u) \sigma_g(dt),$$

where as before $\mathbb{I}_B(\cdot)$ denotes the indicator function, $\nabla f(\cdot)$ the (covariant) gradient of $f(\cdot)$, and $\sigma_g(\cdot)$ the volume form induced by the metric g .

Remark. By taking $k = 1$, $f := \nabla h$ the gradient of h (and hence $\nabla f = \nabla^2 h$ its Hessian) and $u = (0, \dots, 0)$, Theorem 2.1 yields the expected number of critical points with values in B for the scalar random field h . Simple modifications similarly yield the expected values for maxima, minima, and saddle points.

The previous results have all been restricted to vector-valued random fields whose image space has co-dimension zero. However, the results can be similarly generalized to strictly positive co-dimensions. Indeed, under the same setting as before assume instead that $f : M \rightarrow \mathbb{R}^{d'}$ is such that $d' < d$; then ∇X is a $d \times d'$ rectangular matrix, and the following generalization of the expectation metatheorem holds (see [1, 3]).

Theorem 2.2 ([1, 3]). *It holds that*

$$\begin{aligned} \mathbb{E}[\mathcal{H}_u(f, h, M, B)] &= \int_M \mathbb{E}[|\det\{(\nabla f(t))^T (\nabla f(t))\}|^{1/2} \mathbb{I}_B(h) \mid f(t) = u] p_{f(t)}(u) \sigma_g(dt), \end{aligned}$$

where $\mathcal{H}_u(f, h, M, B)$ denotes the $d - d'$ dimensional Hausdorff measure of the set $\{t \in M : f(t) = u \text{ and } h(t) \in B\}$.

Example 2.3. Let $M = \mathbb{S}^2$ be the standard unit-dimensional sphere in \mathbb{R}^3 , let $f : \mathbb{S}^2 \times \Omega \rightarrow \mathbb{R}$ be a random field, and let

$$\text{Len}(f) := \mathcal{H}_0(f, \mathbb{S}^2, 0) = \text{meas}\{t \in \mathbb{S}^2 : f(t) = 0\},$$

i.e., the length of the nodal lines of $f(\cdot)$. Then

$$\begin{aligned} \mathbb{E}[\text{Len}(f)] &= \int_{\mathbb{S}^2} \mathbb{E}[|\det\{(\nabla f(t))^T (\nabla f(t))\}|^{1/2} \mid f(t) = 0] p_{f(t)}(0) \sigma(dt) \\ &= \int_{\mathbb{S}^2} \mathbb{E}[\|\nabla f(t)\| \mid f(t) = 0] p_{f(t)}(0) \sigma(dt), \end{aligned}$$

where $\|\cdot\|$ denotes Euclidean norm and $\sigma(\cdot)$ the standard Lebesgue measure on the unit sphere. In particular, assuming that the law of $f(\cdot)$ is isotropic (that is, invariant with respect to the action of the group of rotations $\text{SO}(3)$), we get

$$\mathbb{E}[\text{Len}(f)] = 4\pi \times \mathbb{E}[\|\nabla f(t)\| \mid f(t) = 0] p_{f(t)}(0).$$

2.2. Intrinsic volumes and Lipschitz–Killing curvatures

In the sequel, as mentioned earlier we will restrict our attention only to Gaussian processes, which have driven the vast majority of research in this area. We need now to

introduce the Gaussian kinematic formula (see [1, 58]); to this aim, let us first recall the notion of *Lipschitz–Killing curvatures*. In the simplest setting of convex subsets of the Euclidean space \mathbb{R}^d , Lipschitz–Killing curvatures (also known as intrinsic volumes) can be defined implicitly by means of *Steiner’s tube formula*; to recall the latter, for any convex d -dimensional set $A \subset \mathbb{R}^d$ define the Tube of radius ρ around A as

$$\text{Tube}(A, \rho) := \{x \in \mathbb{R}^d : d(x, A) \leq \rho\}, \quad d(x, A) = \inf_{y \in A} d(x, y),$$

where $d(\cdot, \cdot)$ is the standard Euclidean distance. Then the following expansion holds:

$$\mu_d \{ \text{Tube}(A, \rho) \} = \sum_{j=0}^d \omega_{d-j} \rho^{d-j} \mathcal{L}_j(A),$$

where $\mathcal{L}_j(A)$ denotes the j th Lipschitz–Killing curvatures, $\mu_d(\cdot)$ denotes the d -dimensional Lebesgue measure, and

$$\omega_j := \frac{\pi^{j/2}}{\Gamma(\frac{j}{2} + 1)}$$

is the volume of the j -dimensional unit ball ($\omega_0 = 1, \omega_1 = 2, \omega_2 = \pi, \omega_3 = \frac{4}{3}\pi$).

Lipschitz–Killing curvatures can be shown to be additive and to scale with dimensionality, in the sense that

$$\begin{aligned} \mathcal{L}_j(\lambda A) &= \lambda^j \mathcal{L}_j(A) \quad \forall \lambda > 0, \\ \mathcal{L}_j(A_1 \cup A_2) &= \mathcal{L}_j(A_1) + \mathcal{L}_j(A_2) - \mathcal{L}_j(A_1 \cap A_2). \end{aligned}$$

For $j = d$, it is immediately seen that $\mathcal{L}_d(A)$ is just the Hausdorff measure of A , whereas for $j = 0$ we obtain $\mathcal{L}_0(A) = \varphi(A)$, the (integer-valued) Euler–Poincaré characteristic of A . A more general definition of $\mathcal{L}_j(\cdot)$ can be given for basic complexes (i.e., disjoint union of complex sets), for which the following characterization (due to Hadwiger, see [1]) holds:

$$\mathcal{L}_j(A) = \frac{\omega_d}{\omega_{d-j} \omega_j} \binom{d}{j} \int_{\mathcal{E}_d} \varphi(A \cap gE_{d-j}) \mu(dg), \tag{2.2}$$

where $\mathcal{E}_d = \mathbb{R}^d \times O(n)$ is the group of rigid motions, E_{d-j} is any $d - j$ dimensional affine subspace, and the volume form $\mu(dg)$ is normalized so that

$$\text{for all } x \in \mathbb{R}^d, A \subset \mathbb{R}^d, \quad \mu\{g : gx \in A\} = \mathcal{H}(A),$$

where as before $\mathcal{H}(\cdot)$ denotes the Hausdorff measure. For instance, for $A = \mathbb{S}^2$ it is well known and easy to check that (2.2) gives

$$\mathcal{L}_0(\mathbb{S}^2) = 2, \quad \mathcal{L}_1(\mathbb{S}^2) = 0, \quad \mathcal{L}_2(\mathbb{S}^2) = 4\pi,$$

which represent, respectively, the Euler–Poincaré characteristic, (half) the boundary length, and the area of the 2-dimensional unit sphere.

2.3. The Gaussian kinematic formula

From now on, we shall restrict our attention to Gaussian processes $f : M \rightarrow \mathbb{R}$, which we shall take to be zero-mean and isotropic, meaning as usual that $\mathbb{E}[f(t)] = 0$ and $f(gt) \stackrel{d}{=} f(t)$ for all $t \in M \subset \mathbb{R}^d$ and $g \in \text{SO}(d)$; more explicitly, the law of the field $f(\cdot)$ will always be taken to be invariant to rotations. In order to present the Gaussian kinematic formula, let us first introduce a Riemannian structure governed by the covariance function of the field $\{f(\cdot)\}$; more precisely, consider the metric induced on the tangent plan $T_t M$ by the following inner product [1, p. 305]:

$$g^f(X_t, Y_t) := \mathbb{E}[X_t f \cdot Y_t f], \quad X_t, Y_t \in T_t M.$$

This metric takes a particular simple form in case the field $f(\cdot)$ is isotropic; in these circumstances, $g^f(\cdot, \cdot)$ is simply the standard Euclidean metric, rescaled by a factor that corresponds to the square root of (minus) the derivative of the covariance density at the origin.

Example 2.4. Consider the random spherical eigenfunction satisfying

$$\Delta f_\ell = -\lambda_\ell f_\ell, \quad f_\ell : \mathbb{S}^2 \rightarrow \mathbb{R}, \quad \ell = 0, 1, 2, \dots,$$

with

$$\mathbb{E}[f_\ell(x)] = 0, \quad \mathbb{E}[f_\ell(x_1)f_\ell(x_2)] = P_\ell(\langle x_1, x_2 \rangle), \quad P'_\ell(1) = -\frac{\ell(\ell + 1)}{2}.$$

Then the induced inner product is simply

$$g^{f_\ell}(X, Y) = \sqrt{\frac{\ell(\ell + 1)}{2}} \langle X, Y \rangle_{\mathbb{R}^3};$$

this change of metric can of course be realized by transforming \mathbb{S}^2 into

$$\mathbb{S}^2_{\sqrt{\lambda_\ell/2}} := \sqrt{\lambda_\ell/2} \mathbb{S}^2.$$

Let us now write $\mathcal{L}_j^f(A)$ for the j th Lipschitz–Killing curvatures of the set A under the metric induced by the zero-mean Gaussian field f ; for instance, in the case of spherical random eigenfunctions we get immediately

$$\mathcal{L}_0^{f_\ell}(\mathbb{S}^2) = \mathcal{L}_0(\mathbb{S}^2_{\sqrt{\lambda_\ell/2}}) = 2, \quad \mathcal{L}_1^{f_\ell}(\mathbb{S}^2) = 0, \quad \mathcal{L}_2^{f_\ell}(\mathbb{S}^2) = 4\pi \frac{\lambda_\ell}{2}.$$

For further notation, as in [1] we shall write

$$\begin{aligned} \rho_j(u) &:= \frac{1}{(2\pi)^{1/2+j/2}} \exp(-u^2/2) H_{j-1}(u), \quad j \geq 1, \\ \rho_0(u) &:= 1 - \Phi(u) = \int_u^\infty \varphi(t) dt, \end{aligned}$$

where as usual $\varphi(t) = (2\pi)^{-1/2} \exp(-t^2/2)$ denotes the standard Gaussian density and we introduced the Hermite polynomials

$$H_k(u) := (-1)^k \exp\left(\frac{u^2}{2}\right) \frac{d^k}{du^k} \exp\left(-\frac{u^2}{2}\right), \quad k = 0, 1, 2, \dots, u \in \mathbb{R}; \quad (2.3)$$

for instance $H_0(u) = 1, H_1(u) = u, H_2(u) = u^2 - 1, \dots$ Finally, we shall introduce the *flag coefficients*

$$\begin{bmatrix} d \\ k \end{bmatrix} := \binom{d}{k} \frac{\omega_d}{\omega_k \omega_{d-k}}, \quad k = 0, 1, \dots, d. \quad (2.4)$$

We are now in the position to state the following.

Theorem 2.5 (Gaussian kinematic formula, [1, Theorem 13.4.1] and [58]). *Under regularity conditions, for all $j = 0, 1, \dots, n$ one has that*

$$\mathbb{E}[\mathcal{L}_j^f(A_u(f; M))] = \sum_{k=0}^{d-j} \begin{bmatrix} k+j \\ k \end{bmatrix} \rho_k(u) \mathcal{L}_{k+j}^f(M). \quad (2.5)$$

Before we proceed with some examples, it is worth discussing formula (2.5). We are evaluating the expected value of a complex geometric functional on a complicated excursion set, in very general circumstances (under minimal regularity conditions on the field and on the manifold on which it is defined). It is clear that the expected value should depend on the manifold, on the threshold level, and on the field one considers, and one may expect these three factors to be intertwined in a complicated manner. On the contrary, formula (2.5) shows that their role is completely decoupled; more precisely

- the threshold u enters the formula merely through the functions $\rho_j(u)$ which are very simple and fully universal (i.e., they do not depend neither on the field nor on the manifold);
- on the left-hand side Lipschitz–Killing curvatures appear, but they are computed on the original manifold, not on the excursion sets, and they are therefore again extremely simple to compute;

- the role of the field f is confined to the new metric $g^f(\cdot, \cdot)$ that it induces and under which the Lipschitz–Killing curvatures are computed on both sides; under the (standard) assumption of isotropy, this implies only a rescaling of the manifold by means of a factor depending only on the derivative of the covariance function at the origin.

Example 2.6. Let us consider a zero-mean isotropic Gaussian field f defined on \mathbb{S}^d (the unit sphere in \mathbb{R}^{d+1}); its covariance function can be written as

$$\mathbb{E}[f(x_1)f(x_2)] = \sum_{\ell=0}^{\infty} \frac{n_{\ell,d}}{s_{d+1}} C_{\ell} G_{\ell; \frac{d}{2}}(\langle x_1, x_2 \rangle),$$

where $s_{d+1} = (d + 1)\omega_{d+1}$ is the surface measure of \mathbb{S}^d , $G_{\ell; \alpha}(\cdot)$ denotes the normalized Gegenbauer polynomials of order α , whereas

$$n_{\ell,d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d - 1)!} \ell^{d-1}, \quad \text{as } \ell \rightarrow \infty,$$

is the dimension of the eigenspace corresponding to the ℓ th eigenvalue $\lambda_{\ell,d} := \ell(\ell + d - 1)$; here $\{C_{\ell}\}$ is a sequence of non-negative weights which represent the so-called angular power spectrum of the random field. The derivative of the covariance function at the origin is

$$\mu := \sum_{\ell=0}^{\infty} \frac{n_{\ell,d}}{s_{d+1}} C_{\ell} \frac{\lambda_{\ell,d}}{d}.$$

Recall that the Lipschitz–Killing curvatures of the manifold $\mathbb{S}_{\lambda}^d := \lambda \mathbb{S}^d$ are given by

$$\mathcal{L}_j(\lambda \mathbb{S}^d) = 2 \binom{d}{j} \frac{s_{d+1}}{s_{d+1-j}} \lambda^j,$$

for $d - j$ even, and 0 otherwise, see [1, p. 179]. Then the Gaussian kinematic formula reads

$$\begin{aligned} \mathbb{E}[\mathcal{L}_j^f(A_u(f; \mathbb{S}^d))] &= \sum_{k=0}^{d-j} \rho_k(u) \begin{bmatrix} k + j \\ k \end{bmatrix} \mathcal{L}_{k+j}(\sqrt{\mu} \mathbb{S}^d) \\ &= \sum_{k=0}^{d-j} \rho_k(u) \begin{bmatrix} k + j \\ k \end{bmatrix} \mathcal{L}_{k+j}(\mathbb{S}^d) \mu^{(k+j)/2}. \end{aligned}$$

Example 2.7. As a special case of the previous example, assume that $f = f_{\ell}$ is actually a unit variance eigenfunction on \mathbb{S}^2 corresponding to the eigenvalue $-\ell(\ell + 1)$, $\ell = 0, 1, 2, \dots$

Then the Gaussian kinematic formula gives

$$\begin{aligned} \mathbb{E}[\mathcal{L}_0^{f_\ell}(A_u(f_\ell; \mathbb{S}^2))] &= \mathbb{E}[\mathcal{L}_0(A_u(f_\ell; \mathbb{S}^2))] \\ &= 2\{1 - \Phi(u)\} + \frac{1}{2\pi}u\phi(u)(4\pi)\frac{\ell(\ell + 1)}{2}, \\ \mathbb{E}[\mathcal{L}_1^{f_\ell}(A_u(f_\ell; \mathbb{S}^2))] &= \rho_1(u) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mathcal{L}_2(\mathbb{S}^2) \left\{ \frac{\ell(\ell + 1)}{2} \right\} \end{aligned}$$

so that

$$\mathbb{E}[\mathcal{L}_1(A_u(f_\ell; \mathbb{S}^2))] = \pi \exp\left(-\frac{u^2}{2}\right) \left\{ \frac{\ell(\ell + 1)}{2} \right\}^{1/2},$$

and finally

$$\mathbb{E}[\mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2))] = \{1 - \Phi(u)\} \mathcal{L}_2(\mathbb{S}^2) = \{1 - \Phi(u)\}4\pi.$$

Example 2.8. In the special case of the nodal volume $\mathcal{L}_{d-1}(A_0(\mathbb{S}^d), f_\ell)$ of random eigenfunctions, i.e., half the Hausdorff measure of the zero-set of the eigenfunction, the Gaussian kinematic formula gives

$$\begin{aligned} \mathbb{E}[\mathcal{L}_{d-1}^f(A_u(f_\ell; \mathbb{S}^d))] &= \left(\frac{\lambda_\ell}{d}\right)^{(d-1)/2} \mathbb{E}[\mathcal{L}_{d-1}(A_u(f_\ell; \mathbb{S}^d))] \\ &= \rho_1(u) \frac{d\omega_d}{\omega_1\omega_{d-1}} \mathcal{L}_d(\mathbb{S}^d) \left(\frac{\lambda_\ell}{d}\right)^{d/2} \end{aligned}$$

so that, recalling $\omega_j = \frac{\pi^{j/2}}{\Gamma(\frac{j}{2}+1)}$ and $\mathcal{L}_d(\mathbb{S}^d) = (d + 1)\omega_{d+1}$, we have

$$\begin{aligned} \mathbb{E}[\mathcal{L}_{d-1}(A_u(f_\ell; \mathbb{S}^d))] &= \frac{1}{2\pi} \exp\left(-\frac{u^2}{2}\right) \frac{d\omega_d}{\omega_1\omega_{d-1}} \mathcal{L}_d(\mathbb{S}^d) \left(\frac{\lambda_\ell}{d}\right)^{1/2} \\ &= \exp\left(-\frac{u^2}{2}\right) \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \left(\frac{\lambda_\ell}{d}\right)^{1/2}. \end{aligned} \tag{2.6}$$

For $u = 0$ equation (2.6) was derived for instance in [6] (see [61]) and it is consistent with a celebrated conjecture in [63], which states that for C^∞ manifolds the nodal volume of any eigenfunction corresponding to the eigenvalue E should belong to the interval $[c_1\sqrt{E}, c_2\sqrt{E}]$ for some constants $0 < c_1 \leq c_2 < \infty$. The conjecture was settled for real analytic manifolds in [22]; for smooth manifolds the lower bound was established much more recently; see [29–31] while the upper bound is addressed in [32]. As a consequence of the results in the next two sections below in the case of the sphere in a probabilistic sense, the upper and lower constants can be taken nearly coincident, in the limit of diverging eigenvalues.

3. Wiener chaos expansions, variances, and correlations

In view of the results detailed in Section 2, the question related to the expectation of intrinsic volumes in the case of Gaussian fields can be considered completely settled. The next step of interest is the computation of the corresponding variances, and the asymptotic laws of fluctuations around the expected values, in the high-frequency regime. The first rigorous results in this area can be traced back to a seminal paper by Igor Wigman [61], where the variance of the nodal length (i.e., $\text{Len}(f_\ell, \mathbb{S}^2) := 2\mathcal{L}_1(A_0(f_\ell, \mathbb{S}^2))$) for random spherical harmonics in dimension 2 is computed and shown to be asymptotic to

$$\text{Var} [\text{Len}(f_\ell, \mathbb{S}^2)] = \frac{\log \ell}{32} + O_{\ell \rightarrow \infty}(1). \tag{3.1}$$

We shall start instead from the derivation of variances and central limit theorems for Lipschitz–Killing curvatures of excursion sets at $u \neq 0$, although these results were actually obtained more recently than (3.1).

Let us recall first the notion of Wiener chaos expansions. In the simplest setting, consider $Y = G(Z)$, i.e., the transform of a zero mean, unit variance Gaussian random variable Z , such that $\mathbb{E}[G(Z)^2] < \infty$; it is well known that the following expansion holds, in the $L^2(\Omega)$ sense:

$$G(Z) = \sum_{q=0}^{\infty} \frac{J_q(G)}{q!} H_q(Z), \tag{3.2}$$

where $\{H_q(\cdot)\}_{q=0,1,2,\dots}$ denotes the family of Hermite polynomials that we introduced earlier in (2.3), and $J_q(G)$ are projection coefficients given by $J_q(G) := \mathbb{E}[G(Z)H_q(Z)]$ (see, e.g., [24, 46]). The summands in (3.2) are orthogonal, because when evaluated on pairs of standard Gaussian variables Z_1, Z_2 , Hermite polynomials enjoy a very simple formula for the computation of covariances:

$$\mathbb{E}[H_{q_1}(Z_1)H_{q_2}(Z_2)] = \delta_{q_1}^{q_2} q_1! \{\mathbb{E}[Z_1 Z_2]\}^{q_1}, \tag{3.3}$$

where $\delta_{q_1}^{q_2}$ denotes the Kronecker delta. Equation (3.3) is just a special case of the celebrated *diagram (or Wick’s) formula*; see [46] for much more discussion and details. We thus have immediately

$$\text{Var} \{G(Z)\} = \sum_{q=0}^{\infty} \frac{J_q^2(G)}{q!}.$$

More generally, let $\{Z_1, \dots, Z_j, \dots\}$ be any array of independent standard Gaussian variables, and consider elements of the form

$$H_{q_1}(Z_1) \cdots H_{q_p}(Z_p), \quad q_1 + \cdots + q_p = q;$$

the linear span (in the $L^2(\Omega)$ sense) of these random variables is usually written as \mathcal{C}_q (denoted by the q th-order Wiener chaos; see again [46]) and we have the orthogonal decomposition

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{C}_q.$$

3.1. Wiener chaos expansions for random eigenfunctions

Let us now explain how these techniques can be pivotal for the investigation of fluctuations of geometric functionals. We start from the simplest case, the excursion volume/area for the 2-dimensional sphere, which we can write as

$$\mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2)) = \int_{\mathbb{S}^2} \mathbb{I}_{[u, \infty)}(f_\ell(x)) \, dx,$$

$\mathbb{I}_{[u, \infty)}(\cdot)$ denoting the indicator function of the semi-interval $[u, \infty)$. It is not difficult to show that

$$\begin{aligned} J_q(\mathbb{I}_{[u, \infty)}(\cdot)) &= \mathbb{E}[\mathbb{I}_{[u, \infty)}(Z) H_q(Z)] \\ &= \int_u^\infty H_q(z) \phi(z) \, dz = (-1)^q H_{q-1}(u) \phi(u), \end{aligned}$$

the last result following by integration by parts, under the convention that

$$(-1)H_{-1}(u)\phi(u) := 1 - \Phi(u).$$

In view of (3.2), we thus have [40, 41]

$$\begin{aligned} \mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2)) &= \int_{\mathbb{S}^2} \sum_{q=0}^{\infty} (-1)^q H_{q-1}(u) \phi(u) \frac{H_q(f_\ell(x))}{q!} \, dx \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} H_{q-1}(u) \phi(u) h_{\ell; q}, \end{aligned}$$

where

$$h_{\ell; q} = \int_{\mathbb{S}^2} H_q(f_\ell(x)) \, dx;$$

as a consequence, we have also

$$\text{Var} \{ \mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2)) \} = \sum_{q=0}^{\infty} \frac{1}{(q!)^2} H_{q-1}^2(u) \phi^2(u) \text{Var} \{ h_{\ell; q} \}. \quad (3.4)$$

The crucial observation to be drawn at this stage is that the variances of the components $\{h_{\ell; q}\}$ exhibit a form of phase transition with respect to their order q , in the

high-frequency/high-energy limit $\ell \rightarrow \infty$. In particular, a simple application of the diagram formula (3.3), isotropy, and a change of variable yield

$$\begin{aligned} \text{Var}\{h_{\ell;q}\} &= \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}\{H_q(f_\ell(x))H_q(f_\ell(y))\} dx dy \\ &= 8\pi^2 q! \int_0^\pi \{P_\ell(\cos \theta)\}^q \sin \theta d\theta; \end{aligned}$$

for instance, for $q = 2$ we obtain exactly

$$\text{Var}\{h_{\ell;2}\} = 2 \times 8\pi^2 \int_0^\pi P_\ell^2(\cos \theta) \sin \theta d\theta = 16\pi^2 \frac{2}{2\ell + 1}.$$

Given two sequences of positive numbers a_n, b_n , we shall write $a_n \approx b_n$ when we have that $a_n/b_n \rightarrow c$ as $n \rightarrow \infty, c > 0$. By means of the so-called Hilb asymptotics [57, 61], it is possible to show that, as $\ell \rightarrow \infty$ [42],

$$\text{Var}\{h_{\ell;q}\} \approx \frac{1}{\ell^2} \times \int_0^{\ell\pi} \frac{1}{\psi^{q/2}} \psi d\psi \approx \begin{cases} \ell^{-1} & \text{for } q = 2 \\ \ell^{-2} \log \ell & \text{for } q = 4 \\ \ell^{-2} & \text{for } q = 3, 5, \dots \end{cases}$$

Note that $h_{\ell;1} \equiv 0$ for all $\ell = 1, 2, \dots$, whereas the term for $q = 3$ requires an ad-hoc argument given in [34, 40]. As a consequence, the dominant terms in the variance expansion correspond to $q = 2$ when $H_1(u)$ is non-zero, i.e., for $u \neq 0$; for $u = 0$ the even-order chaoses vanish and all the remaining terms contribute by the same order of magnitude with respect to ℓ . In conclusion, we have that

$$\begin{aligned} \mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2))] \\ = \frac{1}{2} H_1(u)\phi(u)h_{\ell;2} + O_p(\sqrt{\log \ell / \ell^2}), \end{aligned} \tag{3.5}$$

and for $u \neq 0$

$$\text{Var}\{\mathcal{L}_2(A_u(f_\ell; \mathbb{S}^2))\} \sim \left\{ \frac{1}{2} H_1(u)\phi(u) \right\}^2 \text{Var}\{h_{\ell;2}\}, \quad \text{as } \ell \rightarrow \infty.$$

Because

$$h_{\ell;2} = \int_{\mathbb{S}^2} \{f_\ell^2(x) - 1\} dx = \|f_\ell\|_{L^2(\mathbb{S}^2)}^2 - \mathbb{E}[\|f_\ell\|_{L^2(\mathbb{S}^2)}^2],$$

equation (3.5) is basically stating that the fluctuations in the excursion area for $u \neq 0$ are dominated by the fluctuations in the random norm of the eigenfunctions.

Interestingly, the same behaviour characterizes also the other Lipschitz–Killing curvatures; for the boundary length we have the expansion

$$2\mathcal{L}_1(A_u(f_\ell; \mathbb{S}^2)) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^2} \|\nabla f_\ell(x)\| \delta_\varepsilon(f_\ell(x) - u) dx$$

which holds both ω almost surely and in $L^2(\Omega)$; here we write $\delta_\varepsilon(\cdot) = \frac{1}{2\varepsilon} \mathbb{I}(\cdot)$. Similarly for the Euler–Poincaré characteristic we have

$$\mathcal{L}_0(A_u(f_\ell; \mathbb{S}^2)) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^2} \det \{ \nabla^2 f_\ell(x) \} \delta_\varepsilon(\nabla f_\ell(x)) \mathbb{I}_{[u, \infty)}(f_\ell(x)) dx.$$

Similar arguments can be developed, expanding the integrand function into polynomials evaluated on the random vectors $\{ \nabla^2 f_\ell(\cdot), \nabla f_\ell(\cdot), f_\ell(\cdot) \}$; algebraic simplifications occur and the expansions read as follows.

Theorem 3.1. *As $\ell \rightarrow \infty$, for $j = 0, 1, 2$*

$$\begin{aligned} &\mathcal{L}_j(A_u(f_\ell, \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2))] \\ &= -\frac{1}{2} \left[\begin{matrix} 2 \\ 2-j \end{matrix} \right] u \rho'_{2-j}(u) (\lambda_\ell/2)^{(2-j)/2} \int_{\mathbb{S}^2} H_2(f_\ell(x)) dx + R_{\ell;j}, \end{aligned} \quad (3.6)$$

where

$$\mathbb{E}[R_{\ell;j}^2] = o_{\ell \rightarrow \infty}(\ell^{3-2j});$$

as a consequence, one has also the variance asymptotics

$$\begin{aligned} \text{Var} \{ \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) \} &= \frac{1}{4} \left\{ \left[\begin{matrix} 2 \\ 2-j \end{matrix} \right] u \rho'_{2-j}(u) (\lambda_\ell/2)^{(2-j)/2} \right\}^2 \\ &\times \frac{32\pi^2}{2\ell + 1} + o_{\ell \rightarrow \infty}(\lambda_\ell^{2-j-1}). \end{aligned} \quad (3.7)$$

Some features of the previous result are worth discussing.

- The asymptotic behaviour of all the Lipschitz–Killing curvatures is proportional to a sequence of scalar random variables $\{h_{\ell;2}\}_{\ell \in \mathbb{N}}$. As a consequence, these geometric functionals are fully correlated in the high-energy limit $\ell \rightarrow \infty$.
- For the same reasons, these functionals are also fully correlated, in the high-energy limit, when evaluated across different levels u_1, u_2 : for the boundary length, this correlation phenomenon was first noted in [62].
- The leading terms all disappear in the “nodal” case $u = 0$, where the variances are hence an order of magnitude smaller. This is an instance of the so-called Berry cancellation phenomenon [61], to which we shall return in Section 4. We noted before that the leading terms are proportional to the centred random norm;

it is thus natural that these terms should disappear in the nodal case, which is independent of scaling factors. Note that for $j = 0$ the cancellation of the leading term occurs also at $u = 1$.

Remark. The proof of Theorem 3.1 was given in [11], in the case of the 2-dimensional sphere \mathbb{S}^2 . However, we conjecture the result to hold as stated for spherical eigenfunctions in arbitrary dimension; see below for more details. Extensions have also been given to cover for instance the 2-dimensional torus (see [14]), for which a formula completely analogous to (3.1) holds.

Similar results can be shown to hold for other geometric functionals; let us consider for instance critical values, defined by

$$\mathcal{N}_u(f_\ell; \mathbb{S}^2) = \#\{x \in \mathbb{S}^2 : \nabla f_\ell(x) = 0 \text{ and } f_\ell(x) \geq u\}.$$

The asymptotic variance of $\{\mathcal{N}_u(f_\ell; \mathbb{S}^2)\}_{\ell=1,2,\dots}$ was established in [15, 16], and in particular we have

$$\begin{aligned} \mathbb{E}[\mathcal{N}_u(f_\ell; \mathbb{S}^2)] &= \lambda_\ell g_1(u), \\ g_1(u) &= \frac{1}{\sqrt{2\pi}} \int_u^\infty (2e^{-t^2} + (t^2 - 1)e^{-t^2/2}) dt \\ &= u\phi(u) + \sqrt{2}(1 - \Phi(\sqrt{2}u)), \\ \text{Var}[\mathcal{N}_u(f_\ell; \mathbb{S}^2)] &= \frac{1}{4} \lambda_\ell^2 g_2^2(u) \text{Var} \left\{ \int_{\mathbb{S}^2} H_2(f_\ell(x)) dx \right\} + o_{\ell \rightarrow \infty}(\ell^3) \\ &= \frac{1}{4} \lambda_\ell^2 g_2^2(u) \frac{2(4\pi)^2}{2\ell + 1} + o_{\ell \rightarrow \infty}(\ell^3), \end{aligned}$$

where

$$g_2(u) = \int_u^\infty \frac{1}{\sqrt{8\pi}} e^{-3t^2/2} (2 - 6t^2 - e^{-t^2} (1 - 4t + t^4)) dt.$$

Later in [12] it was shown that the critical values above the threshold level u satisfy the asymptotic

$$\begin{aligned} \mathcal{N}_u(f_\ell; \mathbb{S}^2) - \mathbb{E}[\mathcal{N}_u(f_\ell; \mathbb{S}^2)] \\ = \frac{1}{2} \lambda_\ell g_2(u) \int_{\mathbb{S}^2} H_2(f_\ell(x)) dx + o_p\left(\sqrt{\text{Var}[\mathcal{N}_u(f_\ell; \mathbb{S}^2)]}\right), \end{aligned}$$

As a consequence, one has also, for all $u \neq 0, 1$, the correlation result

$$\begin{aligned} \text{Corr}^2 \{ \mathcal{N}_u(f_\ell; \mathbb{S}^2), \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) \} \\ := \frac{\text{Cov}^2 \{ \mathcal{N}_u(f_\ell; \mathbb{S}^2), \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) \}}{\text{Var} \{ \mathcal{N}_u(f_\ell; \mathbb{S}^2) \} \text{Var} \{ \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) \}} \rightarrow 1, \quad \text{as } \ell \rightarrow \infty; \end{aligned}$$

the value $u = 1$ has to be excluded only for $j = 0$. We also have that

$$\text{Corr}^2 \{ \mathcal{N}_{u_1}(f_\ell; \mathbb{S}^2), \mathcal{N}_{u_2}(f_\ell; \mathbb{S}^2) \} \rightarrow 1, \quad \text{as } \ell \rightarrow \infty,$$

that is, asymptotically full correlation between the number of critical values above any two non-zero thresholds u_1, u_2 .

As for the Lipschitz–Killing curvatures, a form of Berry’s cancellation occurs at $u = 0$ and $u \rightarrow \pm\infty$; the total number of critical points has then a lower-order variance (see [16]), as we shall discuss in Section 4.

3.2. Quantitative central limit theorems

The results reviewed in Section 3.1 can be considered as following from a *reduction principle* (see [20]), where the limiting behaviour of $\{ \mathcal{N}_u(f_\ell; \mathbb{S}^2), \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) \}$ is dominated by a deterministic function of the threshold level u , times a sequence of random variables $\{h_{\ell;2}\}$ which do not depend on u . To derive the asymptotic law of these fluctuations, it is hence enough to investigate the convergence in distribution of $\{h_{\ell;2}\}$, as $\ell \rightarrow \infty$. In fact, it is possible to show a stronger result, namely a *quantitative central limit theorem*; to this aim, let us recall that the Wasserstein distance between two random variables X and Y is defined by

$$d_W(X, Y) := \sup_{h \in \text{Lip}(1)} | \mathbb{E}h(X) - \mathbb{E}h(Y) |,$$

where $\text{Lip}(1)$ denotes the class of Lipschitz functions of constant 1; i.e., $|h(x) - h(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. $D_W(\cdot, \cdot)$ defines a metric on the space of probability distributions (for more details and other examples of probability metrics; see [46, Appendix C]). Taking $Z \sim N(0, 1)$ to be a standard Gaussian random variable, a quantitative central limit theorem is defined as a result of the form

$$\lim_{n \rightarrow \infty} d_W \left(\frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}(X_n)}}, Z \right) = 0.$$

The field of quantitative central limit theorems has been very active in the last few decades; more recently, a breakthrough has been provided by the discovery of the so-called *Stein–Malliavin approach* by Nourdin, Peccati, and Nualart [45, 46, 48]. These results entail that for sequences of random variables belonging to a Wiener chaos, say \mathcal{C}_q , a quantitative central limit theorem for the Wasserstein distance can be given simply controlling the fourth-moment of X_n , as follows:

$$d_W \left(\frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}(X_n)}}, Z \right) \leq \sqrt{\frac{2q-2}{3\pi q}} \sqrt{\mathbb{E} \left[\left(\frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}(X_n)}} \right)^4 \right]} - 3. \quad (3.8)$$

Similar results hold for other probability metrics, for instance the Kolmogorov and total variation distances; see again [46].

Quantitative central limit theorems lend themselves to an immediate application for the sequences $\{h_{\ell;q}\}$ that we introduced above. It should be noted indeed that by construction all these random variables belong to the q th-order Wiener chaos; it is then possible to exploit (3.8) to obtain quantitative central limit theorems for these polyspectra at arbitrary orders: their fourth moment can be computed by means of the diagram formula. These results were first given in [40] and then refined in [37], yielding the following.

Theorem 3.2. *As $\ell \rightarrow \infty$, one has*

$$d_W \left(\frac{h_{\ell;q} - \mathbb{E}[h_{\ell;q}]}{\sqrt{\text{Var}(h_{\ell;q})}}, Z \right) = \begin{cases} O\left(\frac{1}{\sqrt{\ell}}\right) & \text{for } q = 2, 3, \\ O\left(\frac{1}{\log \ell}\right) & \text{for } q = 4, \\ O(\ell^{-1/4}) & \text{for } q = 5, 6, \dots \end{cases}$$

Now, we have just shown that for nonzero thresholds $u \neq 0$ the Lipschitz–Killing curvatures and the critical values are indeed proportional to a term belonging to the second-order chaos, plus a remainder that it is asymptotically negligible. The following quantitative central limit theorem then follows immediately (see [11, 40, 50]).

Theorem 3.3. *As $\ell \rightarrow \infty$, for $u \neq 0$ ($j = 1, 2$) and for $u \neq 0, 1$ (for $j = 0$) one has that*

$$d_W \left(\frac{\mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2))]}{\sqrt{\text{Var}(\mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)))}}, Z \right) = O(\ell^{-1/2}).$$

3.3. A higher-dimensional conjecture

The results we discussed so far have been limited to random-spherical harmonics on the 2-dimensional sphere \mathbb{S}^2 . Research in progress suggests however that further generalizations should hold: to this aim, let us define the set of singular points $P_j := \{u \in \mathbb{R} : u \rho'_j(u) = 0\}$ (for instance, $P_0 = P_1 = \{0\}$, $P_2 = \{0, 1\}$, $P_3 = \{0, \pm\sqrt{3}\}, \dots$). Let us now consider Gaussian random eigenfunctions on the higher-dimensional unit sphere \mathbb{S}^d ; e.g.,

$$\Delta_{\mathbb{S}^d} f_{\ell;d} = -\lambda_{\ell;d} f_{\ell;d}, \quad \lambda_{\ell;d} := \ell(\ell + d - 1);$$

these eigenfunctions are normalized so that (see [37, 51])

$$\mathbb{E}[f_{\ell;d}] = 0, \quad \mathbb{E}[f_{\ell;d}^2] = 1, \quad \mathbb{E}[f_{\ell;d}(x) f_{\ell;d}(y)] = G_{\ell;d/2}(\langle x, y \rangle),$$

where as before $G_{\ell;d/2}(\cdot)$ is the standardized ℓ th Gegenbauer polynomial of order $\frac{d}{2}$

(normalized with $G_{\ell;d/2}(1) = 1$); it is convenient to recall that

$$G'_{\ell;d/2}(1) = \frac{\lambda_{\ell;d}}{d}.$$

We recall also that the dimension of the corresponding eigenspaces is

$$n_{\ell;d} = \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d - 1)!} \ell^{d-1}, \quad \text{as } \ell \rightarrow \infty.$$

By means of Parseval's equality we have also as a consequence

$$\begin{aligned} \text{Var} \left[\int_{\mathbb{S}^d} H_2(f_{\ell;d}(x)) \, dx \right] &= \frac{2s_d^2}{n_{\ell;d}} = \frac{2(d + 1)^2 \omega_{d+1}^2}{n_{\ell;d}} \\ &\sim \frac{(d + 1)^2 \omega_{d+1}^2 (d - 1)!}{\ell^{d-1}} \quad \text{as } \ell \rightarrow \infty. \end{aligned}$$

We then propose the following.

Conjecture 3.4. *As $\ell \rightarrow \infty$, for all $k = 0, 1, \dots, d$ one has that*

$$\begin{aligned} &\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d)) - \mathbb{E}[\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d))] \\ &= -\frac{1}{2} \binom{d}{k} \rho'_{d-k}(u) u \left(\frac{\lambda_{\ell;d}}{d} \right)^{(d-k)/2} \int_{\mathbb{S}^d} H_2(f_{\ell;d}(x)) \, dx \\ &\quad + o_p(\sqrt{\ell^{d-2k+1}}). \end{aligned}$$

Remark. An immediate consequence of this conjecture would be

$$\begin{aligned} \frac{\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d)) - \mathbb{E}[\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d))]}{\sqrt{\text{Var}[\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d))]} } &= \frac{h_{\ell;q}}{\sqrt{\text{Var}[h_{\ell;d}(2)]}} + o_p(1), \\ h_{\ell;q} &= \int_{\mathbb{S}^d} H_2(f_{\ell;d}(x)) \, dx. \end{aligned}$$

Remark. The remainder term in Conjecture 3.4 is expected to be $O(\sqrt{\ell^{d-2k}})$, in the $L^2(\Omega)$ sense.

Three further consequences of Conjecture 3.4 would be the following.

- (Variance asymptotics). As $\ell \rightarrow \infty$, for all $k = 0, 1, \dots, d$ and for non-singular points $u \notin P_{d-k}$, one has

$$\begin{aligned} &\text{Var} \{ \mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d)) \} \\ &= \frac{H_{d-k}^2(u) \phi^2(u) u^2}{(2\pi d)^{(d-k)}} \frac{d!}{(d-k)! k!} \frac{\omega_d^2 \omega_{d+1}^2}{\omega_k^2 \omega_{d-k}^2} \frac{(d+1)^2 \lambda_{\ell;d}^{d-k}}{2n_{\ell;d}} + o(\ell^{d-2k+1}). \end{aligned}$$

- (Central limit theorem). As $\ell \rightarrow \infty$, for all $k = 0, 1, \dots, d$ and for non-singular points $u \notin P_{d-k}$, one has

$$d_W \left(\frac{\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d)) - \mathbb{E}[\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d))]}{\sqrt{\text{Var}[\mathcal{L}_k(A_u(f_\ell; \mathbb{S}^d))]}}, Z \right) = o(1),$$

where $Z \sim \mathcal{N}(0, 1)$.

- (Correlation asymptotics). As $\ell \rightarrow \infty$, for all $k_1, k_2 = 0, 1, \dots, d$ and all u_1, u_2 such that $u_1 u_2 H_{d-k_1}(u_1) H_{d-k_2}(u_2) \neq 0$, one has

$$\lim_{\ell \rightarrow \infty} \text{Corr}^2(\mathcal{L}_{k_1}(A_u(f_\ell; \mathbb{S}^d)), \mathcal{L}_{k_2}(A_u(f_\ell; \mathbb{S}^d))) = 1.$$

The driving rationale behind these conjectures is the *ansatz* that the asymptotic variance of the geometric functionals should be governed by fluctuations in the random $L^2(\mathbb{S}^d)$ norm of the eigenfunctions, for non-singular points $u \notin P_j$. In this sense, we believe the result has even greater applicability, for instance to cover combinations of random eigenfunctions defined on more general submanifolds of \mathbb{R}^n , such as *Berry’s random waves* or “short windows” averages of isotropic random eigenfunctions on general manifolds (see [7, 8, 18, 21, 47, 64]). These issues are the object of currently ongoing research.

4. Nodal cases: Berry cancellation and the role of the fourth-order chaos

Section 4 has discussed the behaviour of geometric functionals for non-zero threshold levels $u \neq 0$; under isotropy, it has been shown that all these functionals are asymptotically proportional, in the $L^2(\Omega)$ sense, to a single random variable representing the (centred) random $L^2(\mathbb{S}^2)$ -norm of the eigenfunction. This dominant term has been shown to disappear in the nodal case $u = 0$ (and, more generally, for $\rho'_{d-k}(u)u = 0$, i.e., for the singular points $u \in P_j$); the asymptotic behaviour must then be derived by a different route in these circumstances.

As mentioned above, the first paper to investigate the variance of the nodal length for random spherical harmonics was the seminal work by Igor Wigman [61], which made rigorous an *ansatz* by Michael Berry in the physical literature [8]. In particular, by using a higher-order version of the expectation metatheorem (see again [1, 3]) the following representation for the second moment of the nodal length can be given:

$$\mathbb{E}[\{\text{Len}(f_\ell; \mathbb{S}^2)\}^2] = \int_{\mathbb{S}^2 \times \mathbb{S}^2} \mathbb{E}[\|\{\nabla f_\ell(t_1)\}\| \|\{\nabla f_\ell(t_2)\}\| \mid f_\ell(t_1) = 0, f_\ell(t_2) = 0] \\ \times p_{f_\ell(t_1), f_\ell(t_2)}(0, 0) \sigma_g(dt_1) \sigma_g(dt_2),$$

where as before we write $\text{Len}(f_\ell; \mathbb{S}^2) = 2\mathcal{L}_1(A_0(f_\ell; \mathbb{S}^2))$ for the nodal length. The integrand in the previous formula is denoted by the *2-point correlation function of the nodal length* and generalizes the Kac–Rice argument to second-order moments; analogous generalizations are possible for the other geometric functionals we considered and for higher-order moments as well (see [1]). By means of a challenging and careful expansion of this correlation function and a deep investigation of its behaviour for $\ell \rightarrow \infty$, Wigman was able to investigate the asymptotic for the variance of the nodal length and to show that (3.1) holds.

A natural question which was investigated shortly after this seminal paper was the possibility to derive the asymptotic variances of nodal statistics, and further characterizations such as the law of the asymptotic fluctuations, in terms of the Wiener chaos expansions that we discussed in Section 3. The first efforts were devoted to the analysis of the “nodal area” $\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2))$, for which it is easily shown that all even-order terms vanish at $u = 0$; from (3.4) we are then left with (see [42])

$$\text{Var} \{ \mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) \} = \frac{1}{\ell^2} \sum_{q=1}^{\infty} \frac{c_{2q+1}}{2\pi q!} H_{2q}^2(0) + o(\ell^{-2}),$$

where

$$\begin{aligned} c_{2q+1} &= \lim_{\ell \rightarrow \infty} \ell^2 \int_0^\pi P_\ell^{2q+1}(\cos \theta) \sin \theta d\theta \\ &= \int_0^\infty J_0^{2q+1}(\psi) \psi d\psi, \quad J_0(\psi) := \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (x/2)^{2k}}{(k!)^2}. \end{aligned}$$

The computation of the variance and the results in Theorem 3.2 lead easily also to a central limit theorem, which was given first in [40] and then extended to higher dimensions in [50].

Theorem 4.1 ([40]). *As $\ell \rightarrow \infty$, one has*

$$d_W \left(\frac{\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2))]}{\sqrt{\text{Var} \{ \mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) \}}}, Z \right) = o(1),$$

and hence

$$\frac{\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2))]}{\sqrt{\text{Var} \{ \mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) \}}} \rightarrow_d \mathcal{N}(0, 1).$$

The proof of the previous result is standard; in short, the idea is to write

$$\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2)) - \mathbb{E}[\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2))] = \sum_{k=1}^M \frac{(-1)^{2k+1}}{(2k+1)!} H_{2k}(u) \phi(u) h_{\ell;2k+1} + R_M,$$

where the remainder term is such that, as $M \rightarrow \infty$,

$$R_M = \sum_{k=M+1}^{\infty} \frac{(-1)^{2k+1}}{(2k+1)!} H_{2k}(u) \phi(u) h_{\ell;2k+1} = o_p\left(\sqrt{\text{Var}\{\mathcal{L}_2(A_0(f_\ell; \mathbb{S}^2))\}}\right).$$

It is then enough to show that the central limit theorem holds for M (sufficiently large but) finite; this can be achieved by an application of the multivariate fourth moment theorem to the terms $(h_{\ell;3}, \dots, h_{\ell;2M+1})$ (see [46]). It should be noted that in the case of the defect the limiting behaviour depends on the full sequence $\{h_{\ell;2k+1}\}_{k=1,2,\dots}$; this is due to the exact disappearance of the two natural candidates to be leading terms, that is, $\{h_{\ell;2}\}$ and $\{h_{\ell;4}\}$, both whose coefficients vanish for $u = 0$.

It is thus even more remarkable that for the nodal lines the situation simplifies drastically to yield the following result.

Theorem 4.2 ([39]). *As $\ell \rightarrow \infty$, one has*

$$\text{Len}(f_\ell; \mathbb{S}^2) - \mathbb{E}[\text{Len}(f_\ell; \mathbb{S}^2)] = -\frac{1}{4} \sqrt{\frac{\lambda_\ell}{2}} \frac{1}{4!} h_{\ell;4} + o_p\left(\sqrt{\text{Var}\{h_{\ell;4}\}}\right), \tag{4.1}$$

and hence, in view of (3.2)

$$d_W\left(\frac{\text{Len}(f_\ell; \mathbb{S}^2) - \mathbb{E}[\text{Len}(f_\ell; \mathbb{S}^2)]}{\sqrt{\text{Var}\{\text{Len}(f_\ell; \mathbb{S}^2)\}}}, Z\right) = o(1).$$

The most notable aspect of Theorem 4.2 is that the limiting behaviour of nodal lines is asymptotically fully correlated with the sequence of random variables $\{h_{\ell;4}\}$, so that in principle it would be possible to “predict” nodal lengths by simply computing the integral of a fourth-order polynomial of the eigenfunctions over the sphere.

A natural question that arises is the structure of correlation among functionals evaluated at different thresholds and those considered for the nodal case $u = 0$. Focussing for instance on the boundary length, it is immediate to understand that the latter, which is dominated by the second-order chaos term $\{h_{\ell;2}\}$ when $u \neq 0$, must be independent from the nodal length, which is asymptotically proportional to $\{h_{\ell;4}\}$. A more refined analysis, however, should take into account the fluctuations of the boundary length when the effects of the random norm $\|f_\ell\|_{L^2(\mathbb{S}^2)}$ is subtracted, that is, dropping the second-order chaos term from the Wiener expansion. This corresponds to the evaluation of the so-called partial correlation coefficients Corr^* , for which it was shown in [38] that

$$\lim_{\ell \rightarrow \infty} \text{Corr}^*\left(\text{Len}(f_\ell; \mathbb{S}^2), \mathcal{L}_1(A_u(f_\ell; \mathbb{S}^2))\right) = 1.$$

More explicitly, when compensating the effect of random norm fluctuations, the boundary length at any threshold $u \neq 0$ can be fully predicted on the basis of the

	$\mathcal{L}_j(u_1)$	$\mathcal{L}_j(u_2)$	$\text{Len}(0)$	$\text{Len}^*(u)$	$\mathcal{L}_2(0)$	\mathcal{N}_u	$\mathcal{N}_{-\infty}$
$\mathcal{L}_j(u_1)$	1	1	0	0	0	1	0
$\mathcal{L}_j(u_2)$	1	1	0	0	0	1	0
$\text{Len}(0)$	0	0	1	1	0	0	1
$\text{Len}^*(u)$	0	0	1	1	0	0	1
$\mathcal{L}_2(0)$	0	0	0	0	1	0	0
\mathcal{N}_u	1	1	0	0	0	1	0
$\mathcal{N}_{-\infty}$	0	0	1	1	0	0	1

Table 1. The limiting value of $\text{Corr}^2(\cdot, \cdot)$, as $\ell \rightarrow \infty$.

knowledge of the nodal length, up to a remainder term which is asymptotically negligible in the limit $\ell \rightarrow \infty$. It is interesting to note that a similar phenomenon occurs also for the total number of critical points, for which (building on earlier computations in [16]) it was shown in [13] that

$$\mathcal{N}_{-\infty}(f_\ell; \mathbb{S}^2) - \mathbb{E}[\mathcal{N}_{-\infty}(f_\ell; \mathbb{S}^2)] = -\frac{\lambda_\ell}{2^3 3^2 \sqrt{3}\pi} h_{\ell;4} + o_p(\ell^2 \log \ell);$$

as a consequence, the nodal length of random spherical harmonics and the number of their critical points are perfectly correlated in the high-energy limit:

$$\lim_{\ell \rightarrow \infty} \text{Corr}^2(\text{Len}(f_\ell; \mathbb{S}^2), \mathcal{N}_{-\infty}(f_\ell; \mathbb{S}^2)) = 1.$$

Let us now denote by $\text{Len}^*(u)$ the boundary length at level u after the fluctuations induced by the random norm have been subtracted (e.g., after removing its projection on the second-order chaos); moreover, for brevity’s sake we write

$$\begin{aligned} \mathcal{L}_j(A_u(f_\ell; \mathbb{S}^2)) &= \mathcal{L}_j(u), \quad j = 0, 1, 2, \\ \mathcal{N}_u(f_\ell; \mathbb{S}^2) &= \mathcal{N}_u, \quad \text{Len}(f_\ell; \mathbb{S}^2) = \text{Len}(0), \end{aligned}$$

so that $\mathcal{N}_{-\infty}$ is the total number of critical points and $\mathcal{L}_2(0)$ is the excursion area for $u = 0$. The correlation results that we discussed so far can be summarized in Table 1; here, we denote by $u_1, u_2 \neq 0, 1$ any two non-singular threshold values.

5. Eigenfunctions on different domains

For brevity and simplicity’s sake, this survey has focussed only on the behaviour of random eigenfunctions on the sphere. Of course, as mentioned in Section 1, this is just a special case of a much broader research area, including for instance eigenfunctions

on \mathbb{R}^d and on the standard flat torus $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$. We do not even attempt to do justice to these developments, but it is important to mention some of them which are particularly close to the results we discussed for \mathbb{S}^2 .

5.1. Eigenfunctions on the torus: arithmetic random waves

Eigenfunctions on the torus were first introduced in [52] and have then been studied by several other authors; see for instance [10, 23, 26, 33, 36, 53, 54] and the references therein. In dimension 2 these eigenfunctions (*arithmetic random waves*) are defined by the equations

$$\Delta_{\mathbb{T}^2} f_n + E_n f_n = 0, \quad E_n = 4\pi n, \quad n = a^2 + b^2,$$

for $a, b \in \mathbb{Z}$; the dimension of the n th eigenspace is $\mathcal{N}_n := \text{Card}\{a, b \in \mathbb{Z} : a^2 + b^2 = n\}$, while the expected value of nodal lengths is [52]

$$\mathbb{E}[\text{Len}(f_n; \mathbb{T}^2)] = \frac{\sqrt{E_n}}{2\sqrt{2}}.$$

A major breakthrough was then obtained with the derivation of the variance in [26]. In this paper, the authors introduce a probability measure on \mathbb{S}^1 defined by

$$\mu_n(\cdot) := \frac{1}{\mathcal{N}_n} \sum_{a,b:a^2+b^2=n} \delta_{(a,b)}(\cdot),$$

$\delta_{(a,b)}(\cdot)$ denoting the Dirac measure; its k th-order Fourier coefficients are defined by $\widehat{\mu}_n(k) := \int_{\mathbb{S}^1} \exp(ik\theta) \mu_n(d\theta)$. In [26] it is then shown that the variance of nodal lengths has a non-universal behaviour and is proportional to

$$\text{Var} \{ \text{Len}(f_n; \mathbb{T}^2) \} = \frac{1 + \widehat{\mu}_n(4)^4}{512} \frac{E_n}{N_n^2} + o\left(\frac{E_n}{N_n^2}\right), \quad \text{as } n \rightarrow \infty \text{ s.t. } \mathcal{N}_n \rightarrow \infty.$$

It was later shown in [36] that the behaviour of $\text{Len}(\mathbb{T}^2, f_n)$ is dominated by its fourth-order chaos component, similarly to what we observed above for random spherical harmonics (the result on the torus was actually established earlier than the corresponding case for the sphere). More precisely, we have that

$$\begin{aligned} & \text{Len}(f_n; \mathbb{T}^2) - \mathbb{E}[\text{Len}(f_n; \mathbb{T}^2)] \\ &= \sum_{q=2}^{\infty} \text{Proj} [\text{Len}(f_n; \mathbb{T}^2) | 2q] \\ &= \text{Proj} [\text{Len}(f_n; \mathbb{T}^2) | 4] + o_p\left(\sqrt{\text{Var} \{ \text{Len}(f_n; \mathbb{T}^2) \}}\right), \end{aligned}$$

where $\text{Proj}[\cdot|q]$ denotes projection on the q -th-order chaos. On the contrary of what we observed for the case of the sphere, here it is not possible to express the fourth-order chaos as a polynomial functional of the random eigenfunctions $\{f_n\}$ alone. Moreover, the limiting distribution is non-Gaussian and non-universal; i.e., it depends on the asymptotic behaviour of $\lim_{j \rightarrow \infty} \widehat{\mu}_{n_j}(4)$ which varies along different subsequences $\{n_j\}_{j=1,2,\dots}$ (the attainable measures for the weak convergence of the sequences $\{\mu_{n_j}(\cdot)\}_{n \in \mathbb{N}}$ have been investigated in [26, 27]). Further results in this area include [10, 44] for arithmetic random waves in higher dimension and [28] for the excursion area on subdomains of \mathbb{T}^2 ; as mentioned earlier, an extension of Theorem 3.1 to the torus has been given in [14]. It should be noted that arithmetic random waves can be viewed as an instance of random trigonometric polynomials, whose zeroes have been studied, among others, in [2, 4].

5.2. The Euclidean case: Berry’s random waves

Spherical harmonics on the sphere \mathbb{S}^2 are known to exhibit a scaling limit; i.e., after a change of coordinates they converge locally to a Gaussian random process on \mathbb{R}^2 which is isotropic, zero mean, and has covariance function

$$\mathbb{E}[f(x)f(y)] = J_0(2\pi\|x - y\|), \quad x, y \in \mathbb{R}^2, \quad J_0(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(k!)^2 2^{2k}};$$

here $J_0(\cdot)$ corresponds to the standard Bessel functions, for which the following scaling asymptotics hold:

$$P_\ell\left(\cos \frac{\psi}{\ell}\right) \rightarrow_{\ell \rightarrow \infty} J_0(\psi), \quad \psi \in \mathbb{R}.$$

The behaviour of nodal lines $\mathcal{L}_E(f) = \{x \in \mathbb{R}^2 : f(x) = 0, \|x\| < 2\pi\sqrt{E}\}$ can then be studied in the asymptotic regime $E \rightarrow \infty$; this is indeed the physical setting under which Berry first investigated cancellation phenomena in his pioneering paper [8]. The topology of nodal sets for Berry’s random waves was studied in [17, 43, 55] and others. Concerning nodal lengths, a (quantitative) central limit theorem was established in [47], where intersections of independent random waves were also investigated; more recently, [60] proved a result analogous to Theorem 4.2, namely that, as $E \rightarrow \infty$,

$$\begin{aligned} &\mathcal{L}_E(f) - \mathbb{E}[\mathcal{L}_E(f)] \\ &= -\frac{1}{4} \frac{2\pi}{4!} \sqrt{\frac{E}{2}} \int_{\|x\| < 2\pi\sqrt{E}} H_4(f(x)) dx + o_p\left(\sqrt{\text{Var}\{\mathcal{L}_E(f)\}}\right). \end{aligned} \quad (5.1)$$

We expect that results analogous to (4.1) and (5.1) will hold for more general Riemannian waves on 2-dimensional manifolds [64]; extensions to random waves in \mathbb{R}^3

have been studied, among others, in [18], but in these higher-dimensional settings it is no longer the case that nodal volumes are dominated by a single chaotic component.

5.3. Shrinking domains

As a final issue, we recall how some of the previous results can be extended to shrinking subdomains of the torus and of the sphere. In this respect, a surprising result was derived in [5] concerning the asymptotic behaviour of the nodal length on a suitably shrinking subdomain $B_n \subset \mathbb{T}^2$; indeed it was shown that, for density one subsequences in n ,

$$\lim_{n \rightarrow \infty} \text{Corr}(\text{Len}(\mathbb{T}^2, f_n), \text{Len}(\mathbb{T}^2 \cap B_n, f_n)) = 1,$$

entailing that the behaviour of the nodal length on the whole torus is fully determined by its behaviour on any shrinking disk B_n , provided the radius of this disk is not smaller than $n^{-1/2+\varepsilon}$, some $\varepsilon > 0$. Of course, the asymptotic variance and distributions of the nodal length in this shrinking domain are then immediately shown to be the same as those for the full torus, up to a normalizing factor. Interestingly, the same phenomenon does not occur on the sphere, where on the contrary it was shown in [59] that

$$\lim_{\ell \rightarrow \infty} \text{Corr}(\text{Len}(\mathbb{S}^2, f_\ell), \text{Len}(\mathbb{S}^2 \cap B_\ell, f_\ell)) = 0,$$

so that the nodal length when evaluated on a shrinking subset B_ℓ of the 2-dimensional sphere is actually asymptotically independent from its global value; in the same paper, it is indeed shown that (4.1) generalizes to

$$\begin{aligned} & \text{Len}(\mathbb{S}^2 \cap B_\ell, f_\ell) - \mathbb{E}[\text{Len}(\mathbb{S}^2 \cap B_\ell, f_\ell)] \\ &= -\frac{1}{4} \sqrt{\frac{\lambda_\ell}{2}} \frac{1}{4!} h_{\ell;4}(B_\ell) + o_p\left(\sqrt{\text{Var}\{h_{\ell;4}(B_\ell)\}}\right), \end{aligned} \tag{5.2}$$

$$h_{\ell;4}(B_\ell) = \int_{B_\ell} H_4(f_\ell(x)) dx;$$

from this characterization, a central limit theorem follows easily along the same lines that we discussed in Section 4; see [59] for more details and discussion.

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References

- [1] R. J. Adler and J. E. Taylor, *Random Fields and Geometry*. Springer Monogr. Math., Springer, New York, 2007 Zbl [1149.60003](#) MR [2319516](#)
- [2] J. Angst, V.-H. Pham, and G. Poly, Universality of the nodal length of bivariate random trigonometric polynomials. *Trans. Amer. Math. Soc.* **370** (2018), no. 12, 8331–8357 Zbl [1407.42001](#) MR [3864378](#)
- [3] J.-M. Azaïs and M. Wschebor, *Level Sets and Extrema of Random Processes and Fields*. John Wiley & Sons, Hoboken, NJ, 2009 Zbl [1168.60002](#) MR [2478201](#)
- [4] V. Bally, L. Caramellino, and G. Poly, Non universality for the variance of the number of real roots of random trigonometric polynomials. *Probab. Theory Related Fields* **174** (2019), no. 3-4, 887–927 Zbl [07081459](#) MR [3980307](#)
- [5] J. Benatar, D. Marinucci, and I. Wigman, Planck-scale distribution of nodal length of arithmetic random waves. *J. Anal. Math.* **141** (2020), no. 2, 707–749 Zbl [1458.81020](#) MR [4179775](#)
- [6] P. Bérard, Volume des ensembles nodaux des fonctions propres du Laplacien. In *Bony-Sjöstrand-Meyer Seminar, 1984–1985*, p. Exp. No. 14, École Polytech., Palaiseau, 1985 Zbl [0589.58033](#) MR [819780](#)
- [7] M. V. Berry, Regular and irregular semiclassical wavefunctions. *J. Phys. A* **10** (1977), no. 12, 2083–2091 Zbl [0377.70014](#) MR [489542](#)
- [8] M. V. Berry, Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. *J. Phys. A* **35** (2002), no. 13, 3025–3038 Zbl [1044.81047](#) MR [1913853](#)
- [9] J. Buckley and I. Wigman, On the number of nodal domains of toral eigenfunctions. *Ann. Henri Poincaré* **17** (2016), no. 11, 3027–3062 Zbl [1361.35051](#) MR [3556515](#)
- [10] V. Cammarota, Nodal area distribution for arithmetic random waves. *Trans. Amer. Math. Soc.* **372** (2019), no. 5, 3539–3564 Zbl [07089869](#) MR [3988618](#)
- [11] V. Cammarota and D. Marinucci, A quantitative central limit theorem for the Euler-Poincaré characteristic of random spherical eigenfunctions. *Ann. Probab.* **46** (2018), no. 6, 3188–3228 Zbl [1428.60067](#) MR [3857854](#)
- [12] V. Cammarota and D. Marinucci, A reduction principle for the critical values of random spherical harmonics. *Stochastic Process. Appl.* **130** (2020), no. 4, 2433–2470 Zbl [1457.60071](#) MR [4074706](#)
- [13] V. Cammarota and D. Marinucci, A reduction principle for the critical values of random spherical harmonics. *Stochastic Process. Appl.* **130** (2020), no. 4, 2433–2470 Zbl [1457.60071](#) MR [4074706](#)
- [14] V. Cammarota, D. Marinucci, and M. Rossi, Lipschitz-Killing curvatures for arithmetic random waves. 2020, arXiv:[2010.14165](#)
- [15] V. Cammarota, D. Marinucci, and I. Wigman, On the distribution of the critical values of random spherical harmonics. *J. Geom. Anal.* **26** (2016), no. 4, 3252–3324 Zbl [1353.60020](#) MR [3544960](#)

- [16] V. Cammarota and I. Wigman, Fluctuations of the total number of critical points of random spherical harmonics. *Stochastic Process. Appl.* **127** (2017), no. 12, 3825–3869
Zbl [1377.60060](#) MR [3718098](#)
- [17] Y. Canzani and B. Hanin, Local universality for zeros and critical points of monochromatic random waves. *Comm. Math. Phys.* **378** (2020), no. 3, 1677–1712 Zbl [07250102](#)
MR [4150887](#)
- [18] F. Dalmao, A. Estrade, and J. R. León, On 3-dimensional Berry’s model. *ALEA Lat. Am. J. Probab. Math. Stat.* **18** (2021), no. 1, 379–399 Zbl [1465.60022](#) MR [4213863](#)
- [19] F. Dalmao, I. Nourdin, G. Peccati, and M. Rossi, Phase singularities in complex arithmetic random waves. *Electron. J. Probab.* **24** (2019), Paper No. 71 Zbl [1467.60034](#)
MR [3978221](#)
- [20] H. Dehling and M. S. Taqqu, The empirical process of some long-range dependent sequences with an application to U -statistics. *Ann. Statist.* **17** (1989), no. 4, 1767–1783
Zbl [0696.60032](#) MR [1026312](#)
- [21] G. Dierickx, I. Nourdin, G. Peccati, and M. Rossi, Small scale CLTs for the nodal length of monochromatic waves. 2020, arXiv:[2005.06577](#)
- [22] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds. *Invent. Math.* **93** (1988), no. 1, 161–183 Zbl [0659.58047](#) MR [943927](#)
- [23] A. Granville and I. Wigman, Planck-scale mass equidistribution of toral Laplace eigenfunctions. *Comm. Math. Phys.* **355** (2017), no. 2, 767–802 Zbl [1376.58012](#)
MR [3681390](#)
- [24] S. Janson, *Gaussian Hilbert Spaces*. Cambridge Tracts in Math. 129, Cambridge University Press, Cambridge, 1997 Zbl [0887.60009](#) MR [1474726](#)
- [25] M. Kac, On the distribution of values of trigonometric sums with linearly independent frequencies. *Amer. J. Math.* **65** (1943), 609–615 Zbl [0061.13710](#) MR [9061](#)
- [26] M. Krishnapur, P. Kurlberg, and I. Wigman, Nodal length fluctuations for arithmetic random waves. *Ann. of Math. (2)* **177** (2013), no. 2, 699–737 Zbl [1314.60101](#)
MR [3010810](#)
- [27] P. Kurlberg and I. Wigman, On probability measures arising from lattice points on circles. *Math. Ann.* **367** (2017), no. 3-4, 1057–1098 Zbl [1381.11055](#) MR [3623219](#)
- [28] P. Kurlberg, I. Wigman, and N. Yesha, The defect of toral Laplace eigenfunctions and arithmetic random waves. *Nonlinearity* **34** (2021), no. 9, 6651–6684 Zbl [1471.58012](#)
MR [4304493](#)
- [29] A. Logunov, Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. *Ann. of Math. (2)* **187** (2018), no. 1, 221–239 Zbl [1384.58020](#)
MR [3739231](#)
- [30] A. Logunov, Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. *Ann. of Math. (2)* **187** (2018), no. 1, 241–262
Zbl [1384.58021](#) MR [3739232](#)

- [31] A. Logunov and E. Malinnikova, Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimensions two and three. In *50 Years with Hardy Spaces*, pp. 333–344, Oper. Theory Adv. Appl. 261, Birkhäuser/Springer, Cham, 2018 Zbl [1414.31004](#) MR [3792104](#)
- [32] A. Logunov, E. Malinnikova, N. Nadirashvili, and F. Nazarov, The sharp upper bound for the area of the nodal sets of Dirichlet Laplace eigenfunctions. *Geom. Funct. Anal.* **31** (2021), no. 5, 1219–1244 MR [4356702](#)
- [33] R. W. Maffucci, Nodal intersections for arithmetic random waves against a surface. *Ann. Henri Poincaré* **20** (2019), no. 11, 3651–3691 Zbl [1441.58009](#) MR [4019200](#)
- [34] D. Marinucci, A central limit theorem and higher order results for the angular bispectrum. *Probab. Theory Related Fields* **141** (2008), no. 3-4, 389–409 Zbl [1141.60028](#) MR [2391159](#)
- [35] D. Marinucci and G. Peccati, *Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications*. London Math. Soc. Lecture Note Ser. 389, Cambridge University Press, Cambridge, 2011 Zbl [1260.60004](#) MR [2840154](#)
- [36] D. Marinucci, G. Peccati, M. Rossi, and I. Wigman, Non-universality of nodal length distribution for arithmetic random waves. *Geom. Funct. Anal.* **26** (2016), no. 3, 926–960 Zbl [1347.60013](#) MR [3540457](#)
- [37] D. Marinucci and M. Rossi, Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on S^d . *J. Funct. Anal.* **268** (2015), no. 8, 2379–2420 Zbl [1333.60033](#) MR [3318653](#)
- [38] D. Marinucci and M. Rossi, On the correlation between nodal and nonzero level sets for random spherical harmonics. *Ann. Henri Poincaré* **22** (2021), no. 1, 275–307 Zbl [1469.60165](#) MR [4201595](#)
- [39] D. Marinucci, M. Rossi, and I. Wigman, The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020), no. 1, 374–390 Zbl [1465.60044](#) MR [4058991](#)
- [40] D. Marinucci and I. Wigman, The defect variance of random spherical harmonics. *J. Phys. A* **44** (2011), no. 35, 16 Zbl [1232.60039](#)
- [41] D. Marinucci and I. Wigman, On the area of excursion sets of spherical Gaussian eigenfunctions. *J. Math. Phys.* **52** (2011), no. 9, 093301, 21 Zbl [1272.82017](#) MR [2867816](#)
- [42] D. Marinucci and I. Wigman, On nonlinear functionals of random spherical eigenfunctions. *Comm. Math. Phys.* **327** (2014), no. 3, 849–872 Zbl [1322.60030](#) MR [3192051](#)
- [43] F. Nazarov and M. Sodin, On the number of nodal domains of random spherical harmonics. *Amer. J. Math.* **131** (2009), no. 5, 1337–1357 Zbl [1186.60022](#) MR [2555843](#)
- [44] M. Notarnicola, Fluctuations of nodal sets on the 3-torus and general cancellation phenomena. *ALEA Lat. Am. J. Probab. Math. Stat.* **18** (2021), no. 2, 1127–1194 Zbl [1468.60062](#) MR [4282185](#)
- [45] I. Nourdin and G. Peccati, Stein’s method on Wiener chaos. *Probab. Theory Related Fields* **145** (2009), no. 1-2, 75–118 Zbl [1175.60053](#) MR [2520122](#)

- [46] I. Nourdin and G. Peccati, *Normal Approximations with Malliavin Calculus. From Stein's Method to Universality*. Cambridge Tracts in Math. 192, Cambridge University Press, Cambridge, 2012 Zbl [1266.60001](#) MR [2962301](#)
- [47] I. Nourdin, G. Peccati, and M. Rossi, Nodal statistics of planar random waves. *Comm. Math. Phys.* **369** (2019), no. 1, 99–151 Zbl [1431.60025](#) MR [3959555](#)
- [48] D. Nualart and G. Peccati, Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** (2005), no. 1, 177–193 Zbl [1097.60007](#) MR [2118863](#)
- [49] S. O. Rice, Mathematical analysis of random noise. *Bell System Tech. J.* **24** (1945), 46–156 Zbl [0063.06487](#) MR [11918](#)
- [50] M. Rossi, The defect of random hyperspherical harmonics. *J. Theoret. Probab.* **32** (2019), no. 4, 2135–2165 Zbl [07120214](#) MR [4020703](#)
- [51] M. Rossi, Random nodal lengths and Wiener chaos. In *Probabilistic Methods in Geometry, Topology and Spectral Theory*, pp. 155–169, Contemp. Math. 739, Amer. Math. Soc., Providence, RI, 2019 Zbl [1458.60058](#) MR [4033918](#)
- [52] Z. Rudnick and I. Wigman, On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. *Ann. Henri Poincaré* **9** (2008), no. 1, 109–130 Zbl [1142.60029](#) MR [2389892](#)
- [53] Z. Rudnick and I. Wigman, Nodal intersections for random eigenfunctions on the torus. *Amer. J. Math.* **138** (2016), no. 6, 1605–1644 Zbl [1373.58017](#) MR [3595496](#)
- [54] Z. Rudnick, I. Wigman, and N. Yesha, Nodal intersections for random waves on the 3-dimensional torus. *Ann. Inst. Fourier (Grenoble)* **66** (2016), no. 6, 2455–2484 Zbl [1360.60081](#) MR [3580177](#)
- [55] P. Sarnak and I. Wigman, Topologies of nodal sets of random band-limited functions. *Comm. Pure Appl. Math.* **72** (2019), no. 2, 275–342 Zbl [1414.58019](#) MR [3896022](#)
- [56] M. Stecconi, Kac-Rice formula for transverse intersections. *Anal. Math. Phys.* **12** (2022), no. 2, Paper No. 44 Zbl [07493058](#) MR [4386457](#)
- [57] G. Szegő, *Orthogonal Polynomials*. 4th edn., Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975 Zbl [0305.42011](#) MR [0372517](#)
- [58] J. E. Taylor, A Gaussian kinematic formula. *Ann. Probab.* **34** (2006), no. 1, 122–158 Zbl [1094.60025](#) MR [2206344](#)
- [59] A. P. Todino, Nodal lengths in shrinking domains for random eigenfunctions on S^2 . *Bernoulli* **26** (2020), no. 4, 3081–3110 Zbl [07256169](#) MR [4140538](#)
- [60] A. Vidotto, A note on the reduction principle for the nodal length of planar random waves. *Statist. Probab. Lett.* **174** (2021), Paper No. 109090 Zbl [07425242](#) MR [4237481](#)
- [61] I. Wigman, Fluctuations of the nodal length of random spherical harmonics. *Comm. Math. Phys.* **298** (2010), no. 3, 787–831 Zbl [1213.33019](#) MR [2670928](#)
- [62] I. Wigman, On the nodal lines of random and deterministic Laplace eigenfunctions. In *Spectral Geometry*, pp. 285–297, Proc. Sympos. Pure Math. 84, Amer. Math. Soc., Providence, RI, 2012 Zbl [1317.60013](#) MR [2985322](#)

- [63] S.-T. Yau, Open problems in geometry. In *Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA, 1990)*, pp. 1–28, Proc. Sympos. Pure Math. 54, Amer. Math. Soc., Providence, RI, 1993 Zbl [0801.53001](#) MR [1216573](#)
- [64] S. Zelditch, Real and complex zeros of Riemannian random waves. In *Spectral Analysis in Geometry and Number Theory*, pp. 321–342, Contemp. Math. 484, Amer. Math. Soc., Providence, RI, 2009 Zbl [1176.58021](#) MR [1500155](#)

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